


**COVER PAGE**

***Document downloaded by @DAEL***

***Thu Nov 26 23:17:37 2026***
***For personal use***

When automatic English translation is provided, only the original document is authentic.
The EAA cannot be held responsible of any translation error

Bibliographical reference

*Three-Dimensional Theory of Longitudinal-Radial Coupled Vibration for Annular Elastic Cylinder*, Xiaoli Zhang, Shuyu Lin, Yong Wang and Zhiqiang Fu, Acta Acustica **vol. 100** (Number 2), 2014, pp. 254-258

DOI
https://doi.org/10.3813/AAA.918704

**Acta Acustica united with Acustica**
**Vol. 100 (2014) 254 – 258**

DOI 10.3813/AAA.918704

# Three-Dimensional Theory of Longitudinal-Radial Coupled Vibration for Annular Elastic Cylinder

Xiaoli Zhang, Shuyu Lin, Yong Wang, Zhiqiang Fu
Institute of Applied Acoustics, Shaanxi Normal University, Xian, Shaanxi 710062, China. sylin@ snnu.edu.cn

**Summary**
In this paper, the resonance frequencies of longitudinal-radial coupled vibration, and the relationships between the resonance frequencies and the dimensions of the cylinders with comparable longitudinal and radial dimensions are studied. Based on elastic dynamic equation and according to the boundary conditions of free-end annular elastic cylinder, the resonance frequency equation of annular cylinder of longitudinal-radial coupled vibration is derived. The first two order resonance frequencies are calculated by using MATLAB software and their vibrational distributions are simulated by using ANSYS software. The results show that the analytical resonance frequencies are in good agreement with the numerical results. Besides, compared with the analytical, numerical or experimental results in references [1] and [2], it can be also seen that the present theory is suitable for not only annular cylinders but also solid cylinders. So, it is expected that this theory can serve as reference for the design of the longitudinal-radial composite vibrational systems.

**PACS no. 43.25.Ed, 43.40.+s, 43.20.Rz**

## 1. Introduction

Along with the development of ultrasonic technology, transducers with high efficiency and large output power are increasingly needed and widely used in underwater sound and power ultrasonic applications [3, 4, 5]. As one of the basic components of ultrasonic transducers [6, 7], annular elastic cylinder's vibration characteristics have been widely studied by many researchers over a long period of time. Early investigations of the vibrations of solid or hollow cylinders and cylindrical shell structures based on the finite element method were made by Gladwell [8] and Thambiratnam [9]. After that, Gladwell and Vijay further studied the general problem of three-dimensional vibrations of a finite length circular cylinder with traction free surfaces [10]. By using a series solution, the natural frequencies for the vibrations of hollow elastic circular cylinders were found by Hutchinson and El-Azhari [11]. Singal and Williams [12] gave a description for the mode shapes of thick-walled hollow cylinders by using the energy method based on the three-dimensional theory of elasticity. In recent years, based on Kirchhoff basic assumption or elastic thin shell theory, vibration characteristics of short cylinders, long tubes, pipes, and cylindrical shells with large geometrical dimensions have been also given [13, 14, 15, 16, 1]. In the literature [1], using the equivalent mechanical coupling coefficient, the coupled vibration of thin-wall cylinder is reduced to two equivalent one-dimensional (1D) vibrations in radial and axial directions, and then the approximate resonance frequency is obtained. In some other literatures [17, 18, 2, 19, 20], vibration responses of solid cylinders also have accurate theoretical analysis. For example, reference [19] provided extensive and very accurate frequency data determined by the 3-D Ritz analysis for both free-free and fixed-free solid cylinders. However, for annular cylinders with comparable longitudinal and radial dimensions, resonance frequencies of coupled vibration both in its longitudinal and radial directions, and the relationship between the resonance frequencies and its dimensions still have not been obtained.

