

COVER PAGE

Document downloaded by @DAEL

Sat Nov 28 02:37:22 2026
For personal use

When automatic English translation is provided, only the original document is authentic.
The EAA cannot be held responsible of any translation error

Bibliographical reference

Comparing Approaches to the Spherical and Planar Single Layer Potentials for Interior Sound Field Synthesis, Frank Schultz and Sascha Spors, Acta Acustica **vol. 100** (Number 5), 2014, pp. 900-911

DOI
https://doi.org/10.3813/AAA.918769

Acta Acustica united with Acustica
Vol. 100 (2014) 900 – 911

Scientific Papers
DOI 10.3813/AAA.918769

Comparing Approaches to the Spherical and Planar Single Layer Potentials for Interior Sound Field Synthesis

Frank Schultz, Sascha Spors
Institute of Communications Engineering, University of Rostock, R.-Wagner-Str. 31 (Haus 8), 18119 Rostock, Germany. {frank.schultz, sascha.spors}@uni-rostock.de

Summary
Practical approaches to sound field synthesis are described by the single layer potential for which spherical monopoles are driven by appropriate driving functions. This article reviews analytic derivations of known driving functions in the sound field synthesis literature in a consistent framework. It will be shown that the same fundamental methods hold for planar and spherical problems considering a continuous layer of spherical monopoles. The resulting single layer potential can be either solved explicitly as an inverse problem. On the other hand, the Helmholtz integral equation postulates the superposition of a single and a double layer potential which can be simplified to a single layer potential. This is achieved by imposing homogeneous boundary conditions on the involved scalar fields. The equivalent sound-soft scattering approach using the Dirichlet boundary condition on the pressure leads to similar driving functions as derived by the explicit solution. The well known Rayleigh integral equation using the Neumann Green's function incidentally yields an equivalent single layer potential driving function for the planar geometry only.

PACS no. 43.60.Fg, 43.60.Gk, 43.60.Pt, 43.60.Sx, 43.60.Tj

1. Introduction

Sound field synthesis (SFS) aims at synthesizing a virtual, so-called primary source with a so-called secondary source distribution (SSD), which in practice is realized by densely spaced loudspeaker arrays with different geometries. In the last decades research on SFS was undertaken from different viewpoints. Near-Field Compensated Higher-Order Ambisonics (NFC-HOA) [1] and Wave Field Synthesis (WFS) [2] became two well established approaches which were derived differently. While NFC-HOA originally was formulated as a spatially discretized inverse problem, that is numerically solved for a finite number of spherical monopoles located on a sphere, the derivation of WFS was initially performed by imposing a boundary condition on the Helmholtz integral equation (HIE) for a planar SSD using spherical monopoles. Both approaches for SFS were seen as different physical problems in the research community.

In the last decade it was shown that the SFS problem can be formalized using a continuous formulation [3, 4] and introducing spatial discretization and truncation in a second step [5, 6]. The modern formulation of WFS [5] represents an implicit solution, and therefore can be applied also to spherical SSDs [7]. It was shown that an explicit solution applied to the planar geometry, termed Spectral Division Method (SDM) [8] yields equivalent sound fields to WFS. Hence, WFS, SDM and analytic formulations of NFC-HOA [9, 10] are equivalent approaches for a planar and spherical geometry respectively when considering a continuous SSD. The parallels regarding the fundamental solutions were reviewed in [11, 12]. From this framework other known Ambisonics methods, such as Higher-Order Ambisonics and Ambisonics Amplitude Panning, which originate from numerical inversion, can be analytically deduced, cf. [4, Ch. 3.3.4, 3.3.5].

The main differences between WFS and NFC-HOA occur due to the spatial discretization. While WFS is a spatially fullband technique, NFC-HOA is commonly employed with spatial bandwidth limitation, which yields different characteristics of the synthesized sound fields in the spatial and temporal domain [6, 13]. Furthermore, for 2.5D-SFS [14, Ch. 3] the implicit solution of WFS inherently includes a farfield-/high frequency approximation [15], whereas the explicit solution of NFC-HOA does not.

The main concern of this article is to give a compact review on analytic methods for three-dimensional sound field synthesis using the single layer integral representation of a continuous SSD. This includes the explicit solution of the single layer potential (SLP) and alternatively boundary conditions that may be imposed onto the HIE to derive the single layer potential. The review is mainly based on the dissertations of Fazi [3] and Ahrens [4] and

Received 5 May 2014,
accepted 14 July 2014.

© S. Hirzel Verlag · EAA

their subsequent articles [9, 8, 10, 16], as well as the paper [17], that deals with non-uniqueness. The extension of the framework to 2D- and 2.5D-SFS is straightforward.

The numerical solution of the spatially discretized inverse problem, e.g. [18, 19, 20, 21, 22, 23, 24] is not considered in the article. Furthermore, the alternative Ambisonics formulation [25, 26] using the inhomogeneous wave equation, rather than the integral representation is not discussed.

After introducing the conventions and nomenclature in section 2, the article reviews the HIE in section 3. The single layer potential and the applied boundary conditions to the HIE will be recollected in section 4. The approaches that lead to an exact solution for the unknown driving function are discussed in section 5 and 6 for a spherical and a planar SSD respectively. The article concludes in section 7 and recollects sound field expansions for spherical and planar waves and their corresponding driving functions for the two discussed geometries in section A1.

2. Nomenclature

This section defines conventions and notations that are used. With the angles azimuth $\phi \in [0, 2\,\pi)$ and colatitude $\theta \in [0, \pi]$ the position vector

$$\mathbf{x} = (x, y, z) = \|\mathbf{x}\| \cdot (\cos\phi \sin\theta, \sin\phi \sin\theta, \cos\theta)$$

with $\|\mathbf{x}\| = r = \sqrt{x^2 + y^2 + z^2}$ and the wave propagation vector $\mathbf{k}$ is used with $(\omega/c)^2 = \langle \mathbf{k}, \mathbf{k} \rangle$ denoting the scalar product as $\langle \cdot, \cdot \rangle$ and the speed of sound c in m/s. Freefield conditions are assumed. The dispersion relation $(\omega/c)^2 = k_x^2 + k_y^2 + k_z^2$ and the scalar product $\langle \mathbf{k}, \mathbf{x} \rangle = k_x\, x + k_y\, y + k_z\, z$ are used for the description of plane waves in cartesian coordinates. The unit inward normal $\mathbf{n}$ is used. The imaginary number is denoted as j with $\mathrm{j}^2 = -1$. The complex conjugate notation $(\mathrm{e}^{-\mathrm{j}\,\omega\,t})^* = \mathrm{e}^{+\mathrm{j}\,\omega\,t}$ is used. The temporal angular frequency $\omega = 2\,\pi\,f$ in rad/s is linked to the temporal frequency f in Hz.

The temporal Fourier transform sign and normalization convention

$$\begin{aligned} p(\mathbf{x}, t) &= \frac{1}{2\,\pi} \int_{-\infty}^{+\infty} P(\mathbf{x}, \omega)\, \mathrm{e}^{+\mathrm{j}\,\omega\,t}\, \mathrm{d}\omega, \\ P(\mathbf{x}, \omega) &= \int_{-\infty}^{+\infty} p(\mathbf{x}, t)\, \mathrm{e}^{-\mathrm{j}\,\omega\,t}\, \mathrm{d}t, \end{aligned} \tag{1}$$

is used for the relationship of the sound pressure $p(\mathbf{x}, t)$ in the time domain and its temporal spectrum $P(\mathbf{x}, \omega)$.

The spatial Fourier transform sign and normalization convention

$$P(k_x, y, k_z, \omega) = \iint_{-\infty}^{+\infty} P(x, y, z, \omega)\, \mathrm{e}^{+\mathrm{j}(k_x x + k_z z)} \mathrm{d}x\, \mathrm{d}z \tag{2}$$

$$\begin{aligned} P(x, y, z, \omega) = \frac{1}{4\,\pi^2} \iint_{-\infty}^{+\infty} & P(k_x, y, k_z, \omega) \\ & \cdot\, \mathrm{e}^{-\mathrm{j}(k_x x + k_z z)}\, \mathrm{d}k_x\, \mathrm{d}k_z \end{aligned} \tag{3}$$

is used. This implies that the wave vector $\mathbf{k}$ denotes the propagation direction of the plane wave. Therefore, $\mathrm{e}^{-\mathrm{j}\,\langle \mathbf{k}_{\mathrm{PW}}, \mathbf{x} \rangle}\, \mathrm{e}^{+\mathrm{j}\,\omega_{\mathrm{PW}}\,t}$ describes a monochromatic plane wave that propagates into direction of $\mathbf{k}_{\mathrm{PW}}$.

For the chosen conventions the three-dimensional, freefield Green's function with the source position $\mathbf{x_0}$ is defined by [27, Ch. 27.4]

$$G(\mathbf{x}, \mathbf{x_0}, \omega) = \frac{\mathrm{e}^{-\mathrm{j}\,\frac{\omega}{c}\,\|\mathbf{x} - \mathbf{x_0}\|}}{4\,\pi\,\|\mathbf{x} - \mathbf{x_0}\|}. \tag{4}$$

It models the outgoing field of a spherical monopole. Although the notation $G(\mathbf{x}, \mathbf{x_0}, \omega)$ is used for brevity, the symmetry $G(\|\mathbf{x} - \mathbf{x_0}\|, \omega) = G(\|\mathbf{x_0} - \mathbf{x}\|, \omega)$ holds. This inherently includes the invariance of $G(\mathbf{x}, \mathbf{x_0}, \omega)$ regarding spatial translation and rotation for freefield conditions.

