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Discrete Acoustical Resonators with Harmonic Eigenfrequencies

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Summary

The present paper deals with the calculation of resonators built as a concatenation of cylinders with equal lengths, named discrete resonators hereafter, and having harmonic eigenfrequencies when closed at the input. It is shown that for a given harmonic series there is a large variety of solutions due to the periodicity and symmetry of the impedance spectrum. Some resolution methods are proposed. Solutions are presented and discussed for resonators made with one to six cylinders. Results give some insight useful for the optimum design of reed or brass instruments.

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1. Introduction

For most, if not all, musical instruments, harmonicity of eigenfrequencies is known to be of great importance. For instance, setting aside for a short while the family of wind instruments that are dealt with here, high quality bells have approximately harmonic resonance frequencies and some drums such as the tabla are modified in order to obtain harmonicity of the first modes [1, 2]. The main objective is to get quasiperiodic vibrations that are perceived rather more as sounds than as noises. This constraint of harmonicity is the reason for the large use of strings that can be plucked, struck or bowed. For wind instruments, harmonicity of the eigenfrequencies of the pipe has an even more crucial importance, especially for reed instruments [3]. These last ones can have mainly two geometries: the cylinder for clarinets and the cone for instruments in the oboe and saxophone families [4]. For practical reasons only truncated cones can be used, for which resonances are only approximately harmonic at low frequencies. This makes the optimization of the embouchure rather tricky because poor harmonicity affects the playability of the instruments, some notes being unstable or hard to play [5]. Aside cylinders and cones, other geometries have been found: the so-called “stepped cones” [6], approximating a cone as a discrete sequence of an arbitrary number N of cylinders with equal lengths and cross-sections defined by the series $a_i = i(i+1)/2a_1$, a_i being the i th cylinder cross-section area. Some bamboo saxophones exemplify it, with a geometry close to this last theoretical shape [7].

In their paper, Dalmont and Kergomard [6] conjectured that the above sequence of cross-sections would define the only geometry with the wanted harmonicity property but recently many other sequences [8, 9], actually an infinity, appeared sharing this property. It is thus interesting to try to classify this set of resonators, by imposing other properties. Early investigations [8] unveiled the existence of such resonators, relying upon a graph modelling approach of wind instruments [9, 10]: the scope was to illustrate the non uniqueness of harmonic piecewise cylindrical resonators for a given fundamental frequency, while making appear the possibility to have large gaps in the series of partials. Recently [11], it was shown that imposing orthogonality of characteristic polynomials [9] forces the discrete resonators to be stepped-cones only, when harmonicity is looked for.

With this additional condition, the Dalmont Kergomard [6] conjecture becomes true. The object of the present work is to further investigate, with a different point of view, the possible geometries of piecewise cylindrical resonators. This is done through: 1) focusing instead on a particular class of harmonic series (called *M-natural series* below), that have a missing partial depending on the number of pieces of the resonator. Among these series, only the two known series of all (for cones) or only odd (for cylinders) integer multiples of a fundamental frequency seem to have received attention; 2) deepening the results of previous work [8] on several other discrete resonators having from 2 to 6 pieces. One proceeds differently, using the more familiar transmission matrix method (TMM).

Nevertheless, full connections between, and complementarity of, both approaches are not clear at this time. As a consequence, it is fair to say that the study of this large family of resonators is far from being complete by

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now. As these resonators are built as a discrete sequence of cylinders, from now on they will be called *discrete resonators*.

The following assumptions hold in this work: 1) restriction is done to reed-type instruments for which the pipe is considered as a resonator closed at the input - including the equivalent volume of the mouthpiece - and open at the output; 2) resonators are lossless and corrections for discontinuities and radiation are ignored: in other words, at an end where the pipe is closed, the boundary condition is $U = 0$, with U the volume velocity, while when the end is open the boundary condition is $p = 0$, with p the acoustic pressure. This might appear as a rough approximation but it was shown [6] that neglecting discontinuities and radiation corrections do not affect significantly harmonicity; 3) resonators are made with equal lengths cylindrical pieces (Figure 1) although continuous resonators could be considered as well, as a limiting case when the number of cylinders tends to infinity while their length goes to zero.

After having fixed the notations and defined the series of interest (the so-called M -series), some results are presented in section 2 for the first five M -series and different numbers of cylindrical tubes. In section 3 other resonators with up to 6 cylindrical pieces are studied. Finally, in the conclusion, a discussion is made on some insights resulting of the present investigation. Useful mathematical results for discrete resonators with any number N of equal lengths cylinders are given in the appendix, to which one can refer for the formal computations details.

Notations

Consider a discrete resonator with N pieces such as in Figure 1: its geometry is defined by specifying the cross-sections areas a_i and is, for convenience, denoted with brackets: $[a_1 \ a_2 \ a_3 \ a_4 \ \dots]$, with a_1 arbitrarily set to 1 or another integer. As detailed in the appendix, its eigenfrequencies, denoted f_i , are obtained from the roots of a polynomial of degree N in the variable $t = \tan(kL)$ with $k = 2\pi f/c$ and c the sound velocity in free air. For a given resonator, the fundamental frequency f_1 is related to the length L which is arbitrary. Its series of eigenfrequencies is denoted with parenthesis: $(f_1/f, f_2/f, f_3/f, f_4/f, \dots)f$ with $f = f_1$ or any other frequency. After having defined what is called a M -natural series, several examples are studied in detail.