In order to design ultrasonic transducers with large radiation area in power ultrasonics conveniently, longitudinal-radial coupled vibration for annular elastic cylinder with large dimensions is proposed in the present paper. Based on elastic dynamics equation, according to the free boundary conditions at both ends, inner and outer sides of the annular cylinder, resonance frequency equation of longitudinal-radial coupled vibration for annular cylinder was derived, and the first two order resonance frequencies were given. Then, by using ANSYS software its vibration distribution was simulated. It can be seen that the analytical resonance frequencies are in good agreement with the numerical results.

## 2. Theoretical analysis

An annular elastic cylinder for longitudinal-radial coupled vibration is schematically shown in Figure 1. Its outer and

Received 4 July 2013,
accepted 26 December 2013.



© S. Hirzel Verlag · EAA

inner radii are $a$ and $b$, and $l$ is its length. In cylindrical coordinates $(r, \theta, z)$, $u_r$, $u_\theta$, $u_z$ are radial, circumferential and axial components of the displacement vector $\boldsymbol{u}$ in the direction $(r, \theta, z)$ respectively. Its elastic dynamic equation is [20],

$$\rho\frac{\partial^2\boldsymbol{u}}{\partial t^2} = \mu\nabla^2\boldsymbol{u} + \big(\lambda+\mu\big)\nabla\big(\nabla\cdot\boldsymbol{u}\big). \tag{1}$$

Here,

$$\begin{aligned}\nabla &= \frac{\partial}{\partial r}\boldsymbol{e}_r + \frac{1}{r}\frac{\partial}{\partial\theta}\boldsymbol{e}_\theta + \frac{\partial}{\partial z}\boldsymbol{e}_z,\\ \nabla\cdot\boldsymbol{u} &= \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_z}{\partial z},\\ \nabla^2 &= \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2},\end{aligned} \tag{2}$$

where $\rho$ is density of annular elastic cylinder; $\lambda$ and $\mu$ are Lamé first and second parameters. $\lambda = \upsilon E/((1+\upsilon)(1-2\upsilon) =$, $\mu = E/(2(1+\upsilon))$, $E$ and $\upsilon$ are Young's modulus and Poisson ratio respectively. The annular elastic cylinder is a space axisymmetric element, so, in cylindrical coordinates, stress and strain can be expressed as four independent variables $S_r$, $S_\theta$, $S_z$, $S_{rz}$ and $T_r$, $T_\theta$, $T_z$, $T_{rz}$ respectively [21]. The relationships between strain and displacement and between stress and strain can be expressed as

$$\begin{aligned}&\big[S_r\ S_\theta\ S_z\ S_{rz}\big] = \left[\frac{\partial u_r}{\partial r}\ \frac{u_r}{r}\ \frac{\partial u_z}{\partial z}\ \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right],\\ &\begin{Bmatrix}T_r\\T_\theta\\T_z\\T_{rz}\end{Bmatrix} = \begin{bmatrix}\lambda+2\mu & \lambda & \lambda & 0\\ \lambda & \lambda+2\mu & \lambda & 0\\ \lambda & \lambda & \lambda+2\mu & 0\\ 0 & 0 & 0 & \mu\end{bmatrix}\begin{Bmatrix}S_r\\S_\theta\\S_z\\S_{rz}\end{Bmatrix}.\end{aligned} \tag{3}$$

At the cylinder ends $z = (0, l)$, with both ends free boundary condition $T_z = 0$, the solution of equation (1) is the harmonic function of coordinate $z$ and time $t$, which can be expressed as

$$\begin{aligned}u_r &= \overline{u}(r)\sin\big(k_z z\big)\mathrm{e}^{\mathrm{j}\omega t},\\ u_z &= \overline{w}(r)\cos\big(k_z z\big)\mathrm{e}^{\mathrm{j}\omega t}.\end{aligned} \tag{4}$$