The definition of the spherical surface harmonics [28, (2.1.59)]

$$\begin{aligned} Y_n^m(\theta, \phi) = (-1)^m & \sqrt{\frac{2n+1}{4\,\pi} \cdot \frac{(n - |m|)!}{(n + |m|)!}} \\ & \cdot\, \mathrm{P}_n^{|m|}(\cos\theta)\, \mathrm{e}^{+\mathrm{j}\,m\,\phi} \end{aligned} \tag{5}$$

with $n \in \mathbb{N}_0$, $m \in \mathbb{Z}$, $-n \leq m \leq +n$ and the associated Legendre polynomials $\mathrm{P}_n^{|m|}(\cdot)$ [28, Ch. 2.1.2.1] is used. The spherical Bessel function of 1st kind of order n is denoted by $j_n(\cdot)$. Due to the chosen time convention the spherical Hankel function of 2nd kind of order n describes outgoing propagating waves and is denoted $h_n(\cdot) := h_n^{(2)}(\cdot)$ for brevity [29, §10.47]. The shorthand notation of the derivatives $j_n'(x) := \mathrm{d}j_n(x)/\mathrm{d}x$, $h_n'(x) := \mathrm{d}h_n(x)/\mathrm{d}x$ is used. The Wronskian equality $h_n'(x)\, j_n(x) - j_n'(x)\, h_n(x) = 1/(\mathrm{j}\,x^2)$ holds.

For brevity, dependence on the temporal angular frequency ω is omitted for all relevant functions $P(\mathbf{x}, \omega)$, $D(\mathbf{x_0}, \omega)$, $G(\mathbf{x}, \mathbf{x_0}, \omega)$, $P(k_x, y, k_z, \omega)$, $\check{P}(k_x, k_z, \omega)$, $P_n^m(r, \omega)$ and $b_n^m(\omega)$.

3. Helmholtz-Integral Equation (HIE)

This section introduces the HIE. With the closed, source-free volume V, its smooth surface $A = \partial V$, the vector $\mathbf{x_0} \in \partial V$, the position vector $\mathbf{x}$, the inward unit normal vector $\mathbf{n}$ and the Green's function $G(\mathbf{x}, \mathbf{x_0})$, the scalar field of the sound pressure $P(\mathbf{x})$ is given by [30, (2.5)], [31, (8.15)]

$$\begin{aligned} \oint_{\partial V} \left[-\frac{\partial P(\mathbf{x_0})}{\partial n}\, G(\mathbf{x}, \mathbf{x_0}) + \frac{\partial G(\mathbf{x}, \mathbf{x_0})}{\partial n}\, P(\mathbf{x_0}) \right] \mathrm{d}A(\mathbf{x_0}) \\ = \begin{cases} +P(\mathbf{x}) & \forall \mathbf{x} \in V \\ +1/2\, P(\mathbf{x}) & \forall \mathbf{x} \in \partial V \\ 0 & \forall \mathbf{x} \notin V \end{cases} \end{aligned} \tag{6}$$

under the assumption that $P(\mathbf{x})$ is continuously differentiable. The directional derivative $\partial P(\mathbf{x_0})/\partial n$ takes the gradient of $P(\mathbf{x})$ with respect to the field variable $\mathbf{x}$, evaluates

it at $\mathbf{x} = \mathbf{x_0}$ and specifies the scalar product of the resulting vector and inward $\mathbf{n}$ at the respective position $\mathbf{x_0}$, hence

$$\frac{\partial P(\mathbf{x_0})}{\partial n} = \left\langle \operatorname{grad}_{\mathbf{x}} P(\mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x_0}}, \mathbf{n}(\mathbf{x_0}) \right\rangle. \tag{7}$$

The term $\partial G(\mathbf{x}, \mathbf{x_0})/\partial n$ takes the gradient of $G(\mathbf{x}, \mathbf{x_0})$ with respect to the variable $\mathbf{x_0}$ and determines the scalar product with $\mathbf{n}$ at the relevant position $\mathbf{x_0}$,

$$\frac{\partial G(\mathbf{x}, \mathbf{x_0})}{\partial n} = \left\langle \operatorname{grad}_{\mathbf{x_0}} G(\mathbf{x}, \mathbf{x_0}), \mathbf{n}(\mathbf{x_0}) \right\rangle. \tag{8}$$

The directional derivative with $r = \|\mathbf{x} - \mathbf{x_0}\|$ is therefore given as

$$\frac{\partial G(\mathbf{x}, \mathbf{x_0})}{\partial n} = \frac{\mathrm{e}^{-\mathrm{j}\frac{\omega}{c}r}}{4\pi r^3} \cdot \left(1 + \mathrm{j}\frac{\omega}{c} r\right) \cdot \left\langle \mathbf{x} - \mathbf{x_0}, \mathbf{n}(\mathbf{x_0}) \right\rangle. \tag{9}$$

SFS aims at reproducing the pressure field $S(\mathbf{x})$ of a virtual source for which $\partial S(\mathbf{x_0})/\partial n$ and $S(\mathbf{x_0})$ have to be known in order to physically realize the HIE, that is $P(\mathbf{x}) = S(\mathbf{x})\ \forall \mathbf{x} \in V$ and $P(\mathbf{x}) = 0\ \forall \mathbf{x} \notin V$. However, the HIE requires spherical monopoles and dipoles located on A, which has not been considered as a feasible solution up to now. Thus, a simplification of the HIE as the general SFS solution is required for which different approaches are discussed in the next section.

4. Single Layer Potential (SLP)

The SLP is used for practical SFS and aims at synthesizing $P(\mathbf{x}) = S(\mathbf{x})$ for $\mathbf{x} \in V$, i.e. only V is controllable. It is given as

$$P(\mathbf{x}) = \oint_{\partial V} D(\mathbf{x_0})\, G(\mathbf{x}, \mathbf{x_0})\, \mathrm{d}A(\mathbf{x_0}), \tag{10}$$

where the driving function is denoted by $D(\mathbf{x_0})$, also known as the source strength density function. A closed loudspeaker approximates a spherical monopole and thus the Green's function.

4.1. Explicit solution

Mathematically, (10) is known as a Fredholm type integral of the first kind and solving for the unknown driving function is treated as an inverse problem. For the discussed SSDs and in the context of SFS this yields analytic and unique solutions under certain assumptions. This is achieved by the expansion of the known acoustic transfer function $G(\mathbf{x}, \mathbf{x_0})$ and the desired sound field $S(\mathbf{x})$ as well as the unknown driving function $D(\mathbf{x_0})$. In the eigenfunction domain a comparison of the eigenvalues then yields the solution. The approach is comparable for the spherical and planar geometry with a fundamental difference, however. For the spherical problem the number of eigenfunctions are infinite, but denumerable (discrete spectrum of eigenvalues), whereas the eigenfunctions for the planar problem are infinite and non-denumerable (continuous spectrum of eigen'values', i.e. the angular spectrum).

4.2. Dirichlet Boundary Condition on Pressure (DBCP)

The HIE (6) can be reduced to the SLP (10) by imposing suitable boundary conditions on the involved scalar fields. For the homogeneous Dirichlet boundary condition (DBCP)

$$P_{\mathrm{T}}(\mathbf{x}) \Big|_{\mathbf{x}=\mathbf{x_0}} = 0 \tag{11}$$

on the total sound field $P_{\mathrm{T}}(\mathbf{x}) = S(\mathbf{x}) + P_{\mathrm{SC}}(\mathbf{x})$ the reduction of HIE yields the SLP

$$P(\mathbf{x}) = \oint_{\partial V} -\frac{\partial P_{\mathrm{T}}(\mathbf{x_0})}{\partial n}\, G(\mathbf{x}, \mathbf{x_0})\, \mathrm{d}A(\mathbf{x_0}). \tag{12}$$

Here the sound-soft scattered field $P_{\mathrm{SC}}(\mathbf{x})$ has to be known on $A(\mathbf{x_0})$. The total pressure field $P_{\mathrm{T}}(\mathbf{x})$ satisfies the DBCP and (12) synthesizes the desired sound field $P(\mathbf{x}) = S(\mathbf{x})\ \forall \mathbf{x} \in V$. This is known as the formulation of an equivalent sound-soft scattering problem [16]. It is worth pointing out, that the equivalent scattering is a conceptual model where the scatterer appears only notional to derive the unknown driving function. The driving function is explicitly given by the directional derivative of the total sound field on the SSD. The driving function may also be formulated as its spectral representation, which may be more elegant when describing sound fields. As a consequence the equivalence of the solution with the explicit one also becomes apparent.

4.3. Neumann Green's Function (NBCG)

For a Green's function G_{N} that satisfies the homogeneous Neumann boundary condition (NBCG)

$$\frac{\partial G_{\mathrm{N}}(\mathbf{x}, \mathbf{x_0})}{\partial n} = 0 \tag{13}$$

the HIE is reduced to

$$P(\mathbf{x}) = \oint_{\partial V} -\frac{\partial S(\mathbf{x_0})}{\partial n}\, G_{\mathrm{N}}(\mathbf{x}, \mathbf{x_0})\, \mathrm{d}A(\mathbf{x_0}). \tag{14}$$

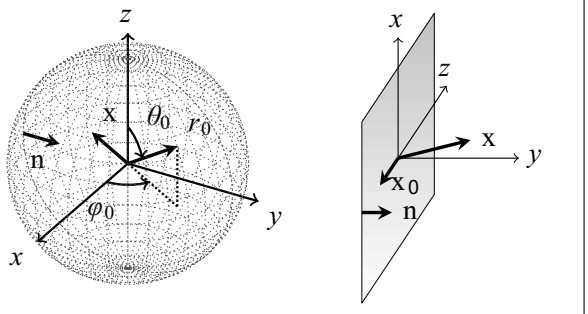
The surface A modeled with Neumann Green's functions can be interpreted as rigid because the Green's function can be interpreted as a velocity potential stemming from a unit source [27, p. 641]. In general $G_{\mathrm{N}}(\mathbf{x}, \mathbf{x_0}) \neq G(\mathbf{x}, \mathbf{x_0})$ and thus, a single layer potential cannot be realized using $G_{\mathrm{N}}(\mathbf{x}, \mathbf{x_0})$. The planar problem can be considered as a special case for which a SLP solution can be derived due to $G_{\mathrm{N}}(\mathbf{x}, \mathbf{x_0}) = 2\, G(\mathbf{x}, \mathbf{x_0})$ [31, Ch. 8.8.3].