2. M -natural series

To begin with, define the “natural” series as the series of all integer multiples of some fundamental frequency f_1 . For a discrete resonator made with N pieces closed at the input and open at the output, any integer multiple of $N + 1$ times the fundamental frequency f_1 cannot be an eigenfrequency (see [8, 9] and Appendix). Thus the “natural” series cannot be obtained with a finite number of cylinders. For this reason, generalizing the natural series, M -natural series are defined as series in which the term of rank $M + 1$

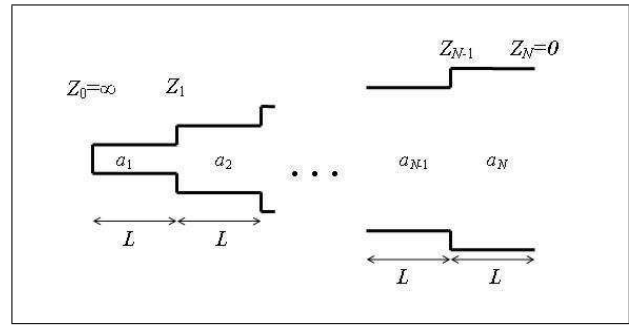


Figure 1. Sketch of a resonator made with cylinders of equal lengths.

is lacking. It can be obtained from a discrete resonator with some integer multiple of M cylinders but from other resonators too. The 1-natural series is the usual series of odd multiples of the fundamental frequency ($M + 1 = 2$ i.e. even multiples of are lacking), the 2-natural series lacks integer multiples of 3 f_1 thus is given as: (1, 2, 4, 5, 7, 8, ...) f_1 and so on. The mere natural series is obtained by letting formally M tend to infinity. To be still more precise, for a discrete resonator to have harmonic resonances, kL/π must be a rational number [9]. Thus, the eigenfrequencies must satisfy: $f_i = p f_1$ with $f_1 = f_{M+1}/q$, p and q integers and $f_{M+1} = c/(2L)$. Choosing $q = M + 1$ corresponds to the eigenfrequencies of the stepped cone family found by Dalmont and Kergomard [6]. This can be generalized by realizing the M -natural series with a discrete resonator having any number N of pieces. The eigenfrequencies are then defined by the equality

$$k_i L = i \frac{M\pi}{(M+1)N} \quad \text{with } i \text{ non multiple of } M+1. \quad (1)$$

N is a multiple of M and $N = M$ corresponds to the eigenfrequencies of the stepped cone family. These M -natural series are investigated below for M from 1 to 6 and some values of N . In each case, the natural frequencies are computed as the roots of the corresponding polynomial (see Appendix, Equation A4) according to the general results in the appendix. Notice that only the mere natural series (for cones) and 1-natural series (for cylinders) seem to have been studied until now in the literature.

2.1. 1-natural series

For this series, i.e. the odd multiples of f_1 , the cylinder is the most well-known instance. Even in that simple case, no other geometry was evidenced before. The series is obtained from the equality (1) that writes: $k_i L = i\pi/(2N)$ and i odd ($i \in \mathbf{N} - 2\mathbf{N}$).

If $N = 1$ the solution is obvious: the condition $D_1 = \cos(kL) = 0$ (see equations (A6) of the Appendix) leads to $kL = \pi/2$. Any cross section a_1 is valid, whereby one retrieves the cylinder. This series can also be obtained with a larger number N of cylinders.

For $N = 2$, $kL = \pi/4$ hence $i^2 = 1$: thanks to (A7), the solution of $D_2 = 0$ leads to $a_1 = a_2$ i.e. the same solution as for $N = 1$ but with a length $2L$.

For $N = 3$, $kL = \pi/6$ and $t^2 = 1/3$. There are 2 degrees of freedom (including a_1) for the three unknowns a_1, a_2, a_3 . According to (A8), one finds an infinite number of possible geometries including a cylinder of length $3L$. Choosing, say, a_1 and a_2 as parameters, positive but otherwise arbitrary, a closed-form expression for a_3 is found:

$$a_3 = a_2 \frac{a_1 + a_2}{3a_2 - a_1}, \quad (2)$$

with necessarily: $a_2 > a_1/3$ so that $a_3 > a_1$ as one can check on Figure 2. This last result can be interpreted in terms of reflection coefficients. Setting $R_{ij} = (a_i - a_j)/(a_i + a_j)$ equation (2) is equivalent to

$$R_{23} = \frac{R_{12}}{R_{12} - 1} \quad \text{with } R_{12} < 1/2. \quad (3)$$

Figure 2 illustrate the above results by drawing a_3 versus a_2 , both normalized by a_1 , according to formula (2).

Some input impedance curves of solutions with three resonators are given Figure 3. To avoid infinite amplitude of resonances, viscothermal losses on walls have been included but not the dispersion. In other words wave number k is replaced by a complex wave number $k_{ci} = k - j\alpha_i$ with $\alpha_i = 3 \cdot 10^{-5} \sqrt{f}/r_i$ the damping coefficient where r_i is the radius of i th cylinder. Compared to the single cylinder of length $3L$, it is noticeable that the relative amplitude of the second peak is increased when $a_2 > a_1$ and decreased when $a_2 < a_1$ (Figure 3). Two solutions with the same a_3/a_1 ratio are dual solutions: for example [2 6 3] and [2 1 3]. The minimum a_3/a_1 is unity and corresponds to a cylinder of length $3L$ which is its own dual (see Appendix for the discussion on duality).

For $N = 4$, $i = 1$ and $i = 3$ give two distinct possibilities: $k_1L = \pi/8$ and $k_2L = 3\pi/8$ so that there are two degrees of freedom, i.e. two free parameters to be fixed arbitrarily. Correspondingly, one has: $t_1^2 = 3 - 2\sqrt{2}$ and $t_2^2 = 3 + 2\sqrt{2}$. From equation (A11), one gets

$$\frac{a_2 a_4}{a_1 a_3} = 1 \quad \text{and} \quad (4)$$

$$\left(\frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_1}{a_4} + \frac{a_2}{a_3} + \frac{a_2}{a_4} + \frac{a_3}{a_4} \right) = 6,$$

from which one deduces

$$\frac{a_1}{a_4} = \frac{a_2}{a_3} = K \quad \text{with} \quad K = \frac{8a_1 a_2}{(a_1 + a_2)^2} - 1. \quad (5)$$

As equation (2), equation (5) can be interpreted physically introducing the reflection coefficients

$$R_{34} = -R_{12} \quad \text{and} \quad R_{23} = -\frac{R_{12}^2}{1 - R_{12}^2}, \quad (6)$$

with

$$R_{12}^2 < \frac{1}{2} \quad \text{and} \quad -1 < R_{23} \leq 0. \quad (7)$$

This implies that the third tube has a larger cross section than the second tube. This suggest that the global problem

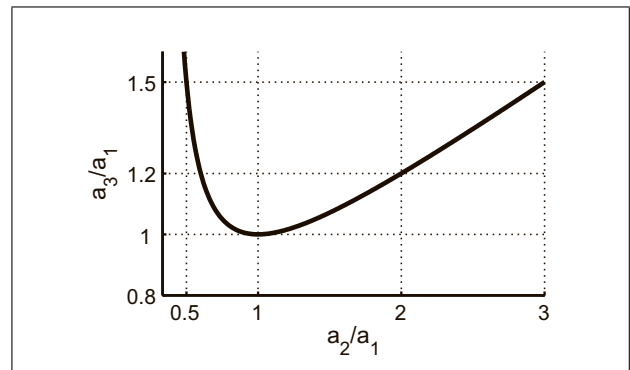


Figure 2. Ratio of cross sections a_3/a_1 as a function of a_2/a_1 for a resonator with 3 cylinders with 1-natural eigenfrequencies (Equation 2).