Here, $\mathrm{j} = \sqrt{-1}$, $\omega$ is angular frequency, and $k_z$ is axial wave number, $k_z = n\pi/l$, where $n$ is an integer. Substituting equation (4) into equation (1), then solving for $\overline{u}(r)$ and $\overline{w}(r)$ yields

$$\begin{aligned}\overline{u}(r) &= -k_e\big[C_1J_1(k_er) + C_2Y_1(k_er)\big]\\ &\quad + k_z\big[C_3J_1(k_er) + C_4Y_1(k_er)\big],\\ \overline{w}(r) &= k_z\big[C_1J_0(k_er) + C_2Y_0(k_er)\big]\\ &\quad + k_s\big[C_3J_0(k_er) + C_4Y_0(k_er)\big],\end{aligned} \tag{5}$$

where $J_0$ and $J_1$, $Y_0$ and $Y_1$ are Bessel functions of first kind and second kind.

$$k_e = \sqrt{\frac{\omega^2\rho}{\lambda+2\mu}},\quad k_s = \sqrt{\frac{\omega^2\rho}{\mu} + k_z^2}.$$

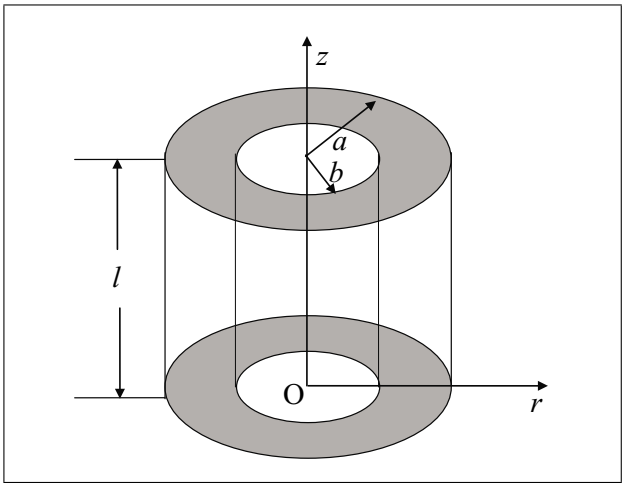



Figure 1. Schematic diagram of annular cylinder for longitudinal-radial coupled vibration.

For annular cylinder of free vibration, the inner and outer boundary conditions can be written as

$$\begin{aligned}T_r\big|_{r=a} &= 0, \qquad T_r\big|_{r=b} = 0,\\ T_{rz}\big|_{r=a} &= 0, \qquad T_{rz}\big|_{r=b} = 0.\end{aligned} \tag{6}$$

Substituting equations (4), (5) and (3) into equation (6) yields resonance frequency equation of longitudinal-radial coupled vibration for annular elastic cylinder,

$$\mathrm{Eqs1}\cdot\mathrm{Eqs4} - \mathrm{Eqs2}\cdot\mathrm{Eqs3} = 0, \tag{7}$$

where

$$\begin{aligned}\mathrm{Eqs1} &= DJ_0(k_ea) + FJ_1(k_ea)/k_ea)\\ &\quad - B\big[J_0(k_sa) - J_1(k_sa)/k_sa)\big]P_3\\ &\quad - B\big[Y_0(k_sa) - Y_1(k_sa)/k_sa)\big]P_4,\\ \mathrm{Eqs2} &= DY_0(k_ea) + FY_1(k_ea)/k_ea)\\ &\quad - B\big[J_0(k_sa) - J_1(k_sa)/k_sa)\big]Q_3\\ &\quad - B\big[Y_0(k_sa) - Y_1(k_sa)/k_sa)\big]Q_4,\\ \mathrm{Eqs3} &= DJ_0(k_eb) + FJ_1(k_eb)/k_eb)\\ &\quad - B\big[J_0(k_sb) - J_1(k_sb)/k_sb)\big]P_3\\ &\quad - B\big[Y_0(k_sb) - Y_1(k_sb)/k_sb)\big]P_4,\\ \mathrm{Eqs3} &= DY_0(k_ea) + FY_1(k_ea)/k_eb)\\ &\quad - B\big[J_0(k_sb) - J_1(k_sb)/k_sb)\big]Q_3\\ &\quad - B\big[Y_0(k_sb) - Y_1(k_sb)/k_sb)\big]Q_4,\end{aligned}$$