The derivation of appropriate driving functions using the spectral representation and the different boundary conditions is presented next for the spherical and planar geometry of the SSD.

5. Spherical secondary source distribution

For interior SFS ($\|\mathbf{x}\| < \|\mathbf{x_0}\|$) the SLP for a spherical volume V with radius $r_0 = \|\mathbf{x_0}\|$ is rewritten in spherical coordinates to

$$P(\mathbf{x}) = \int_0^{2\pi} \int_0^{\pi} D(\mathbf{x_0})\, G(\mathbf{x}, \mathbf{x_0})\, r_0^2\, \sin\theta_0\, \mathrm{d}\theta_0\, \mathrm{d}\phi_0. \tag{15}$$



(a) Spherical SSD (b) Planar SSD

Figure 1. Position of the discussed secondary source distributions in the used coordinate systems.

The continuous SSD is schematically depicted in Figure 1(a). Recall that an explicit solution for the unknown driving function $D(\mathbf{x_0})$ is sought by using the predefined sound field $S(\mathbf{x}) = P(\mathbf{x})$ and the given Green's function $G(\mathbf{x}, \mathbf{x_0})$.

5.1. Explicit solution

The SLP for spherical problems (15) is known as a Fredholm type integral of the first kind. Reference [33] is presumably one of the first monographs treating solution methods of Fredholm integrals by recollecting and enhancing the research of Fredholm, Hilbert, Schmidt and others. This already includes the fundamental solution approach in terms of the spectral representations for the involved functions in (15), cf. [33, Pt. 5]. For that the functions are expanded into the orthonormal set of spherical surface harmonics as eigenfunctions and the problem becomes solvable by comparing the eigenvalues [34, Ch. 6.9]. This is in accordance with the results of [35, p. 940,994] which were recollected in [36],[4, Ch. 3.2]. Later on in mathematics the problem was more formalized in terms of operator theory by representing (15) as

$$P(\mathbf{x}) = (\mathcal{A}D)(\mathbf{x}) = \oint_{\partial V} D(\mathbf{x_0})\, G(\mathbf{x}, \mathbf{x_0})\, \mathrm{d}A(\mathbf{x_0}), \tag{16}$$

meaning that the operator $\mathcal{A}$ acts on $D(\mathbf{x})$ [32, 37]. Due to the bounded, regular spherical surface the linear operator is defined as compact and by the characteristics of the so-called kernel $G(\mathbf{x}, \mathbf{x_0})$ as nonself-adjoint [37, p. 173]. For this operator a spectral expansion is known [37, Ch. 4.3.4] and the solution of the inverse problem and its existence is given by Picard's theorem [38, Ch. VI §4],[32, Ch. 15.4]. This framework was used in [3, Ch. 3.5]. The operator can be expanded into an infinite series of eigenfunctions (modes) and thus has infinite however denumerable eigenvalues.

The essence of the spectral representation is given in what follows, cf. [10, sec. 2.4]. The sound field is decomposed into the spherical harmonics expansion coefficients by the spherical harmonics transform $P_n^m(r) = \mathcal{SHT}\{P(\mathbf{x})\}$ for $r = \|\mathbf{x}\|$ [31, (6.49)], [28, (2.1.68)]

$$P_n^m(r) = \int_0^{2\pi} \int_0^{\pi} P(\mathbf{x})\, Y_n^m(\theta, \phi)^*\, \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi, \tag{17}$$

whereas the expansion is given by its inverse transformation [31, (6.48)], [28, (2.1.65)]

$$P(\mathbf{x}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{+n} P_n^m(r)\; Y_n^m(\theta, \phi). \tag{18}$$

In the remainder of this article the double sum of the Laplace series [39, (12.155)] is denoted by

$$\sum_{n=0}^{\infty} \sum_{m=-n}^{+n} := \sum_{nm}. \tag{19}$$

The expansion of the kernel into the eigenfunctions $Y_n^m(\theta_0, \phi_0)^* \cdot Y_n^m(\theta, \phi)$ and its corresponding eigenvalues for the chosen conventions is given as [17, (12)], [9], [31, (8.22)]

$$G(\mathbf{x}, \mathbf{x_0}) = \sum_{nm} \left[-\mathrm{j}\, \frac{\omega}{c}\, h_n\!\left(\frac{\omega}{c} r_0\right) j_n\!\left(\frac{\omega}{c} r\right)\right] \cdot Y_n^m(\theta_0, \phi_0)^*\, Y_n^m(\theta, \phi) \quad \forall r \le r_0,\ \mathbf{x} \neq \mathbf{x_0}. \tag{20}$$

The complete decomposition of (15) for the desired interior sound field $P(\mathbf{x}) = S(\mathbf{x})$ leads to the equation

$$\underbrace{b_n^m\, j_n(\frac{\omega}{c} r)}_{S_n^m(r)} = D_n^m(r) \left[-\mathrm{j}\, \frac{\omega}{c}\, h_n\left(\frac{\omega}{c} r_0\right)\, j_n\left(\frac{\omega}{c} r\right)\right] r_0^2, \tag{21}$$

which must be satisfied for all coefficients n, m. Equation (21) is evaluated for $r = r_0$. If $j_n(\omega/c\, r_0) \neq 0$ the term cancels out and the solution is unique. For cases $j_n(\omega/c\, r_0) = 0$ equation (21) becomes undetermined (i.e. a Dirichlet eigenvalue). The CHIEF point method as a further constraint to the problem [40] allows for setting up another radius $r_C < r_0$ in order to obtain $j_n(\omega/c\, r_C) \neq 0$ and hence finding the unique solution [17]. However, note that $j_n(\omega/c\, r_C) \neq 0$ requires an appropriate, frequency dependent radius r_C. In [3, Ch. 6.3], [10, sec. 4] this method was referred to as *changing the shape of the control region*. In the limiting case $r_C = 0$ only one CHIEF point in the coordinate system's origin is chosen. It was shown that this choice leads to a frequency independent unique solution [9, p. 991],[4, p. 64]. In [3, Ch. 6.3],[10, sec. 4] this method was referred to as *CHIEF points* as a strategy for overcoming the non-uniqueness of the solution.

All approaches lead to the same unique solution of (21), which is independent of the receiver position $\mathbf{x}$ and reads

$$b_n^m = D_n^m \left[-\mathrm{j}\, \frac{\omega}{c}\, h_n\left(\frac{\omega}{c} r_0\right)\right] \cdot r_0^2. \tag{22}$$

Equation (22) can be solved for D_n^m by division of the known spectral representations of the sound field and the Green's function, provided that they are different from zero. This is the essence of using the spectral expansion

of the integral operator. The procedure is also referred to as mode matching in literature [20]. The expansion (18) yields the driving function on the SSD, cf. e.g. [21, sec. V],[10, (10)]

$$D(\mathbf{x_0}) = \sum_{nm} \frac{b_n^m}{-\mathrm{j}\,\frac{\omega}{c}\,r_0^2\,h_n(\frac{\omega}{c}\,r_0)}\,Y_n^m(\theta_0,\phi_0), \tag{23}$$

which is often referred to as the Ambisonics solution.

An approach for the explicit solution by interpreting (15) as a full 3D-rotational, spatial convolution was introduced in [9, (10)],[4, (3.17)],[12, (3)]. The derivation of the convolution theorem inherently uses the characteristics of the compact, nonself-adjoint operator and of the orthonormal set of the spherical surface harmonics. Therefore, the approach is fully equivalent. The spatial convolution can be expressed in terms of a multiplication in the domain of the spherical harmonics expansion coefficients as [41, (82)], [42, p. 210]

$$S_n^m(r) = r_0^2\,\sqrt{\frac{4\,\pi}{2\,n+1}}\,D_n^m(r)\cdot G_n^0(r). \tag{24}$$

Note that the convolution is originally derived for a unit sphere, therefore the problem at hand is normalized with r_0^2 for arbitrary radii. $G_n^0(r)$ describes the spherical harmonics expansion of a secondary source that is located at the north pole of the sphere $\mathbf{x_0} = (0, 0, r_0)$, i.e. $z = r_0$, $\theta_0 = 0$, $\phi_0 = 0$ and reads

$$G_n^0(r) = -\mathrm{j}\,\frac{\omega}{c}\,h_n\Big(\frac{\omega}{c}\,r_0\Big)\,Y_n^0(\theta_0,\phi_0)^*\,j_n\Big(\frac{\omega}{c}\,r\Big). \tag{25}$$

With [31, (6.63)], [31, (6.25)]

$$Y_n^0(\theta_0,\phi_0)^* = \sqrt{\frac{2\,n+1}{4\,\pi}}\,\mathrm{P}_n(\cos\theta_0), \quad \mathrm{P}_n(1) = 1 \tag{26}$$

it follows that

$$Y_n^0(0,0)^* = \sqrt{\frac{2\,n+1}{4\,\pi}} \tag{27}$$

and (24) can be written as

$$S_n^m(r) = r_0^2\;D_n^m(r)\cdot\Big[-\mathrm{j}\,\frac{\omega}{c}\,h_n\Big(\frac{\omega}{c}\,r_0\Big)\,j_n\Big(\frac{\omega}{c}\,r\Big)\Big], \tag{28}$$

proving the equivalence with (21). The driving functions [9, (14)], [4, (3.21)], [12, (6)] are therefore equivalent to (23).

For convenience the coefficients b_n^m and $S_n^m(r)$ and the resulting driving functions for simple model-based primary source characteristics are given in section A1 using the chosen conventions. For data-based SFS the coefficients $S_n^m(r_0)$ of the desired sound field $S(\mathbf{x})$ may be derived from measurements of $S(\mathbf{x_0})$ with a spherical microphone array by numerical evaluation of (17).