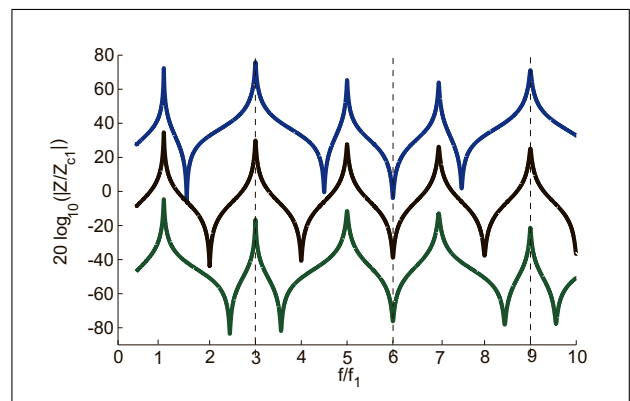


Figure 3. Impedances of resonators. From top to bottom [2 6 3] (blue), [2 2 2] (black) and [2 1 3] (green) with viscothermal losses on walls ($L = 10$ cm, input radius 1 cm). The curves have been vertically shifted. Dashed vertical lines correspond to $kL = \pi/2$ and multiples. $Z_{ci} = \rho c/a_i$ and $f_1 = c/(12L)$.

of harmonic resonators can probably be addressed in the time domain as is it the case for the stepped cones [6].

Here again there is an infinite number of possible geometries including a cylinder of length $4L$ ($K = 1$). Equations (5) show that cross sections $[a_1 a_2]$ and $[a_3 a_4]$ can be permuted leading to pairs of solutions $[a_1 a_2 a_3 a_4]$ and $[a_2 a_1 a_4 a_3]$. These two solutions with the same a_4/a_1 ratio are dual solutions: for example [1 3 6 2] and [3 1 2 6].

Some input impedance curves of solutions with four resonators are given Figure 4. Compared to the single cylinder of length $4L$, it is noticeable that the relative amplitudes of second and third peak are increased when $a_2 > a_1$ and decreased when $a_2 < a_1$ (Figure 4). The limit cases are obtained for $K = 0$ that is $a_1/a_2 = \tan^2(\pi/8)$ and $a_1/a_2 = \tan^2(3\pi/8)$ (Equation A13), in accordance with the computations and the conjecture about bounds [8]. The two limit cases correspond to the same resonator inverted and give respectively the series (1, 7, 9, 15, 17, ...) and (3, 5, 11, 13, ...). The minimum a_4/a_1 ratio is unity and corresponds to a cylinder of length $4L$ which is the autodual solution.

For larger and larger N , one obtains more and more possible discrete resonators having odd integer multiples of the fundamental frequency as partials. One conceives that

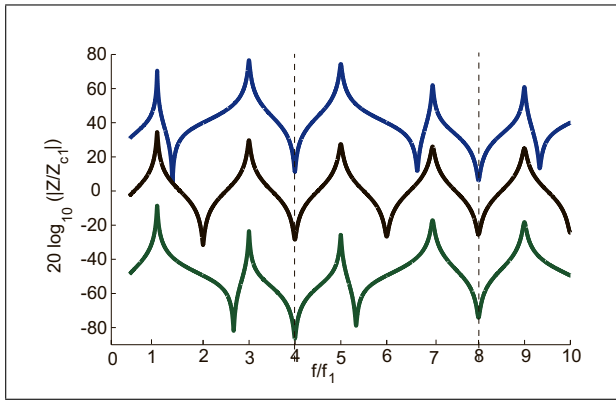


Figure 4. (Colour online) Impedances of resonators. From top to bottom [1 3 6 2] (blue), [2 2 2 2] (black) and [3 1 2 6] (green) with viscothermal losses on walls ($L = 10$ cm, input radius 1 cm). The curves have been shifted vertically. Dashed vertical lines correspond to $kL = \pi/2$ and $kL = \pi Z_{cl} = \rho c/a_1$, and $f_1 = c/(16L)$.

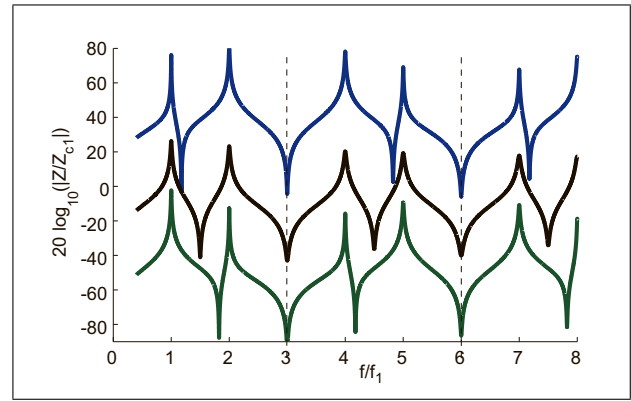


Figure 5. (Colour online) Impedances of resonators. From top to bottom [5 10 54 27] (blue), [1 1 3 3] (black) and [10 5 27 54] (green) with viscothermal losses on walls ($L = 10$ cm, input radius 1 cm). The curves have been shifted vertically. Dashed vertical lines correspond to $kL = \pi/2$ and $kL = \pi$. $Z_{cl} = \rho c/a_1$ and $f_1 = c/(12L)$.

this will be true, all the more, for a growing number of pieces constituting the discrete resonator.

2.2. 2-natural series

Contrary to the above series of odd integer multiples of the fundamental frequency f_1 , the present and following M -natural series ($M > 1$) have not been studied as such in other works, since their appearing for stepped-cones [6]. This series is defined for any N by: $k_i L = i 2\pi/(3N)$ and i non multiple of 3 ($i \in \mathbf{N} - 3\mathbf{N}$).

There is no solution when N is odd as N is necessarily a multiple of $M = 2$.