$$\begin{aligned}P_3 &= \frac{-A_1\big[J_1(k_ea)Y_1(k_sb) - J_1(k_eb)Y_1(k_sa)\big]}{A_2\big[J_1(k_sa)Y_1(k_sb) - J_1(k_sb)Y_1(k_sa)\big]},\\ Q_3 &= \frac{-A_1\big[Y_1(k_ea)Y_1(k_sb) - Y_1(k_eb)Y_1(k_sa)\big]}{A_2\big[J_1(k_sa)Y_1(k_sb) - J_1(k_sb)Y_1(k_sa)\big]},\\ P_4 &= \frac{-A_1\big[J_1(k_ea)J_1(k_sb) - J_1(k_eb)J_1(k_sa)\big]}{A_2\big[Y_1(k_sa)J_1(k_sb) - Y_1(k_sb)J_1(k_sa)\big]},\\ Q_4 &= \frac{-A_1\big[Y_1(k_ea)J_1(k_sb) - Y_1(k_eb)J_1(k_sa)\big]}{A_2\big[Y_1(k_sa)J_1(k_sb) - Y_1(k_sb)J_1(k_sa)\big]},\end{aligned}$$

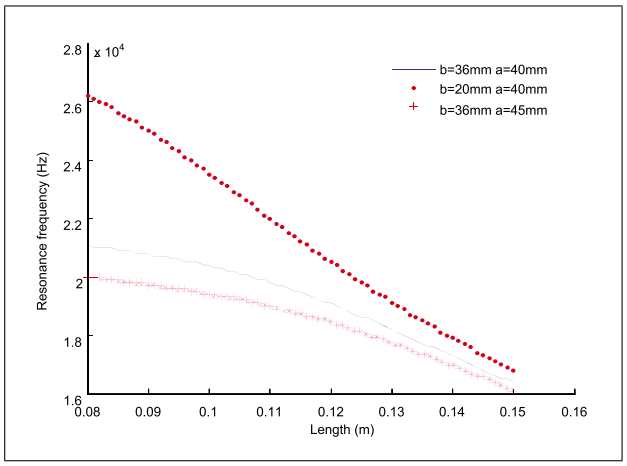



Figure 2. Relation curve between 1st order resonance frequency and length $l$.

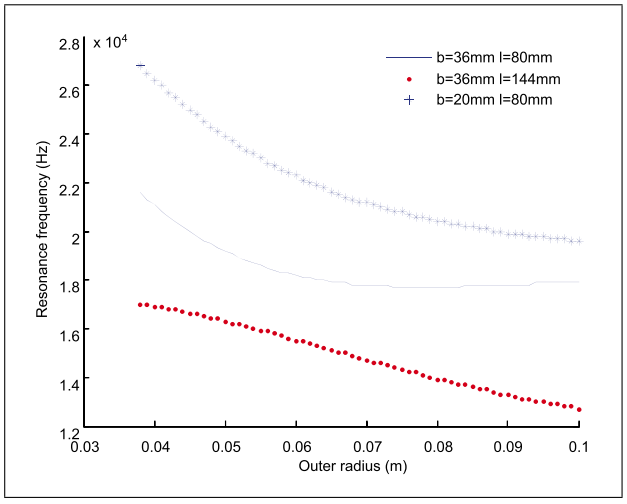



Figure 3. Relation curve of 1st order resonance frequency and outer radius $a$.

$$A_1 = -2\mu k_e k_z, \quad A_2 = -\mu(k_s^2 - k_z^2),$$
$$B = -2\mu k_s k_z, \quad D = -\left[(\lambda + 2\mu)k_e^2 + \lambda k_z^2\right],$$
$$F = 2\mu k_e^2$$

It is obvious that equation (7) depends on the material parameters and geometrical dimensions of the cylinder. The values of angular frequency $\omega$ can be obtained from equation (7). According to $f = \omega/(2\pi)$, the first two frequencies are 1st order resonance frequency and 2nd order resonance frequency from the theoretical analysis of coupled vibration respectively.