5.2. Dirichlet Boundary Condition on Pressure (DBCP)

In [4, 17, 16] the DBCP was explicitly formulated by an equivalent sound-soft scattering problem [30, (3.33)], [31, Ch. 8.5]

$$\int_0^{2\,\pi}\int_0^{\pi}\underbrace{\Big(-\frac{\partial S(\mathbf{x_0})}{\partial n}-\frac{\partial P_{\mathrm{SC}}(\mathbf{x_0})}{\partial n}\Big)}_{D(\mathbf{x_0})} \cdot G(\mathbf{x},\mathbf{x_0})\,r_0^2\,\sin\theta_0\,\mathrm{d}\theta_0\,\mathrm{d}\phi_0 = \begin{cases} +S(\mathbf{x}), & \mathbf{x}\in V \\ +S(\mathbf{x}), & \mathbf{x}\in\partial V \\ -P_{\mathrm{SC}}(\mathbf{x}), & \mathbf{x}\notin V \end{cases} \tag{29}$$

where $S(\mathbf{x})$ denotes the desired sound field within V and $P_{\mathrm{SC}}(\mathbf{x})$ the scattered field of the imaginary sound-soft sphere onto which $S(\mathbf{x})$ impinges from $\mathbf{x}\notin V$. The driving function is implicitly given, however, the scattered sound field has to be known for this approach, which is elegantly deduced in the eigenfunction domain. Hence, the involved pressure fields are expanded to

$$\begin{Bmatrix} S(\mathbf{x}) \\ P_{\mathrm{SC}}(\mathbf{x}) \end{Bmatrix} = \sum_{nm}\begin{Bmatrix} b_n^m\,j_n(\frac{\omega}{c}\,r) \\ c_n^m\,h_n(\frac{\omega}{c}\,r) \end{Bmatrix} Y_n^m(\theta,\phi). \tag{30}$$

The required boundary condition (11) is imposed onto the decomposed sound fields [16, (21)]

$$\Big[b_n^m\,j_n\Big(\frac{\omega}{c}\,r\Big)+c_n^m\,h_n\Big(\frac{\omega}{c}\,r\Big)\Big]\Big|_{r=r_0} = 0, \tag{31}$$

for which the coefficients of the scattered sound field are determined

$$c_n^m = -b_n^m\,\frac{j_n(\frac{\omega}{c}r_0)}{h_n(\frac{\omega}{c}r_0)}. \tag{32}$$

The required directional derivative in (29) is given as

$$\frac{\partial(\cdot)}{\partial n} = \Big\langle \mathrm{grad}_{\mathbf{x}}(\cdot)\big|_{\mathbf{x}=\mathbf{x_0}}, \mathbf{n}(\mathbf{x_0})\Big\rangle = -\frac{\partial(\cdot)}{\partial r}\Big|_{r=r_0} \tag{33}$$

and is applied to the expansions in (30). This yields the eigenfunction decomposition of (29)

$$\Big[\frac{\omega}{c}\,b_n^m\,j_n'\Big(\frac{\omega}{c}\,r_0\Big)+\frac{\omega}{c}\,c_n^m\,h_n'\Big(\frac{\omega}{c}\,r_0\Big)\Big] \Big[-\mathrm{j}\,\frac{\omega}{c}\,j_n\Big(\frac{\omega}{c}\,r\Big)\,h_n\Big(\frac{\omega}{c}\,r_0\Big)\Big]r_0^2 = b_n^m\,j_n\Big(\frac{\omega}{c}\,r\Big) \tag{34}$$

using the chain rule

$$\frac{\partial j_n(\frac{\omega}{c}\,r)}{\partial r} = \frac{\mathrm{d}\big(\frac{\omega}{c}\,r\big)}{\mathrm{d}r}\,\frac{\mathrm{d}j_n\big(\frac{\omega}{c}\,r\big)}{\mathrm{d}\big(\frac{\omega}{c}\,r\big)} = \frac{\omega}{c}\,j_n'\Big(\frac{\omega}{c}\,r\Big). \tag{35}$$

By evaluating (34) at $r = r_0$ or at a CHIEF point $r < r_0$ to ensure uniqueness as discussed in the previous section, the coefficients of the scattered field in (32) inherently are given when inserting the Wronskian to (34), cf. [16, sec. V],[17, sec. 5]. The driving function can be expanded to

$$D(\mathbf{x_0}) = \sum_{nm}\Big(+\frac{\omega}{c}\,b_n^m\,j_n'(\frac{\omega}{c}\,r_0)+\frac{\omega}{c}\,c_n^m\,h_n'(\frac{\omega}{c}\,r_0)\Big) \cdot Y_n^m(\theta_0,\phi_0). \tag{36}$$

With the coefficients c_n^m and a further simplification with the Wronskian the resulting driving function becomes [4, (3.44)], [17, (21)], [16, (22)]

$$D(\mathbf{x_0}) = -\sum_{nm} \frac{b_n^m \, Y_n^m(\theta_0, \phi_0)}{\mathrm{j}\,\frac{\omega}{c}\, r_0^2 \, h_n\!\left(\frac{\omega}{c}\, r_0\right)}, \tag{37}$$

which is identical to the explicit solution (23). The same driving function following an equivalent so called *simple source approach* [31, Ch. 8.7],[4, Ch. 3.4],[16] was derived in [20, (65)].

5.3. Neumann Green's Function (NBCG)

The reduced HIE under NBCG reads

$$P(\mathbf{x}) = \int_0^{2\pi} \int_0^{\pi} \underbrace{-\frac{\partial S(\mathbf{x})}{\partial n}}_{D_\mathrm{N}(\mathbf{x_0})} G_\mathrm{N,i}(\mathbf{x}, \mathbf{x_0})\, r_0^2 \, \sin\theta_0 \, \mathrm{d}\theta_0 \, \mathrm{d}\phi_0 \tag{38}$$

in spherical coordinates and is referred to as a Rayleigh-like integral [31, (6.150)]. For an interior problem $\|\mathbf{x}\| < r_0$ under NBCG the appropriate interior Neumann Green's function must be used and is expanded to

$$\begin{aligned} G_\mathrm{N,i}(\mathbf{x}, \mathbf{x_0}) \;=\; & \sum_{nm} Y_n^m(\theta_0, \phi_0)^* \, Y_n^m(\theta, \phi) \\ & - \mathrm{j}\,\frac{\omega}{c} \left[h_n\!\left(\frac{\omega}{c}\, r_0\right) j_n\!\left(\frac{\omega}{c}\, r\right) \right. \\ & \left. - j_n\!\left(\frac{\omega}{c}\, r\right) \frac{h_n'\!\left(\frac{\omega}{c}\, r_0\right)}{j_n'\!\left(\frac{\omega}{c}\, r_0\right)} \, j_n\!\left(\frac{\omega}{c}\, r_0\right) \right], \end{aligned} \tag{39}$$

and simplified with the Wronskian

$$\begin{aligned} G_\mathrm{N,i}(\mathbf{x}, \mathbf{x_0}) \;=\; & \sum_{nm} Y_n^m(\theta_0, \phi_0)^* \, Y_n^m(\theta, \phi) \\ & \cdot \frac{1}{\frac{\omega}{c}\, r_0^2} \frac{j_n\!\left(\frac{\omega}{c}\, r\right)}{j_n'\!\left(\frac{\omega}{c}\, r_0\right)} \end{aligned} \tag{40}$$

following [31, Ch. 8.8.1], [43, p. 354]. The decomposition of (38) leads to the equality

$$\underbrace{b_n^m \, j_n\!\left(\frac{\omega}{c}\, r\right)}_{S_n^m(r)} = \underbrace{b_n^m \, \frac{\omega}{c} \, j_n'\!\left(\frac{\omega}{c}\, r_0\right)}_{D_n^m(r)} \cdot \frac{1}{\frac{\omega}{c}\, r_0^2} \, \frac{j_n\!\left(\frac{\omega}{c}\, r\right)}{j_n'\!\left(\frac{\omega}{c}\, r_0\right)} r_0^2, \tag{41}$$

from which the expansion of the driving function for NBCG is directly deduced to

$$D_\mathrm{N}(\mathbf{x_0}) = \sum_{nm} \frac{\omega}{c} \, b_n^m \, j_n'\!\left(\frac{\omega}{c}\, r_0\right) Y_n^m(\theta_0, \phi_0), \tag{42}$$

observing the same issues regarding uniqueness (here further constraints must be imposed for the Neumann eigenvalues). While (42) is satisfying from a theoretical point of view, it is not of practical relevance, since it cannot be used for SFS using the spherical SLP. This is due to the scattering characteristics of the interior Neumann Green's function [31, Ch. 8.8.2] that does not represent a SLP anymore.

An approximation by using the SLP with a secondary source selection (i.e. spatial windowing of the driving function with $w(\mathbf{x_0})$, $w \in \mathbb{R}$) [5, 16, 7]

$$\begin{aligned} P(\mathbf{x}) \;\approx\; & \int_0^{2\pi} \int_0^{\pi} \underbrace{-w(\mathbf{x_0}) \, \frac{\partial S(\mathbf{x_0})}{\partial n}}_{D(\mathbf{x_0})} \\ & \cdot G(\mathbf{x}, \mathbf{x_0})\, r_0^2 \, \sin\theta_0 \, \mathrm{d}\theta_0 \, \mathrm{d}\phi_0 \end{aligned} \tag{43}$$

is commonly used in NBCG-based WFS (cf. sec. 6.3) for arbitraryly shaped SSDs. It is worth realizing that for three-dimensional SFS the result of (43) is comparable with Near-Field Compensated *Infinite* Order Ambisonics (i.e. (23) in (15)).