For $N = 2$, $kL = \pi/3$. So the only solution is obtained for $a_2 = 3a_1$ i.e. the stepped cone with two cylinders.

For $N = 4$, $k_1 > L = \pi/6$ and $k_2 L = \pi/3$, so $t_1^2 = 1/3$ and $t_2^2 = 3$. From Equation (A11), we get

$$\frac{a_2 a_4}{a_1 a_3} = 1 \quad \text{and} \quad \left(\frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_1}{a_4} + \frac{a_2}{a_3} + \frac{a_2}{a_4} + \frac{a_3}{a_4} \right) = t_1^2 + t_2^2 = \frac{4}{3},$$

which leads to

$$\frac{a_4}{a_1} = \frac{a_3}{a_2} = 3 \frac{(a_1 + a_2)^2}{(3a_1 - a_2)(3a_2 - a_1)} \quad \text{with} \quad (8) \quad 1/3 < a_2/a_1 < 3.$$

If $a_1 = a_2$ the solution is the same as for $N = 2$ with two cylinders of length $2L$. Indeed the solution with four different cylinders can be seen as a division of the first and second cylinder of length $2L$ into two cylinders of length L . Equation (8) can also be interpreted physically introducing the reflection coefficients

$$R_{34} = -R_{12} \quad \text{and} \quad R_{23} = -\frac{1/2 + R_{12}^2}{1 - R_{12}^2} \quad \text{with} \quad (9) \quad |R_{12}| < \frac{1}{2} \quad \text{and} \quad -1 < R_{23} \leq -\frac{1}{2}.$$

So, as for the case of the 1-natural series, when halving the tubes, the reflection coefficient at the junction between third and fourth cylinder has to be opposite to that at the junction between the first two. Also, the third tube has a cross section at least three times larger than the second one.

For a given ratio a_3/a_2 , there are two solutions, one such that $a_1 < a_2$ and the other such that $a_1 > a_2$. As for the 1-natural series, two solutions with the same a_4/a_1 ratio are dual solutions, second being obtained from the first by swapping the first and second and third and fourth cylinders. The autodual resonator is [1 1 3 3] which the second stepped cone made with 2 cylinders of length $2L$.

Some input impedance curves of solutions with four resonators are given Figure 5. It is noticeable that compared to the two cylinders solution, the relative amplitude of the second peak is increased when $a_2 > a_1$ and decreased when $a_2 < a_1$. The two limit cases correspond to the same resonator inverted [1 3] and [3 1] which give respectively the series (2, 4, 8, 10, ...) and (1, 5, 7, 11, ...).

The four cylinders resonators, for example [1.5 2.7 5.4], can be seen as a division of the stepped cone [1 3] of length $4L$ with fluctuations which preserve harmonicity.

2.3. 3-natural series

Following the same path as above, the eigenfrequencies are defined by: $k_i L = i 3\pi/(4N)$ and i non multiple of 4 ($i \in \mathbf{N} - 4\mathbf{N}$).

For $N = 3$, $k_L = \pi/4$ and $t^2 = 1$. The geometry $a_2 = 3a_1$ and $a_3 = 6a_1$ is the well known stepped-cone solution, but this is not the only one. All geometries such that

$$a_2 > a_1 \quad \text{and} \quad a_3 = a_2 \frac{a_2 + a_1}{a_2 - a_1} \quad \text{i.e.} \quad (10)$$

$$R_{23} = -\frac{1 + R_{12}}{1 - R_{12}} \quad \text{with} \quad R_{12} < 0$$

are solutions.

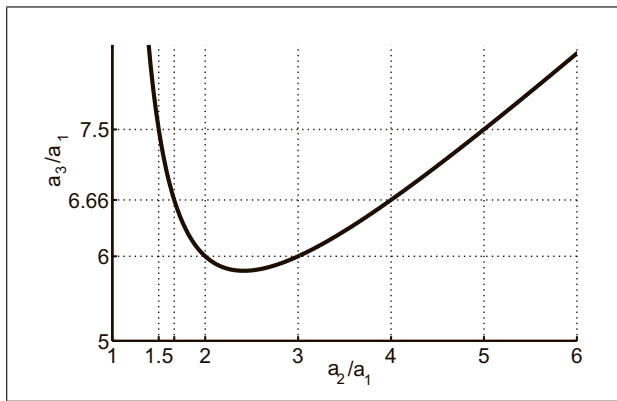


Figure 6. Ratio of cross sections a_3/a_1 as a function of a_2/a_1 for a resonator with 3 cylinders with 3-natural eigenfrequencies (Equation 9).

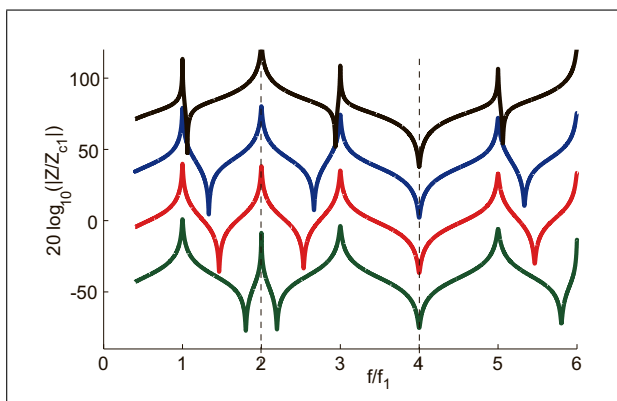


Figure 7. (Colour online) Impedances of resonators. From top to bottom [1 20 22.1] (black), [1 3 6] (blue), [1 2 6] (red) and [1 1.1 23.1] (green) with viscothermal losses on walls ($L = 10$ cm, input radius 1 cm). The curves have been shifted vertically. Dashed vertical lines correspond to $kL = \pi/2$ and $kL = \pi$. $Z_{cl} = \rho c/a_1$ and $f_1 = c/(8L)$.

It is noticeable that $a_3 \geq (3+2\sqrt{2})a_1 = (1+\sqrt{2})^2 a_1$, the limit value being obtained for $a_2 = (1+\sqrt{2})a_1$ which is the autodual solution and a stepped exponential horn (Figure 6). Two solutions having the same a_3/a_1 ratio are dual solutions (see definition in the Appendix A3).