Figures 2–4 illustrate the relationships between 1st order resonance frequency and its length, outer and inner radii, respectively. It can be seen from Figure 2 that 1st order resonance frequency decreases with the increase of its length when the inner and outer radii of annular cylinder are fixed. When $2a < l$, along with the increase of its length $l$, three curves almost become a curve, because 1st order vibration gives first place to longitudinal vibration. The vibration mainly depends on the length of the cylinder, when the length is much larger than its radius. From

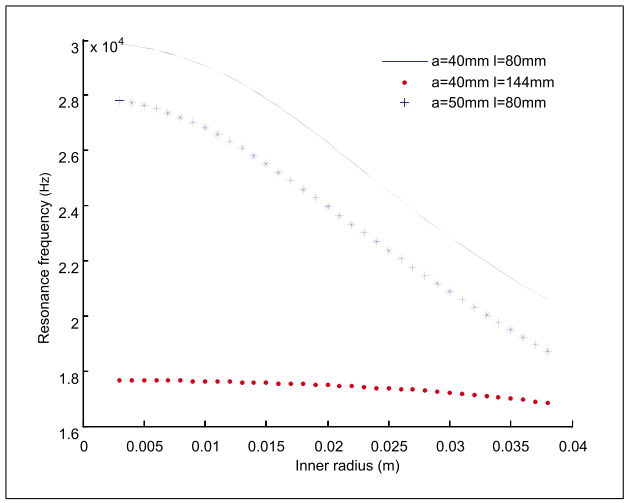



Figure 4. Relation curve of 1st order resonance frequency and inner radius $b$.

Figure 3, when inner radius and length are fixed, 1st order resonance frequency decreases along with the increase of outer radius, namely the increase of its wall thickness. When $2a > l$, along with the increase of its outer radius $a$, the curves of the same length cylinders tend to be a curve, because 1st order vibration gives first place to radial vibration. This vibration mainly depends on the outer radius. As shown in Figure 4, when outer radius and length are fixed, along with the increase of inner radius, namely the decrease of wall thickness, 1st order resonance frequency gradually decreases. When $b \to 0$, the frequency is 1st order resonance frequency of solid cylinder. When $2a < l$, the change of inner radius has little effect on the resonance frequency, because 1st order vibration gives first place to longitudinal vibration which mainly depends on the length of the cylinder.

## 3. Theoretical calculation and numerical simulation

In the above section, the longitudinal-radial coupled vibration of the annular elastic cylinder is studied by using the analytical method. As a supplementary means to verify the analytical results, the numerical method is used to simulate the vibration of the cylinder by using ANSYS software in the following analysis.

The materials used for the annular cylinder is stainless steel. Its standard material parameters are density $\rho = 7800\,\mathrm{kg/m^3}$, Poisson ratio $\sigma = 0.28$, Young modulus $E = 2.09 \cdot 10^{11}\,\mathrm{N/m^3}$. The geometrical dimensions of the annular cylinders are listed in Table I. Based on resonance frequency equation (7) and ANSYS software, the first two order resonance frequencies of the annular cylinders with free boundary condition are given. $f_r$ is the analytical frequency from equation (7) and $f_n$ is the numerical result from the finite element software. Corresponding mode with the theoretical analysis is the major vibrational modal existing in longitudinal-radial coupled vibration.