The exterior Neumann Green's function can be used to model a notional rigid scatterer in order to solve for a unique solution of the SLP driving function. The driving function is derived for the rigid scattering problem [3, (6.61)]

$$\begin{aligned} & S(\mathbf{x}) + P_\mathrm{SC}(\mathbf{x}) = P_\mathrm{T}(\mathbf{x}) = \\ & \int_0^{2\pi} \int_0^{\pi} D(\mathbf{x_0})\, G(\mathbf{x}, \mathbf{x_0})\, r_0^2 \, \sin\theta_0 \, \mathrm{d}\theta_0 \, \mathrm{d}\phi_0 + P_\mathrm{SC}(\mathbf{x}) \end{aligned} \tag{44}$$

using $\|\mathbf{x}\| = r = \mathrm{const} \leq r_0$. With [3, (6.64)] it was shown that (44) can be reformulated to [10, (22)]

$$P_\mathrm{T}(\mathbf{x}) = \int_0^{2\pi} \int_0^{\pi} D(\mathbf{x_0})\, G_\mathrm{N,e}(\mathbf{x}, \mathbf{x_0})\, r_0^2 \, \sin\theta_0 \, \mathrm{d}\theta_0 \, \mathrm{d}\phi_0, \tag{45}$$

using the exterior Neumann Green's function $G_\mathrm{N,e}(\mathbf{x}, \mathbf{x_0})$ [10, (24)], [31, (8.85)]. The imposed boundary condition as a further constraint to the inverse problem yields a unique solution of $D_n^m(r)$ within the spectral representation of (45). The resulting driving function [3, (6.66)], [10, (28)] is precisely equivalent to those from the sections above, cf. (23), (37). A similar approach can be found in [4, Ch. 3.10].

6. Planar secondary source distribution

This section discusses SFS for the planar geometry. The SSD is now defined as an infinite plane $\langle \mathbf{n}, \mathbf{x_0} \rangle = 0$ that includes the coordinate system's origin for convenience. It is assumed that no waves with propagating direction towards the SSD exist, i.e. the Sommerfeld radiation condition [44] for the chosen conventions must be fulfilled. The target half-space of interest $\mathbf{x} \in V$ is in direction of $\mathbf{n}$. For comparability with [8] the problem formulation using $\mathbf{n} = (0, 1, 0)$ and $\mathbf{x_0} = (x_0, 0, z_0)$ is chosen. The SLP is then given by

$$P(\mathbf{x}) = \iint_{-\infty}^{+\infty} D(\mathbf{x_0})\, G(\mathbf{x}, \mathbf{x_0}) \, \mathrm{d}x_0 \, \mathrm{d}z_0 \tag{46}$$

using $\mathbf{x} = (x, y, z)$ with $y > 0$ and the SSD on the xz-plane. The continuous SSD is schematically depicted in Figure 1(b). Propagating and evanescent waves $\langle \mathbf{n}, \mathbf{k} \rangle > 0$

are considered. For the region of interest, cf. [45, Ch. 2.2], [31, Ch. 2.6]

$$k_y = \begin{cases} +\sqrt{(\frac{\omega}{c})^2 - (k_x^2 + k_z^2)} & \text{for } \left(\frac{\omega}{c}\right)^2 > (k_x^2 + k_z^2) \\ -\mathrm{j}\sqrt{(k_x^2 + k_z^2) - (\frac{\omega}{c})^2} & \text{for } (k_x^2 + k_z^2) > \left(\frac{\omega}{c}\right)^2 \end{cases} \tag{47}$$

is defined for propagating, i.e. 1st case in (47) and evanescent (2nd case) wave propagation. Note that for ease of discussion $(\omega/c)^2 \neq (k_x^2 + k_z^2)$, i.e. $k_y \neq 0$ is assumed.

6.1. Explicit solution

The SLP (46) for planar problems is also referred to as a Fredholm type integral of the first kind. Due to the infinite integration limits the SLP is treated as a singular integral equation [35, p. 993],[34, p. 14] and cannot be solved with the methods of compact operators. However, since $G(\mathbf{x}, \mathbf{x_0}) = G(\|\mathbf{x} - \mathbf{x_0}\|) = G(\|\mathbf{x_0} - \mathbf{x}\|)$ holds for the problem at hand, the explicit solution is derived from continuous convolution operator theory in this special case. There the spatial Fourier transform is used [46, Ch. I §6], [34, p. 273]. In this case the number of eigenvalues is infinite, however non-denumerable and therefore describe a continuous function [35, p. 993], the so-called angular spectrum [31, Ch. 2.9], [45, Ch. 2], [4, Ch. 2.2.7]. This is in accordance with the (plane wave) expansion of the functions in (46) with respect to suitable eigenfunctions $\mathrm{e}^{\pm\mathrm{j}\langle\mathbf{kx}\rangle}$ and solving by comparison of the eigenvalues. Note that in contrast to the spherical problem for which an orthonormal set was used, the eigenfunctions here are orthogonal, however Dirac-normalized. The solution framework is presented such that the fundamental parallels to the spherical geometry become obvious. A rigorous treatment can be found in [3, section 4.3.2].

With the spatial Fourier transform pair, the decomposition (2) and the expansion (3) is given similar to the spherical problem formulation in (17) and (18). The existence of the Fourier transforms for arbitrary sound fields must follow the conditions in [45, Ch. 2.5], [47, 48]. The plane wave expansion of the kernel into the eigenfunctions $\mathrm{e}^{-\mathrm{j}(k_x x + k_z z)}$ / $\mathrm{e}^{+\mathrm{j}(k_x x_0 + k_z z_0)}$ and its corresponding eigenvalues is given as

$$G(\mathbf{x}, \mathbf{x_0}) = \frac{1}{4\pi^2} \iint_{-\infty}^{+\infty} \left[-\mathrm{j}\, \frac{\mathrm{e}^{-\mathrm{j}\, k_y |y - y_0|}}{2\, k_y} \cdot \mathrm{e}^{+\mathrm{j}(k_x x_0 + k_z z_0)}\, \mathrm{e}^{-\mathrm{j}(k_x x + k_z z)} \right] \mathrm{d}k_x\, \mathrm{d}k_z \tag{48}$$

using the definitions for k_y from (47) for propagating and evanescent radiation, $|y - y_0| > 0$ and $\mathbf{x} \neq \mathbf{x_0}$. This expansion is well known as the Weyl representation [49, (2.9.10)] or Weyl integral [47, (B2)],[31, (2.64)]. For $(\frac{\omega}{c})^2 = k_x^2 + k_z^2$ singularities arise within the integration path which is either solvable by circumpassing them with infinitesimal semicircles or infinitesimal shifting the singularities from the original integration path [49, Ch. 2.5, 2.9], [27, Ch. 27.9], [43, p. 365]. Equation (48) can also be interpreted as the spatial Fourier transform of $G(\mathbf{x}, \mathbf{0})$ with the shift theorem applied to it. Similar to (21) the complete decomposition of (46) for a sound field that propagates into the target half-space of interest is given as

$$\underbrace{\check{S}(k_x, k_z)\, \mathrm{e}^{-\mathrm{j}\, k_y y}}_{S(k_x, y, k_z)} = D(k_x, y, k_z) \cdot \left[-\mathrm{j}\, \frac{\mathrm{e}^{-\mathrm{j}\, k_y (y - y_0)}}{2\, k_y} \right]. \tag{49}$$

Note the parallels between the modal expansion coefficients $\check{S}(k_x, k_z)$ and b_n^m in planar and spherical geometry respectively. Equation (49) can be evaluated for an arbitrary plane $y_0 < y < \infty$. This cancels $\mathrm{e}^{-\mathrm{j}\, k_y y}$ on both sides of (49). For the planar SSD ($y_0 = 0$) a unique and receiver independent solution of the driving function's angular spectrum, cf. (22)

$$D(k_x, k_z) = \frac{\check{S}(k_x, k_z)}{-\mathrm{j}\, \frac{1}{2\, k_y}} \tag{50}$$

is obtained by division of the known spectral representations of the sound field and the Green's function. Hence the explicit solution for the driving function is given by its expansion on the SSD

$$D(\mathbf{x_0}) = \frac{1}{4\pi^2} \iint_{-\infty}^{+\infty} \frac{\check{S}(k_x, k_z)}{-\mathrm{j}\, \frac{1}{2\, k_y}}\, \mathrm{e}^{-\mathrm{j}(k_x x_0 + k_z z_0)}\, \mathrm{d}k_x\, \mathrm{d}k_z. \tag{51}$$

This solution was derived in e.g. [3, (4.83)], [8, (5)], [4, (3.65)], [12, (10)]. In the latter contributions this approach was introduced as so-called Spectral Division Method (SDM) by interpreting (46) as a spatial convolution integral using $G(\mathbf{x}, \mathbf{0})$ as the reference monopole. The convolution theorem of the Fourier transform is directly applied under the assumption that all involved integrals converge (i.e. $k_y \neq 0$) [8, (53)],[4, (C.11)]. In [50] a similar approach was used for the helical wave spectrum domain emplpoying a cylindrical SSD.

In section A1 the coefficients $\check{S}(k_x, k_z)$ and the resulting driving functions for simple model-based primary source characteristics are given for the chosen conventions. It is worth realizing that analytical Fourier transforms $\check{S}(k_x, k_z)$ may not be available for arbitrary sound fields.

6.2. Dirichlet Boundary Condition on Pressure (DBCP)

The DBCP is formulated by the equivalent sound-soft scattering problem for cartesian coordinates by analogy with (29)

$$\iint_{-\infty}^{-\infty} \underbrace{\left(-\frac{\partial S(\mathbf{x_0})}{\partial n} - \frac{\partial P_{\mathrm{SC}}(\mathbf{x_0})}{\partial n} \right)}_{D(\mathbf{x_0})} G(\mathbf{x}, \mathbf{x_0})\, \mathrm{d}x_0\, \mathrm{d}z_0 = \begin{cases} +S(\mathbf{x}), & \mathbf{x} \in V \\ +S(\mathbf{x}), & \mathbf{x} \in \partial V \\ -P_{\mathrm{SC}}(\mathbf{x}), & \mathbf{x} \notin V. \end{cases} \tag{52}$$

In [16, sec. VI] the scattering was described with the directional derivatives due to the trivial characteristics of scattered sound fields on planar boundaries. In order to draw further parallels to the spherical geometry in section 5.2 the problem is solved again in the eigenfunction domain.