Some input impedance curves are given in Figure 7. When a_2 tends to a_1 , a_3 tends to infinity: the second peak vanishes and the resonator tends to a cylinder of length $2L$. Likewise, when a_2 (resp. a_3) tends to infinite, the first and third peaks vanish and the resonator tends to a cylinder of length L .

From the curves in Figure 7, it is interesting to notice that between the solutions [1 2 6] and [1 3 6] slightly different solutions exist for which resonances remain strictly harmonic (including [1 1+ $\sqrt{2}$ 3+2 $\sqrt{2}$]). This indicates a large tolerance to small deviations in geometry on the point of view of the harmonicity which is much lower elsewhere, especially near the bounds of the domain.

It is interesting to notice that the two dual resonators [1 2 6] and [1 3 6] can be realized with cylinders of same diameter and same length, as a tree of tubes. Indeed, the

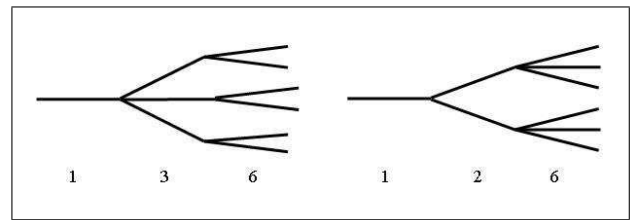


Figure 8. Sketch of resonators made with cylinders of equal cross section and length connected together. Left [1 3 6]. Right [1 2 6].

impedance of a given number of identical tubes in parallel is equal to that of a single tube which cross section is the sum of their cross sections. As an example the resonator [1 3 3] can be made with one cylinder connected to three cylinders of same cross section and same length. As well, resonator [1 2 6] can be realized by connecting the first cylinder to 2 cylinders which are themselves connected each to 3 others. Resonator [1 3 6] can be made by connecting the first cylinder to 3 cylinders which are themselves connected each to 2 others (see Figure 8). From a practical point of view, the junction between the tubes can be done by using T or X-fittings, such as those conventionally used in plumbing.

3-natural series can be found with six cylinders with 3 degrees of freedom, that is to say, a wide variety of geometries. More generally we can get the series with $3N$ cylinders but, in practice, solutions are difficult to derive analytically.

2.4. 4-natural series

Continuing in the same vein, the series now has eigenfrequencies defined by: $k_i L = i 4\pi/(5N)$ and i non multiple of 5 ($i \in \mathbf{N} - 5\mathbf{N}$).

For $N = 4$, $t_1^2 = \tan^2(\pi/5)$ and $t_2^2 = \tan^2(2\pi/5)$ are roots of $p(t) = 1 - 2t^2 + t^4/5$. So

$$\frac{a_2 a_4}{a_1 a_3} = 5 \quad \text{and} \quad \left(\frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_1}{a_4} + \frac{a_2}{a_3} + \frac{a_2}{a_4} + \frac{a_3}{a_4} \right) = 2,$$

which leads to

$$\frac{a_4}{a_3} = 5 \frac{a_1}{a_2} \quad \text{and} \quad \frac{a_2}{a_3} = \frac{16a_1 a_2}{(a_2 + a_1)(a_2 + 5a_1)} - 1. \quad (11)$$

This gives a_3 and a_4 knowing a_1 and a_2 , with the condition $5 - 2\sqrt{5} < a_2/a_1 < 5 + 2\sqrt{5}$ (Figure 9). We find the well known stepped-cone solution [1 3 6 10] but also, for example, the two duals [1 1 3 15] and [1 5 15 15]. It is noticeable that the two latter can be made with cylinders of same diameter similarly to Figure 8. Notice that a_1 can be larger than a_2 , which is not the case for the 3-natural series with $N = 3$. In that case a_3 and a_4 are much larger and the solution tends to the solution with two cylinders for series (1, 4, 6, 9, 11, ...) (see section 3) when a_2/a_1 tends to $5 - 2\sqrt{5}$. On the other hand when a_2/a_1 tends to $5 + 2\sqrt{5}$ the solution tends to the solution for series (2, 3, 7, 8, 12, ...) (see section 3). Two solutions having the same

Table I. Some resonators with 2 cylinders.

Series	kL	a_1	a_2	Comment
2-natural series (1, 2, 4, 5, 7, 8, ...)	$\pi/3$	1	3	Stepped cone
1-natural series (1, 3, 5, 7, 9, ...)	$\pi/4$	1	1	Cylinder of length $2L$
(1, 4, 6, 9, 11, ...)	$\pi/5$	1	$5-2\sqrt{5}$	a_2/a_1 not rational
(2, 3, 7, 8, 12, ...)	$2\pi/5$	1	$5+2\sqrt{5}$	a_2/a_1 not rational
(1, 5, 7, 11, 13, ...)	$\pi/6$	3	1	Inverted stepped cone
(1, 7, 9, 15, 17, ...)	$\pi/8$	1	$3-2\sqrt{2}$	a_2/a_1 not rational
(3, 5, 11, 13, ...)	$3\pi/8$	$3-2\sqrt{2}$	1	Previous one inverted
(1, 29, 31, 59, 61, ...)	$\pi/30$	90.52...	1	A Helmholtz resonator

a_4/a_1 ratio are dual solutions. The autodual resonator is $[1 \sqrt{5} \ 2 + \sqrt{5} \ 5 + 2\sqrt{5}]$. Some input impedance curves are given in Figure 10.

2.5. 5-natural series

As a last example in the same spirit, the eigenfrequencies for the 5-natural series are given by: $k_i L = i 5\pi/(6N)$ and i non multiple of 6 ($i \in \mathbf{N} - 6\mathbf{N}$).

Equation (A16) leads to $\sum_{1 \leq i < j \leq 5} a_i/a_j = 10/3$ and equation (A18) to $a_1 a_3/a_2 a_4 + a_1 a_3/a_2 a_5 + a_1 a_4/a_2 a_5 + a_1 a_4/a_3 a_5 + a_2 a_4/a_3 a_5 = 1$. Simple algebra allows to calculate a_5/a_4 and a_4/a_3 knowing a_2/a_1 and a_3/a_2 . Some examples are given in Table IV. The autodual solution is $[1 \ 2.1547 \ 3.7319 \ 6.4637 \ 13.9274]$.

For higher values of N it becomes difficult to derive solutions analytically by hand.