Table I. Resonance frequencies of annular cylinders.

| $a$ (mm) | $b$ (mm) | $l$ (mm) | Theoretical order | $f_r$ (kHz) | $f_n$ (kHz) | Corresponding mode | $\Delta_r$ (%) |
|---|---|---|---|---|---|---|---|
| 40 | 36 | 80 | 1 | 21.113 | 21.171 | 1st radial vibration | 0.27 |
| | | | 2 | 34.774 | 34.904 | 1st longitudinal vibration | 0.37 |
| 40 | 36 | 144 | 1 | 16.957 | 16.998 | 1st longitudinal vibration | 0.24 |
| | | | 2 | 23.956 | 24.021 | 1st radial vibration | 0.27 |
| 40 | 36 | 50 | 1 | 21.770 | 21.588 | 1st radial vibration | 0.84 |
| | | | 2 | 54.346 | 54.458 | 1st longitudinal vibration | 0.21 |
| 50 | 36 | 144 | 1 | 16.363 | 16.444 | 1st longitudinal vibration | 0.49 |
| | | | 2 | 22.086 | 22.225 | 1st radial vibration | 0.63 |
| 40 | 4 | 80 | 1 | 29.850 | 30.099 | 1st longitudinal vibration | 0.83 |
| | | | 2 | 45.630 | 45.214 | 1st radial vibration | 0.92 |

Table II. Resonance frequencies of solid or hollow cylinders.

| Reference | $a$ (mm) | $b$ (mm) | $l$ (mm) | Theoretical order | $f_r$ (kHz) | $f_{r\Re e(f)}$ (kHz) | $f_n$ or $f_e$ (kHz) |
|---|---|---|---|---|---|---|---|
| [1] | 44.5 | 38.5 | 120 | 1 | 18.193 | 18.193 | 18.303 |
| | | | | 2 | 24.553 | 24.570 | 24.667 |
| | 54 | 48 | 120 | 1 | 15.534 | 15.528 | 15.567 |
| | | | | 2 | 23.397 | 23.407 | 23.440 |
| | 56 | 52.5 | 120 | 1 | 14.702 | 14.699 | 14.756 |
| | | | | 2 | 23.219 | 23.221 | 23.248 |
| | 57 | 51 | 120 | 1 | 14.777 | 14.768 | 14.902 |
| | | | | 2 | 23.232 | 23.240 | 23.348 |
| [2] | 24.95 | 0 | 159.68 | 1 | 15.657 | 15.651 | 15.550 |
| | | | | 2 | 30.709 | 30.729 | 30.510 |

In Table I, $\Delta_r = |f_n - f_r|/f_n$, which can show that the results from the theoretical analysis and the numerical method are in a good agreement with each other. The main reasons of the error are as follows: equation (7) is a transcendental equation, which has no analytical solution, and the accuracy of its numerical solution is related to the step size; when the vibrational order of the cylinder with the same dimensions increases, there can be other vibrational modes, but the theory above is applicable to longitudinal-radial coupled vibration mode. It can be seen from Table I that, when $2a > l$, 1st order vibration gives first place to radial vibration, and 2nd order resonance frequency from theoretical calculation is 1st order longitudinal vibration frequency; when $2a < l$, 1st order vibration gives first place to longitudinal vibration, and 2nd order resonance frequency from theoretical calculation is 1st order radial vibration frequency. However, these curves from section 2 reflect the relationships only between 1st order resonance frequencies from the theoretical analysis and geometric dimensions of the cylinders. As shown in Figure 3, when $b = 36\,\text{mm}$, $l = 80\,\text{mm}$, along with the increase of outer radius, 1st order radial resonance frequency gradually decreases; when $b = 36\,\text{mm}$, $l = 144\,\text{mm}$, along with the increase of outer radius, 1st order longitudinal resonance frequency decreases gradually. From Figure 4 and Table I, the theory is also suitable for solid cylinder, namely $b \to 0$.