The incident wave field $S(\mathbf{x})$ is decomposed into its angular spectrum

$$S(k_x, y, k_z) = \check{S}(k_x, k_z)\, \mathrm{e}^{-\mathrm{j}\, k_y\, y}. \tag{53}$$

The scattered wave field is given as

$$P_{\mathrm{SC}}(k_x, y, k_z) = \check{P}_{\mathrm{SC}}(k_x, k_z)\, \mathrm{e}^{+\mathrm{j}\, k_y\, y}, \tag{54}$$

since the angular spectrum characteristics regarding k_x and k_z are not changed by the scatterer and only the propagating direction changes into $-y$-direction for planar problems. The boundary condition (11) requires

$$\left[S(k_x, y, k_z) + P_{\mathrm{SC}}(k_x, y, k_z)\right]\Big|_{y=y_0=0} = 0, \tag{55}$$

which is solved to

$$\check{P}_{\mathrm{SC}}(k_x, k_z) = -\check{S}(k_x, k_z). \tag{56}$$

The expansions of (53) and (54), i.e. the inverse spatial Fourier transforms (3), read

$$\begin{Bmatrix} S(\mathbf{x}) \\ P_{\mathrm{SC}}(\mathbf{x}) \end{Bmatrix} = \frac{1}{4\,\pi^2} \iint_{-\infty}^{+\infty} \left(\begin{Bmatrix} \check{S}(k_x, k_z)\, \mathrm{e}^{-\mathrm{j}\, k_y y} \\ \check{P}_{\mathrm{SC}}(k_x, k_z)\, \mathrm{e}^{+\mathrm{j}\, k_y y} \end{Bmatrix} \cdot \mathrm{e}^{-\mathrm{j}\,(k_x x + k_z z)} \right) \mathrm{d}k_x\, \mathrm{d}k_z, \tag{57}$$

by analogy with (30). The driving function is derived by inserting (57) into $D(\mathbf{x_0})$ of (52), calculating the directional derivatives

$$\frac{\partial(\cdot)}{\partial n} = \left\langle \mathrm{grad}_{\mathbf{x}}(\cdot)\Big|_{\mathbf{x}=\mathbf{x_0}}, \mathbf{n}(\mathbf{x_0}) \right\rangle = \frac{\partial(\cdot)}{\partial y}\Big|_{y=y_0=0} \tag{58}$$

and applying the required Dirichlet condition (56). This yields the SLP driving function under DBCP condition

$$D(\mathbf{x_0}) = \frac{1}{4\,\pi^2} \iint_{-\infty}^{+\infty} 2\,\mathrm{j}\, k_y\, \check{S}(k_x, k_z) \cdot \mathrm{e}^{-\mathrm{j}\,(k_x x_0 + k_z z_0)}\, \mathrm{d}k_x\, \mathrm{d}k_z. \tag{59}$$

The remarkable coherence of the DBCP approach (59) with the explicit solution (51) is easily observed. This is precisely in line with the equivalence of the explicit solution (23) and the DBCP approach (37) for the spherical SSD.

The driving function (59) can also be rewritten to

$$D(\mathbf{x_0}) = \frac{\partial}{\partial y} \left(-2\, \frac{1}{4\,\pi^2} \iint_{-\infty}^{+\infty} \overbrace{\check{S}(k_x, k_z)\, \mathrm{e}^{-\mathrm{j}\, k_y\, y}}^{S(k_x, y, k_z)} \cdot \mathrm{e}^{-\mathrm{j}\,(k_x\, x_0 + k_z\, z_0)}\, \mathrm{d}k_x\, \mathrm{d}k_z \right) \Bigg|_{y=y_0=0} \tag{60}$$

by reintroducing the directional derivative. The integral with the $\frac{1}{4\,\pi^2}$ normalization term constitutes an inverse spatial Fourier transform and (60) can be reformulated to

$$D(\mathbf{x_0}) = -2\, \frac{\partial S(\mathbf{x})}{\partial y}\Bigg|_{y=y_0=0} = -2\, \frac{\partial S(\mathbf{x_0})}{\partial n}. \tag{61}$$

Furthermore, the more general notation of the directional derivative $\partial(\cdot)/\partial y|_{y=0} = \partial(\cdot)/\partial n$ can be used for arbitrarily located planes, which is confirmed by observing $\partial P_{\mathrm{SC}}(\mathbf{x_0})/\partial n = \partial S(\mathbf{x_0})/\partial n$ [16, (32)]. The resulting driving function (61) is identical to that derived from the NBCG approach, which is discussed in the next subsection.

6.3. Neumann Green's Function (NBCG)

The reduced HIE under NBCG reads

$$P(\mathbf{x}) = \iint_{-\infty}^{+\infty} -\frac{\partial S(\mathbf{x_0})}{\partial n}\, G_{\mathrm{N}}(\mathbf{x}, \mathbf{x_0})\, \mathrm{d}x_0\, \mathrm{d}z_0. \tag{62}$$

A Green's function that fulfills the NBCG (13) for planar problems is derived by the considerations in [43, Ch. 7.4] and reads

$$G_{\mathrm{N}}(\mathbf{x}, \mathbf{x_0}) = 2 \cdot G(\mathbf{x}, \mathbf{x_0}). \tag{63}$$

By rearranging the factor 2 to the dedicated driving function the SLP under NBCG is given by

$$P(\mathbf{x}) = \iint_{-\infty}^{+\infty} \underbrace{-2 \cdot \frac{\partial S(\mathbf{x_0})}{\partial n}}_{D(\mathbf{x_0})}\, G(\mathbf{x}, \mathbf{x_0})\, \mathrm{d}x_0\, \mathrm{d}z_0, \tag{64}$$

and as a consequence an implicit solution for the unknown driving function is derived, which is well known as a Rayleigh integral equation [51, p. 107],[27, Ch. 23.5], [31, (2.75)], [31, Ch. 8.3.3].

A modern formulation of WFS [5] originated from (64). It is worth realizing that – contrary to 2.5D-SFS – no far-field/high-frequency approximations for this NBCG-derived WFS-method must be applied [5, 15] in three-dimensional problems since the solution is exact provided that $\partial S(\mathbf{x_0})/\partial n$ is known. The initial derivation of WFS in [2] imposes the homogeneous Dirichlet boundary condition for the Green's function, that does not yield the SLP representation in the first instance. With various assumptions and far-field/high-frequency approximations w.r.t the primary source and the SSD a SLP driving function for 2.5D-SFS is obtained, which is referred to as a 2.5D-Rayleigh operator.

By analogy with (44),(45), the problem for planar geometry can also be reformulated to

$$P_{\mathrm{T}}(\mathbf{x}) = \iint_{-\infty}^{+\infty} D(\mathbf{x_0})\, G_{\mathrm{N}}(\mathbf{x}, \mathbf{x_0})\, \mathrm{d}x_0\, \mathrm{d}z_0, \tag{65}$$

which should be fulfilled for a rigid xz-plane at $y = y_0$, $\mathbf{x} \neq \mathbf{x_0}$. By employing the rigid scatterer characteristics $\check{P}_{\mathrm{SC}}(k_x, k_z) = \check{S}(k_x, k_z)$, (65) can be decomposed and solved precisely for $D(k_x, k_z)$ in (50).

It is worth realizing that Rayleigh's initial derivation [51] towards the integral (64) started from the inhomogeneous, linearized wave equation

$$\operatorname{div}\operatorname{grad} P(\mathbf{x}) + \left(\frac{\omega}{c}\right)^2 P(\mathbf{x}) = -Q(\mathbf{x}), \tag{66}$$

with the source distribution $Q(\mathbf{x})$ and imposing homogeneous boundary conditions on the HIE. The integral representation of (66) therefore reads, cf. [27, Ch. 27.4]

$$P(\mathbf{x}) = \int_V Q(\mathbf{x_0})\, G(\mathbf{x}, \mathbf{x_0})\, \mathrm{d}V(\mathbf{x}), \tag{67}$$

and was then simplified towards an infinite planar source distribution of infinitesimal thickness yielding a surface integral

$$P(\mathbf{x}) = \int_S Q_{\text{Plane}}(\mathbf{x_0})\, G(\mathbf{x}, \mathbf{x_0})\, \mathrm{d}A(\mathbf{x_0}). \tag{68}$$

By observing the symmetry of the sound field $P(\mathbf{x})$ regarding the planar SSD and applying the special characteristics of the directional derivatives, the Rayleigh integral was derived in terms of the velocity potential and it was concluded that the plane may be interpreted as rigid, which however is not mandatory [51, p. 104 ff.]. Hence the implicit solution (64) is incidentally equivalent to the explicit solution (51) and therefore also with the DBCP solution (59), which is only valid for the planar geometry. The direct link between the Rayleigh integral equation and the explicit solution using the angular spectrum is well known from Fourier optics and acoustics [31, Ch. 2.10],[45, Ch. 6.5]. It is based on wave field extrapolation using the Weyl integral (48) as a wave field propagator.

The wave field extrapolation characteristics of the Rayleigh integral equation was used in [52, (9)]. The derived driving function's angular spectrum [52, (10)] is equivalent to the representation in (59). The equivalence of the Neumann Rayleigh integral and thus NBCG-WFS and the SDM solution for planar problems was exemplarily confirmed by comparing plane wave driving functions (7) and (22) in [8].

7. Discussion and conclusion

This article recollected methods for the single layer potential in the context of analytic, three-dimensional sound field synthesis using continuous planar and spherical secondary source distributions.

Direct and explicit solutions of the single layer potential are well known from a strict mathematical treatment using the spectral expansion/representation method. For both geometries this also allows for the interpretation of a spatial convolution integral that is solved in the corresponding eigenfunctions' domain. Therefore, the treatments follow the same strategy for planar and spherical secondary source distributions. This implies that the Spectral Division Method and Near-Field Compensated Higher-Order Ambisonics using infinite spatial bandwidth are identical approaches for different coordinate systems, provided that three-dimensional sound field synthesis with a continuous secondary source distribution is considered.

The explicit solution using the spectral representation (section 5.1,(23) and 6.1,(51)) and the equivalent sound-soft scattering approach by imposing the homogeneous Dirichlet boundary condition onto a total sound pressure field in the Helmholtz integral equation (section 5.2,(37) and 6.2,(59)) yield equivalent results for the unknown driving function.