3. Various resonators overview

3.1. Resonators with 2 cylinders

From (A4.2), the roots of $D_2(t)$ are given from: $t^2 = a_2/a_1$. For a given series, the solution is unique, the only degree of freedom being the input cross section a_1 . Among the possible solutions, resonators for which $kL = \pi/m$ are of specific interest. Different cases are summarized and briefly commented in Table I.

The stepped cone [1 3] is particularly interesting as it has harmonic eigenfrequencies for mixed as well as for homogeneous boundary conditions: the series (1, 2, 3, 4, ...) $c/(4L)$ when open-open or closed-closed, series (1, 5, 7, 11, 12, ...) $c/(12L)$ when closed at large end and the 3-natural series (1, 2, 4, 5, 7, 8, ...) $c/(6L)$ when is closed at small end [12].

3.2. Resonators with 3 cylinders

Computing the roots of $D_3(t)$ from (A4.3) leads to $t^2 = 1/(a_1/a_2 + a_1/a_3 + a_2/a_3)$. For a given series, the solution is not unique as there are 2 degrees of freedom including the input cross section a_1 .

Resonators for which $kL = \pi/m$ are also of specific interest. Table II gives several cases with comments.

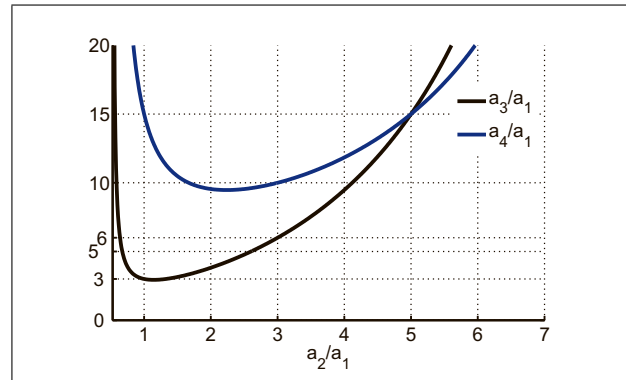


Figure 9. Ratio of cross sections a_3/a_2 and a_4/a_1 as a function of a_2/a_1 for a resonator with 4 cylinders with 4-natural eigenfrequencies.

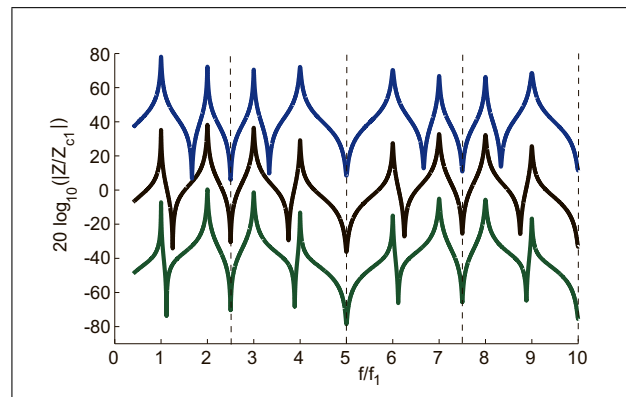


Figure 10. (Colour online) Impedances of resonators. From top to bottom [1 1 3 15] (blue), [1 3 6 10] (black) and [1 5 15 15] (green) with viscothermal losses on walls ($L = 10$ cm, input radius 1 cm). The curves have been shifted vertically. Dashed vertical lines correspond to $kL = \pi/2$ and multiples. $Z_{cl} = \rho c/a_1$ and $f_1 = c/(10L)$.

3.3. Resonators with 4 cylinders

Along, the same line as above, for 4 cylindrical pieces, resonators such that $k_1 L = \pi/(2m)$ and $k_2 L = (m - 1)\pi/(2m)$ are of specific interest and in that case one has the special relation: $t_1^2 t_2^2 = 1$. This leads to $a_4 a_2/a_1 a_3 = 1$ and to $a_1/a_4 = a_2/a_3 = K$ with $1 + K = (1 + t_1^2 + t_2^2) a_1 a_2 / (a_1 + a_2)^2$. For a given series, the solution is not unique as a_2/a_1 can be arbitrarily chosen, albeit respect-

Table II. Some resonators with 3 cylinders.

Series	kL	a_1	a_2	a_3	Comment
(2, 3, 4, 8, 9, 10, 14, ...)	$\pi/3$	1	6	42	
3-natural series (1, 2, 3, 5, 6, 7, 9, ...)	$\pi/4$	1	3	6	Stepped cone
		1	2	6	Dual of the stepped cone
		1	$1 + \sqrt{2}$	$3 + 2\sqrt{2}$	Autodual / Minimal value of a_3/a_1
		2	3	15	Dual resonators
		2	10	15	Dual resonators
		3	12	20	Dual resonators
		3	5	20	Dual resonators
(2, 5, 8, 12, 15, 18, ...)	$\pi/5$	1	1.426...	2.033...	
1-natural series (1, 3, 5, 7, 9, ...)	$\pi/6$	1	1	1	Cylinder of length $3L$
		2	6	3	Dual resonators
		2	1	3	Dual resonators
		5	10	6	Dual resonators
		5	3	6	Dual resonators
(1, 4, 7, 9, 12, ...)	$\pi/8$	1	3	$9\sqrt{2} - 12$	a_3/a_1 not rational as t^2 is not

ing the condition $t_1^2 < a_2/a_1 < t_2^2$. Dual resonators are obtained by swapping the first and second cylinders and the third and fourth cylinders. The same way, several cases are summarized and commented in Table III.

A pair of resonators is worth noticing: the dual resonators [1 3 6 2] and [3 1 2 6], that give the 1-natural series when closed at the input and open at the end (see above section 2.1). The same resonators reversed ([2 6 3 1] and [6 2 1 3]), give the series (1, 5, 7, 11, 13...) $c/(24L)$ where the multiples of 2 and 3 are missing (see Figure 11). It is remarkable that the eigenmodes of the open-open resonator (given by the minima of the impedance) [3 1 2 6] (identical to [6 2 1 3]) give the series (2, 3, 4, 6, 8, 9, 10, 12,...) $c/(12L)$ whereas those of resonator [2 6 3 1] (identical to [1 3 6 2]) give the series (1, 3, 5, 6, 7, 9, 11, 12,...) $c/(12L)$ (see Figure 11). The two resonators being dual, the eigenfrequencies of resonator [3 1 2 6] closed at each end are those of resonator [2 6 3 1] open at each end and vice-versa. We are therefore in the presence of resonators having harmonic modes whatever the boundary conditions (open-open, closed-open, open-closed and closed-closed). This defines a singular family of resonators. It should be noted that, except [1 3] (see 3.1), the stepped cone family do not belong to this family as they do not have harmonic modes when closed at each end, or when reversed.