For $a = 40\,\text{mm}$, $b = 36\,\text{mm}$, $l = 80\,\text{mm}$, and $a = 40\,\text{mm}$, $b = 36\,\text{mm}$, $l = 144\,\text{mm}$ annular cylinders, based on ANSYS software, 3D finite element models of annular cylinders are established by using Solid45 entity unit, 3D vibrational distributions are simulated as shown in Figures 5 and 6. In the 1st and 2nd theoretical vibration modes, the dotted line is the edge for the model before deformation, and the blue part with the grid is deformed. It can be seen that, when $2a \geq l$, 1st order vibration gives first place to radial vibration; when $2a < l$, 1st order vibration gives first place to longitudinal vibration.

In Table II, $f_r$ is the analytical frequency from equation (7) and $f_{r,\Re e(f)}$ is the resonance frequency obtained in references [1] or [2]. $f_n$ is the numerical result in reference [1], but $f_e$ is the experimental result in reference [2]. Based on the theoretical analysis above and compared with the analyzed, numerical or experimental results in references [1] and [2], it can be seen from Table II that the present theory is suitable for not only annular cylinders but also solid cylinders.

## 4. Conclusions

In this paper, based on elastic dynamic equation and according to the boundary conditions of free-end annular elastic cylinder, the resonance frequency equation of annular cylinder with large dimensions of longitudinal-radial coupled vibration is derived. The first two order resonance frequencies are calculated by using MATLAB software and their vibrational distributions are simulated by using ANSYS software. We can see that, when $2a \geq l$, 1st order vibration is mainly on radial vibration, and when outer radius gradually increases, the resonance frequencies

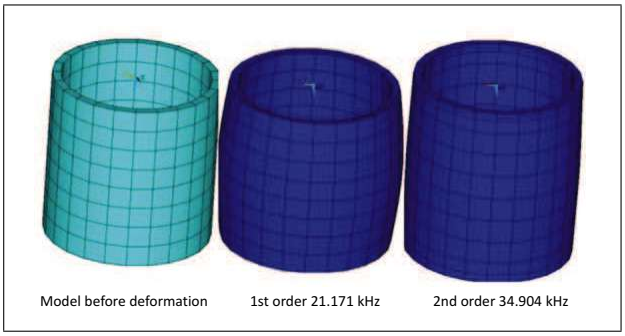



Figure 5. The first two order vibrational modes of a cylinder as $a = 40\,\text{mm}$, $b = 36\,\text{mm}$, $l = 80\,\text{mm}$.

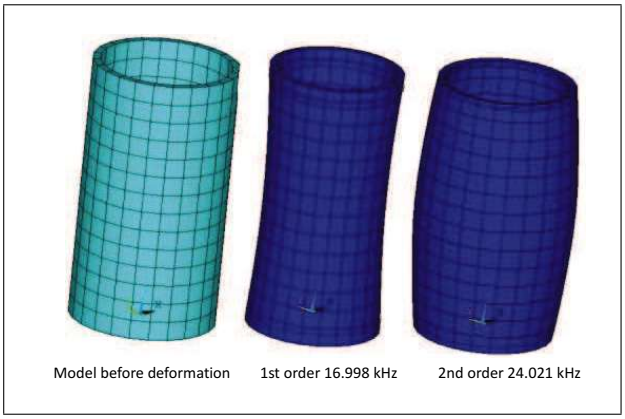



Figure 6. The first two order vibrational modes of a cylinder as $a = 40\,\text{mm}$, $b = 36\,\text{mm}$, $l = 144\,\text{mm}$.

of the same length cylinders converge, because the resonance frequency mainly depends on the outer radius; when $2a < l$, 1st order vibration gives first place to longitudinal vibration, and when the length gradually increases, the resonance frequencies of the same outer radius cylinders converge, because the resonance frequency mainly depends on its length; when the outer radius and length are fixed, along with the increase of inner radius, 1st order resonance frequency gradually decreases. The results show that the analytical resonance frequencies are in good agreement with the numerical results. This theory can serve as reference for the design of the longitudinal-radial composite vibrational systems.