For planar problems the Neumann Green's function leads directly to one of the well-known Rayleigh integral equations, that constitutes an implicit solution (section 6.3,(64)). This solution is incidentally equivalent to the explicit solution and the equivalent sound-soft scattering approach, because the Rayleigh integral can be derived from the angular spectrum representation of the wave field propagator. Therefore, Wave Field Synthesis based on the Neumann Rayleigh integral and the Spectral Division Method are equivalent for three-dimensional sound field synthesis. Note that this holds also for 2D, but not strictly for 2.5D sound field synthesis, since the 2.5D operator is differently derived for the explicit and implicit solution, cf. [4, Ch. 3.9.4], [8, (30)].

The Rayleigh-like integral for interior spherical problems is a solution that cannot be adapted to a single layer potential representation (section 5.3,(38)).

Appendix

A1. Sound Field Expansions and Driving Functions

A comprehensive collection of driving function formulations considering various shaped SSDs (e.g. planar, linear, spherical, cylindrical) and 2D-/2.5D-/3D-SFS can be found in [53], that are also implemented in the Sound Field Synthesis Toolbox [54].

A1.1. Spherical Expansion of a Plane Wave

The spherical expansion of a plane wave with radiating direction into

$$\mathbf{k}_{\text{PW}} = \frac{\omega_{\text{PW}}}{c} \begin{pmatrix} \cos\phi_{\text{PW}} & \sin\theta_{\text{PW}} \\ \sin\phi_{\text{PW}} & \sin\theta_{\text{PW}} \\ & \cos\theta_{\text{PW}} \end{pmatrix}, \tag{A1}$$

is given as

$$\mathrm{e}^{-\mathrm{j}\langle \mathbf{k}_{\text{PW}}, \mathbf{x}\rangle} = \sum_{nm} \overbrace{4\pi\, \mathrm{j}^{-n}\, Y_n^m(\theta_{\text{PW}}, \phi_{\text{PW}})^*}^{b_n^m} \underbrace{j_n\left(\frac{\omega}{c} r\right)}_{S_n^m(r)}\, Y_n^m(\theta, \phi). \tag{A2}$$

Inserting the spectral coefficients b_n^m into (23) yields the driving function [55, (13)],[16, (25)]

$$D(\mathbf{x_0}) = \sum_{nm} \frac{4\pi\, \mathrm{j}^{-n}\, Y_n^m(\theta_{\text{PW}}, \phi_{\text{PW}})^*}{-\mathrm{j}\, \frac{\omega}{c}\, r_0^2\, h_n(\frac{\omega}{c} r_0)}\, Y_n^m(\theta_0, \phi_0). \tag{A3}$$

Note that the coefficients b_n^m are frequency independent for a plane wave.

A1.2. Spherical Expansion of a Spherical Monopole

The spherical interior expansion of a spherical monopole at position

$$\mathbf{x}_{\mathrm{PS}} = r_{\mathrm{PS}} \begin{pmatrix} \cos\phi_{\mathrm{PS}} & \sin\theta_{\mathrm{PS}} \\ \sin\phi_{\mathrm{PS}} & \sin\theta_{\mathrm{PS}} \\ & \cos\theta_{\mathrm{PS}} \end{pmatrix} \tag{A4}$$

with $r_{\mathrm{PS}} = \sqrt{x_{\mathrm{PS}}^2 + y_{\mathrm{PS}}^2 + z_{\mathrm{PS}}^2}$ and $r_{\mathrm{PS}} > r_0$ is given similar to the expansion of the Green's function (20)

$$\frac{\mathrm{e}^{-\mathrm{j}\,\frac{\omega}{c}\,\|\mathbf{x}-\mathbf{x}_{\mathrm{PS}}\|}}{4\,\pi\,\|\mathbf{x}-\mathbf{x}_{\mathrm{PS}}\|} = \tag{A5}$$

$$\sum_{nm} \underbrace{\overbrace{-\mathrm{j}\,\frac{\omega}{c}\,h_n(\frac{\omega}{c}\,r_{\mathrm{PS}})\,Y_n^m(\theta_{\mathrm{PS}},\phi_{\mathrm{PS}})^*}^{b_n^m}\; j_n(\frac{\omega}{c}\,r)}_{S_n^m(r)}\; Y_n^m(\theta,\phi).$$

Inserting the spectral coefficients b_n^m into (23) yields the driving function [4, (5.7)], [17, (16)], [16, (26)]

$$D(\mathbf{x_0}) = \sum_{nm} \frac{h_n(\frac{\omega}{c}\,r_{\mathrm{PS}})\,Y_n^m(\theta_{\mathrm{PS}},\phi_{\mathrm{PS}})^*}{r_0^2\,h_n(\frac{\omega}{c}\,r_0)} Y_n^m(\theta_0,\phi_0). \tag{A6}$$

A1.3. Planar Expansion of a Plane Wave

A monochromatic plane wave with radiation direction $\mathbf{k}_{\mathrm{PW}} = (k_{x,\mathrm{PW}},\, k_{y,\mathrm{PW}},\, k_{z,\mathrm{PW}})$ into the target half-space is decomposed into [8, (49)], [4, (C.6)]

$$S(k_x, y, k_z) = \overbrace{4\,\pi^2\,\delta(k_x - k_{x,\mathrm{PW}})\;\delta(k_z - k_{z,\mathrm{PW}})}^{\check{S}(k_x,k_z)} \cdot \mathrm{e}^{-\mathrm{j}\,k_{y,\mathrm{PW}}\cdot y}. \tag{A7}$$

The driving function (51) is derived to [8, (7)], [4, (3.68)]

$$D(\mathbf{x_0}) = 2\,\mathrm{j}\,k_{y,\mathrm{PW}} \cdot \mathrm{e}^{-\mathrm{j}\,k_{x,\mathrm{PW}}\,x_0}\;\mathrm{e}^{-\mathrm{j}\,k_{z,\mathrm{PW}}\,z_0} \tag{A8}$$

under consideration of the dispersion relation $(\omega_{\mathrm{PW}}/c)^2 = k_{x,\mathrm{PW}}^2 + k_{y,\mathrm{PW}}^2 + k_{z,\mathrm{PW}}^2$.

A1.4. Planar Expansion of a Spherical Monopole

The planar expansion of a spherical monopole at position $\mathbf{x}_{\mathrm{PS}} = (x_{\mathrm{PS}},\, y_{\mathrm{PS}},\, z_{\mathrm{PS}})$ for $y_{\mathrm{PS}} < 0$ and $0 < y < \infty$ is given with (48)

$$\frac{\mathrm{e}^{-\mathrm{j}\,(\omega/c)\,\|\mathbf{x}-\mathbf{x}_{\mathrm{PS}}\|}}{4\,\pi\,\|\mathbf{x}-\mathbf{x}_{\mathrm{PS}}\|} = \frac{1}{4\,\pi^2} \iint_{-\infty}^{+\infty} \Big[\mathrm{e}^{-\mathrm{j}(k_x x + k_y y + k_z z)} \cdot \underbrace{\frac{-\mathrm{j}}{2\,k_y}\,\mathrm{e}^{+\mathrm{j}(k_x x_{\mathrm{PS}} + k_y y_{\mathrm{PS}} + k_z z_{\mathrm{PS}})}}_{\check{S}(k_x,k_z)}\Big]\,\mathrm{d}k_x\,\mathrm{d}k_z \tag{A9}$$

from which $\check{S}(k_x, k_z)$ can be determined. This angular spectrum is applied to (49)

$$\frac{-\mathrm{j}}{2\,k_y}\,\mathrm{e}^{-\mathrm{j}\langle\mathbf{k},\mathbf{x}_{\mathrm{PS}}\rangle}\mathrm{e}^{-\mathrm{j}\,k_y y} = D(k_x, y, k_z) \cdot \left[-\mathrm{j}\,\frac{\mathrm{e}^{-\mathrm{j}\,k_y(y-y_0)}}{2\,k_y}\right]. \tag{A10}$$

Solving for $D(k_x, y, k_z)$ yields

$$D(k_x, y, k_z) = \frac{\frac{-\mathrm{j}}{2\,k_y}\,\mathrm{e}^{-\mathrm{j}\langle\mathbf{k},\mathbf{x}_{\mathrm{PS}}\rangle}\mathrm{e}^{-\mathrm{j}\,k_y y}}{\frac{-\mathrm{j}}{2\,k_y}\,\mathrm{e}^{-\mathrm{j}\,k_y(y-y_0)}}. \tag{A11}$$

By circumpassing the singularity $k_y = 0$ in the integration path or requiring $k_y \neq 0$ the terms $\frac{-\mathrm{j}}{2\,k_y}$ and $\mathrm{e}^{-\mathrm{j}\,k_y y}$ cancel out, which is valid for all $y_0 < y < \infty$, the driving function (51)

$$D(\mathbf{x_0}) = \frac{1}{4\,\pi^2} \iint_{-\infty}^{+\infty} \mathrm{e}^{+\mathrm{j}\langle\mathbf{k},\mathbf{x}_{\mathrm{PS}}-\mathbf{x_0}\rangle}\,\mathrm{d}k_x\,\mathrm{d}k_z \tag{A12}$$

can be described. The explicit solution of this integral is not known in SFS literature. Since it was shown that the explicit solution (51) can be transformed into the form of the implicit solution (64)

$$\begin{aligned} D(\mathbf{x_0}) &= \frac{-2\,\partial S(\mathbf{x_0})}{\partial n} \\ &= -2 \cdot \left\langle \mathrm{grad}_{\mathbf{x}} S(\mathbf{x})\,\big|_{\mathbf{x}=\mathbf{x_0}},\, \mathbf{n}(\mathbf{x_0}) \right\rangle \end{aligned} \tag{A13}$$

the driving function [5, (19)]

$$D(\mathbf{x_0}) = +2\,\frac{\left\langle \mathbf{x_0} - \mathbf{x}_{\mathrm{PS}},\, \mathbf{n}(\mathbf{x_0}) \right\rangle}{\|\mathbf{x_0} - \mathbf{x}_{\mathrm{PS}}\|} \cdot \left(\frac{1}{\|\mathbf{x_0} - \mathbf{x}_{\mathrm{PS}}\|} + \mathrm{j}\,\frac{\omega}{c}\right) \cdot \frac{\mathrm{e}^{-\mathrm{j}\omega/c\,\|\mathbf{x_0}-\mathbf{x}_{\mathrm{PS}}\|}}{4\,\pi\,\|\mathbf{x_0} - \mathbf{x}_{\mathrm{PS}}\|} \tag{A14}$$

was derived in the context of NBCG-based WFS by directly evaluating (A13) for usage with (64).