Finally, it might be noticed that resonator [1 3 6 2] can be made as a tree of branched cylinders of same diameter and same length connected together.

3.4. Resonators with 5 and 6 cylinders

Continuing in the same spirit, some possible geometries, involving rational cross section areas ratios for the 5-natural and 6-natural series are given in Table IV.

Notice that the stepped cone [1 3 6 10 15] can not be built as a tree of branched cylinders of same diameter and

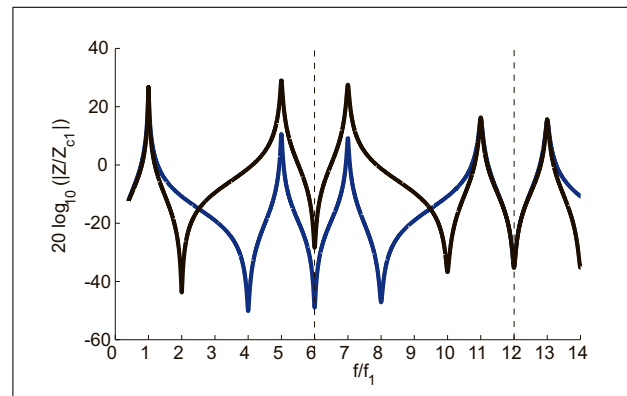


Figure 11. (Colour online) Impedances of resonators. [6 2 1 3] (blue) et [2 6 3 1] (black) with viscothermal losses on walls ($L = 10$ cm, input radius 1 cm). Dashed vertical lines correspond to $kL = \pi/2$ and $kL = \pi$. $Z_{ct} = \rho c/a_1$ and $f_1 = c/(24L)$.

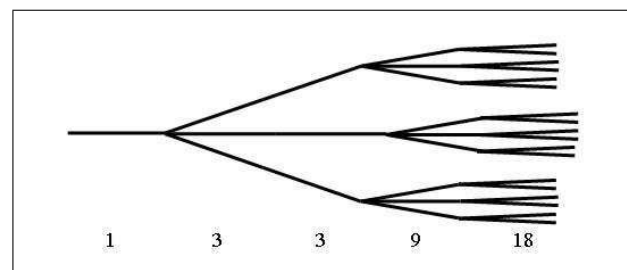


Figure 12. Sketch of resonator [1 3 3 9 18] made with cylinders of equal cross section and length connected together.

same length, as in figure 8, but the dual resonators [1 3 3 9 18] and [1 2 6 6 18] can (see Figure 12). This kind of resonators will be investigated in future work.

Table III. Some resonators with 4 cylinders.

Series	k_1L	k_2L	a_1	a_2	a_3	a_4	Comment
4-natural series (1, 2, 3, 4, 6, 7, 8, 9, ...)	$\pi/5$	$2\pi/5$	1	3	6	10	Stepped cone
			3	5	10	30	Dual of the stepped cone
			1	$\sqrt{5}$	$2 + \sqrt{5}$	$5 + 2\sqrt{5}$	Autodual
			1	1	3	15	$a_2a_4 = 5a_1a_3$ Dual resonators
			1	5	15	15	
2-natural series (1, 2, 4, 5, 7, 8, 10, ...)	$\pi/6$	$2\pi/6$	1	1	3	3	Stepped cone with 2 cylinders of length $2L$
			5	10	54	27	$a_2a_4 = a_1a_3$ Dual resonators
			10	5	27	54	
1-natural series (1, 3, 5, 7, 9, ...)	$\pi/8$	$3\pi/8$	1	1	1	1	Cylinder of length $4L$
			1	3	6	2	$a_2a_4 = a_1a_3$ Dual resonators
			3	1	2	6	
(1, 4, 6, 9, 11, 14, 16, 19, ...)	$\pi/10$	$4\pi/10$	1	3	2.56	0.85	
(1, 5, 7, 11, 13, ...)	$\pi/12$	$5\pi/12$	2	6	3	1	Inverts of the previous dual resonators for $\pi/8$
(6, 7, 35, 36, 48, 49, ...)	$\pi/7$	$\pi/6$	4	1	271.4	83.9	
(1, 2, 28, 29, 31, 32, ...)	$\pi/30$	$2\pi/30$	30	1	132.4	1.98	A so-called bi-chromator (Double Helmholtz resonator with two close eigenmodes)

Table IV. Some resonators with 5 and 6 cylinders.

Series	k_1L	a_1	a_2	a_3	a_4	a_5	a_6	Comment
5-natural series (1, 2, 3, 4, 5, 7, 8, 9, ...)	$\pi/6$	1	3	6	10	15		Stepped cone
		2	3	5	10	30		Dual of the previous
		3	5	10	15	45		
		1	3	3	9	18		
		1	2	6	6	18		
		2	6	5	20	44		
6-natural series (1, 2, 3, 4, 5, 6, 8, 9, ...)	$\pi/7$	6	6	9	39	130		
		1	3	6	10	15	21	Stepped cone
		10	14	21	35	70	210	Dual of the previous
		3	5	20	28	28	84	$a_2a_4a_6 = 7a_1a_3a_5$ Dual resonators
5	15	15	21	84	140			

4. Conclusion

This work is part of a program that aims at investigating the very large family of acoustical resonators, named *discrete resonators*, having moreover harmonically related natural frequencies, uncovered in previous works by the authors. The main result here comes from focusing on the particular case of *M-natural series*, in which exactly the integer multiples of are missing. The stepped-cones family, uncovered two decades ago, appears now as a subset of the whole family. The concept of M-series is interesting in that it allows to organize a lot of these resonators around the stepped-cones, making this way a step towards a classification of discrete resonators: the other resonators made with the same number of cylinders and having the

same series of eigenfrequencies can be considered as variant of stepped cones. It must be understood that if the variant is far from the corresponding stepped cone the amplitudes of the impedance peaks will be highly variable leading to resonators probably poorly usable as musical instruments. The second point in this work was to continue and deepen previous results, for several other resonators having up to 6 pieces, unveiling e.g. singular resonators that are harmonic, whatever the boundary conditions. However, non uniqueness remains and there is room for using the available degrees of freedom in design, especially for the important question of localizing physically feasible solutions. In that respect, the robustness issue when designing such resonators can be studied to a certain extent, as the sensitivity of the unknown cross-sections to

small variations in the free ones: this sensitivity is very high near the bounds of the domain. This is surely useful in an optimization process, for the choice of initial values in an iterative procedure. A preliminary study in that direction has been done in a previous work [13]. Moreover resonators made with a large number of cylinders based on the series of a given stepped cone can be seen as a stepped cone in which the cross sections of the cylinders are not perfectly constant. This is interesting because it can help to interpret the results of some numerical optimization of bore under constraints which lead sometimes to peculiar results [14, 15, 16].