**Acknowledgement**

The author wishes to acknowledge the National Natural Science Foundation of China (113 742 00, 111 741 92), Innovation Funds of Graduate Programs, SNNU. (2013CXB002) and the Research Fund for the Doctoral Program of Higher Education of China (201 102 021 100 10) for their financial support.

**References**

[1] S. Lin: Coupled vibration of isotropic metal hollow cylinders with large geometrical dimensions. Journal of Sound and Vibration **305** (2007) 308–316.

[2] F. J. Nieves, A. Bayon, F. Gascon: Optimization of the Ritz method to calculate axisymmetric natural vibration frequencies of cylinders. Journal of Sound and Vibration **311** (2008) 588–596.

[3] L. Royster: Flextensional underwater acoustic transducer. Journal of the Acoustical Society of America **45** (1969) 671–682.

[4] L. Parrini: New technology for the design of advanced ultrasonic transducers for high-power applications. Ultrasonics **41** (2003) 261–269.

[5] J. Gallego-Juárez, G. Rodriguez, V. Acosta, E. Riera: Power ultrasonic transducers with extensive radiators for industrial processing. Ultrasonics Sonochemistry **17** (2010) 953–964.

[6] S. Lin, L. Xu, W. Xu: A new type of power composite ultrasonic transducer. Journal of Sound and Vibration **330** (2011) 1419–1431.

[7] Z. Liang, G. Xhou, Y. Zhang et al.: Vibration analysis and sound field characteristics of a tubular ultrasonic radiator. Ultrasonics **45** (2006) 146–151.

[8] G. Gladwell, U. Tahbidar: Finite element analysis of the axisymmetric vibrations of cylinders. Journal of Sound and Vibration **22** (1972) 143–157.

[9] D. Thambiratnam, V. Thevendran: Axisymmetric free vibration analysis of cylindrical shell structures using bef analogy. Computers & Structures **43** (1992) 145–150.

[10] G. Gladwell, D. Vijay: Natural frequencies of free finite-length circular cylinders. Journal of Sound and Vibration **42** (1975) 387–397.

[11] J. Hutchinson, S. El-Azhari: Vibration of free hollow circular cylinders. Journal of Applied Mechanics **53** (1986) 641–646.

[12] R. Singal, K. Williams: A theoretical and experiment study of vibrations of thick circular cylindrical shell and rings. Journal of Sound, Acoustics, Stress and Reliability in Design **110** (1988) 533–537.

[13] J. Chandra, K. Ram: Flexural vibrations of finite cylindrical shells of various wall thicknesses. II. Acta Acustica united with Acustica **46** (1980) 283–288.

[14] A. Shermenev: Nonlinear acoustic waves in tubes. Acta Acustica united with Acustica **89** (2003) 426–429.

[15] A. Ghosh: Axisymmetric vibration of a long cylinder. Journal of Sound and Vibration **186** (1995) 711–721.

[16] X. Zhou, J. Sun: Radial-longitudinal coupled vibration of finite thin-walled elastic cylindrical tubes. Acta Acustica with Acustica **21** (1996) 224–230.

[17] F. Nieves, F. Gascon: Estimation of the elastic constants of a cylinder with a length equal to its diameter. Journal of the Acoustical Society of America **104** (1998) 176–179.

[18] D. Ebenezer, K. Ravichandran, C. Padanabhan: Forced vibrations of solid elastic cylinders. Journal of Sound and Vibration **282** (2005) 991–1007.

[19] A. Leissa, J. So: Accurate vibration frequencies of circular cylinders from three dimensional analysis. Journal of the Acoustical Society of America **98** (1995) 2136–2141.

[20] Z. Lin: Principle and design of ultrasonic horn. Science Press, Beijing, 1987. p190.

[21] Z. Xu: Elastic mechanics (Vol. 1). People's Education Press, Beijing, 1982. p204.