Acknowledgement

The authors would like to thank Jens Ahrens, Filippo M. Fazi, Till Rettberg, Franz Zotter and the reviewers for valuable comments and discussions while preparing this manuscript.

References

[1] J. Daniel: Spatial sound encoding including near field effect: Introducing distance coding filters and a viable, new Ambisonic format. Proc. 23rd Audio Eng. Soc. Intl. Conf. on Signal Processing in Audio Recording and Reproduction, Helsingør, 2003, #16.

[2] A. J. Berkhout, D. de Vries, P. Vogel: Acoustic control by Wave Field Synthesis. J. Acoust. Soc. Am. **93** (1993) 2764–2778.

[3] F. M. Fazi: Sound field reproduction. Dissertation. University of Southampton, 2010.

[4] J. Ahrens: Analytic methods of sound field synthesis. Springer, Heidelberg, 2012.

[5] S. Spors, R. Rabenstein, J. Ahrens: The theory of Wave Field Synthesis revisited. Proc. 124th Audio Eng. Soc. Conv., Amsterdam, 2008, #7358.

[6] S. Spors, J. Ahrens: A comparison of Wave Field Synthesis and Higher-Order Ambisonics with respect to physical properties and spatial sampling. Proc. 125th Audio Eng. Soc. Conv., San Francisco, 2008, #7556.

[7] S. Spors: Extension of an analytic secondary source selection criterion for Wave Field Synthesis. Proc. 123rd Audio Eng. Soc. Conv., New York, 2007, #7299.

[8] J. Ahrens, S. Spors: Sound field reproduction using planar and linear arrays of loudspeakers. IEEE Trans. Audio, Speech, Language Process **18** (2010) 2038–2050.

[9] J. Ahrens, S. Spors: An analytical approach to sound field reproduction using circular and spherical loudspeaker distributions. Acta Acust united Ac **94** (2008) 988–999.

[10] F. M. Fazi, P. A. Nelson: Nonuniqueness of the solution of the sound field reproduction problem with boundary pressure control. Acta Acust united Ac **98** (2012) 1–14.

[11] F. M. Fazi, P. A. Nelson, R. Potthast: Analogies and differences between three methods for sound field reproduction. Proc. 1st Ambisonics Symposium, Graz, 2009.

[12] J. Ahrens, S. Spors: Applying the Ambisonics approach to planar and linear distributions of secondary sources and combinations thereof. Acta Acust united Ac **98** (2012) 28–36.

[13] J. Ahrens, S. Spors: Alterations of the temporal spectrum in high-resolution sound field reproduction of different spatial bandwidths. Proc. 126th Audio Eng. Soc. Conv., Munich, 2009, #7742.

[14] E. W. Start: Direct sound enhancement by Wave Field Synthesis. Dissertation. Delft University of Technology, 1997.

[15] S. Spors, J. Ahrens: Analysis and improvement of pre-equalization in 2.5-dimensional Wave Field Synthesis. Proc. 128th Audio Eng. Soc. Conv., London, 2010, #8121.

[16] F. M. Fazi, P. A. Nelson: Sound field reproduction as an equivalent acoustical scattering problem. J. Acoust. Soc. Am. **134** (2013) 3721–3729.

[17] F. Zotter, S. Spors: Is sound field control determined at all frequencies? How is it related to numerical acoustics? Proc. 52nd Audio Eng. Soc. Intl. Conf. on Sound Field Control, Guildford, 2013, #1–3.

[18] O. Kirkeby, P. A. Nelson: Reproduction of plane wave sound fields. J. Acoust. Soc. Am. **94** (1993) 2992–3000.

[19] D. B. Ward, T. D. Abhayapala: Reproduction of a plane-wave sound field using an array of loudspeakers. IEEE Trans. Audio, Speech, Language Process **9** (2001) 697–707.

[20] M. A. Poletti: Three-dimensional surround sound systems based on spherical harmonics. J. Audio Eng. Soc. **53** (2005) 1004–1025.

[21] Y. J. Wu, T. D. Abhayapala: Theory and design of sound-field reproduction using continuous loudspeaker concept. IEEE Trans. Audio, Speech, Language Process **17** (2009) 107–116.

[22] G. N. Lilis, D. Angelosante, G. B. Giannakis: Sound field reproduction using the lasso. IEEE Trans. Audio, Speech, Language Process **18** (2010) 1902–1912.

[23] T. Betlehem, C. Withers: Sound field reproduction with energy constraint on loudspeaker weights. IEEE Trans. Audio, Speech, Language Process **20** (2012) 2388–2392.

[24] N. Radmanesh, I. S. Burnett: Generation of isolated wide-band sound fields using a combined two-stage lasso-ls algorithm. IEEE Trans. Audio, Speech, Language Process **21** (2013) 378–387.

[25] F. Zotter, H. Pomberger, M. Frank: An alternative ambisonics formulation: Modal source strength matching and the effect of spatial aliasing. Proc. 126th Audio Eng. Soc. Conv., Munich, 2009, #7740.

[26] F. Zotter, H. Pomberger, M. Noisternig: Energy-preserving ambisonic decoding. Acta Acust united Ac **98** (2012) 37–47.

[27] E. Skudrzyk: The foundations of acoustics. Springer, New York, 1971.

[28] N. A. Gumerov, R. Duraiswami: Fast multipole methods for the Helmholtz equation in three dimensions. Elsevier, Oxford, 2004.

[29] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, C. W. Clark: NIST Handbook of mathematical functions. Cambridge University Press, 2010.

[30] D. Colton, R. Kress: Inverse acoustic and electromagnetic scattering theory. 3rd ed. Springer, New York, 2013.

[31] E. G. Williams: Fourier acoustics, sound radiation and nearfield acoustic holography. Academic Press, London, 1999.

[32] R. Kress: Linear integral equations. Springer, Berlin, 1989.

[33] D. Hilbert: Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen. Teubner, Leipzig, 1912.

[34] J. Kondo: Integral equations. Oxford University Press, Oxford, 1991.

[35] P. M. Morse, H. Feshbach: Methods of theoretical physics, Part I. McGraw-Hill, New York, 1953.

[36] S. Spors, M. Rath, J. Ahrens: Towards a theory for arbitrarily shaped sound field reproduction systems. Abstract and Slides for Acoustics'08, Paris, 2008, #2991.

[37] G. W. Hanson, A. B. Yakovlev: Operator theory for electromagnetics. Springer, New York, 2002.

[38] W. Pogorzelski: Integral equations and their applications, Vol. I. Pergamon Press, Oxford, 1966.

[39] G. B. Arfken, H. J. Weber: Mathematical methods for physicists. 6th ed. Elsevier, Burlington, 2005.

[40] H. A. Schenck: Improved integral formulation for acoustic radiation problems. J. Acoust. Soc. Am. **44** (1968) 41–58.

[41] N. Baddour: Operational and convolution properties of three-dimensional Fourier transforms in spherical polar coordinates. J. Opt. Soc. Am. A **27** (2010) 2144–2155.

[42] J. Driscoll, D. Healy: Computing Fourier transforms and convolutions on the 2-sphere. Adv. Appl. Math. **15** (1994) 202–250.

[43] P. M. Morse, K. U. Ingard: Theoretical acoustics. Princeton University Press, Princeton, 1986.

[44] S. H. Schot: Eighty years of Sommerfeld's radiation condition. Hist. Math. **19** (1992) 385–401.

[45] M. Nieto-Vesperinas: Scattering and diffraction in physical optics. 2nd ed. World Scientific, Singapore, 2006.

[46] S. G. Mikhlin, S. Prössdorf: Singular integral operators. Akademie-Verlag, Berlin, 1986.

[47] E. Lalor: Inverse wave propagator. J. Math. Phys. **9** (1968) 2001–2006.

[48] E. Lalor: Conditions for the validity of the angular spectrum of plane waves. J. Opt. Soc. Am. **58** (1968) 1235–1237.

[49] J. A. DeSanto: Scalar wave theory, Green's functions and applications. Springer, Berlin, 1992.

[50] S. Koyama, K. Furuya, Y. Hiwasaki, Y. Haneda, Y. Suzuki: Sound field reproduction using multiple linear arrays based on wave field reconstruction filtering in helicalwave spectrum domain. Proc. 38th Intl. Conf. on Acoustics, Speech and Signal Processing (IEEE 38th ICASSP), Vancouver, 2013, 271–275.

[51] J. W. S. 3rd Baron Rayleigh: The theory of sound, Vol. II. 2nd ed. Macmillan & Co., New York, 1896.

[52] S. Koyama, K. Furuya, Y. Hiwasaki, Y. Haneda: Sound field recording and reproduction using transform filter designed in spatio-temporal frequency domain. Proc. 131st Audio Eng. Soc. Conv., New York, 2011, #8544.

[53] H. Wierstorf: Perceptual assessment of sound field synthesis. Dissertation. TU Berlin, 2014.

[54] H. Wierstorf, S. Spors: Sound Field Synthesis Toolbox. Proc. 132nd Audio Eng. Soc. Conv., Budapest, Hungary, 2012. eBrief: #50.

[55] J. Ahrens, S. Spors: Analytical driving functions for higher order Ambisonics. Proc. 33rd Intl. Conf. on Acoustics, Speech and Signal Processing (IEEE 33rd ICASSP), Las Vegas, 2008, 373–376.