The following points appear to be among those that are worth pursuing:

- Address the question of open-open or closed-closed resonators.
- Consider the amplitude of the resonance peak into the constraints towards uniqueness of a solution.
- Consider the problem in the time domain.
- Consider the discrete resonators that can be built as a tree of identical branched cylinders.
- Make the link with inverse methods such as layer peeling algorithm.
- Deepen the connections between the graph and transmission matrix approaches.

Some applications can probably be found, not only in the field of wind instruments but also in the context of noise control in order to treat a harmonic series of frequencies which is usually done using several Helmholtz resonators or quarter wave length resonators.

Appendix: General properties of stepped waveguides and inverse problem

In this appendix we give some general properties of stepped wave guides made with cylinders of same length L and various cross section a_i .

A1. Transfer matrix

We consider the transfer matrix

$$[M] = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

which makes the link between pressure and volume velocity at the input $\begin{pmatrix} p_{in} \\ U_{in} \end{pmatrix}$ and pressure and volume velocity at the output $\begin{pmatrix} p_{out} \\ U_{out} \end{pmatrix}$:

$$\begin{pmatrix} p_{in} \\ U_{in} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} p_{out} \\ U_{out} \end{pmatrix}. \quad (A1)$$

Transfer matrix of a cylinder of cross section a_i and length L is

$$\begin{pmatrix} \cos(kL) & j\rho c \sin(kL)/a_i \\ j \sin(kL)a_i/\rho c & \cos(kL) \end{pmatrix}, \quad (A2)$$

where $k = 2\pi f/c$ is the wave number, with c the sound velocity and ρ the air density. Be careful that, in this appendix, $j = \sqrt{-1}$ (straight) is different from the index j (italic). Because of the continuity of pressure and volume velocity, the transfer matrix of a stepped waveguide is the product of the transfer matrices of the constituting cylinders [4, 17],

$$\begin{aligned} [M_N] &= \begin{pmatrix} A_N & B_N \\ C_N & D_N \end{pmatrix} \\ &= \prod_{i=1}^N \begin{pmatrix} \cos(kL) & j\rho c \sin(kL)/a_i \\ j \sin(kL)a_i/\rho c & \cos(kL) \end{pmatrix} \\ &= x^N \prod_{i=1}^N \begin{pmatrix} 1 & j\rho c t/a_i \\ j t a_i/\rho c & 1 \end{pmatrix}, \quad (A3) \end{aligned}$$

where we set $x = \cos(kL)$ and $t = \tan(kL)$.

The first four matrices are given:

$$[M_1] = x \begin{pmatrix} 1 & j\rho c t/a_1 \\ j t a_1/\rho c & 1 \end{pmatrix}, \quad (A4a)$$

$$[M_2] = x^2 \begin{pmatrix} 1 - \frac{a_2}{a_1} t^2 & j\rho c t \left(\frac{1}{a_1} + \frac{1}{a_2} \right) \\ j t (a_1 + a_2)/\rho c & 1 - \frac{a_1}{a_2} t^2 \end{pmatrix}, \quad (A4b)$$

$$\begin{aligned} [M_3] &= x^3 \begin{pmatrix} 1 - \left(\frac{a_2}{a_1} + \frac{a_3}{a_1} + \frac{a_3}{a_2} \right) t^2 & \dots \\ j t \left[(a_1 + a_2 + a_3) - \frac{a_1 a_3}{a_2} t^2 \right] / \rho c & \dots \\ \dots & j\rho c t \left[\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right) - \frac{a_2}{a_1 a_3} t^2 \right] \\ \dots & 1 - \left(\frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_2}{a_3} \right) t^2 \end{pmatrix}, \quad (A4c) \end{aligned}$$

$$[M_4] = \begin{pmatrix} A_4 & B_4 \\ C_4 & D_4 \end{pmatrix}, \quad (A4d)$$

with

$$\begin{aligned} A_4 &= x^4 \left[1 - \left(\frac{a_2}{a_1} + \frac{a_3}{a_1} + \frac{a_3}{a_2} + \frac{a_4}{a_1} + \frac{a_4}{a_2} + \frac{a_4}{a_3} \right) t^2 + \frac{a_2 a_4}{a_1 a_3} t^4 \right], \\ B_4 &= j\rho c t x^4 \left[\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} \right) - \left(\frac{a_2}{a_1 a_3} + \frac{a_2}{a_1 a_4} + \frac{a_3}{a_1 a_4} + \frac{a_3}{a_2 a_4} \right) t^2 \right], \\ C_4 &= \frac{j t x^4}{\rho c} \left[(a_1 + a_2 + a_3 + a_4) - \left(\frac{a_1 a_3}{a_2} + \frac{a_1 a_4}{a_2} + \frac{a_1 a_4}{a_3} + \frac{a_2 a_4}{a_3} \right) t^2 \right], \\ D_4 &= x^4 \left[1 - \left(\frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_2}{a_3} + \frac{a_1}{a_4} + \frac{a_2}{a_4} + \frac{a_3}{a_4} \right) t^2 + \frac{a_1 a_3}{a_2 a_4} t^4 \right]. \end{aligned}$$

By induction, it can be shown that the coefficients of the transfer matrix $[M_N]$ can be written

$$A_N = x^N \left(1 + \sum_{i=1}^{\lfloor N/2 \rfloor} \left(\sum_{\substack{1 \leq i_1 < \dots < i_i \\ \dots < i_j < \dots < i_N}} \frac{a_{i_1} \dots a_{i_i}}{a_1 \dots a_{i_i}} \right) (-t^2)^i \right), \quad (A5a)$$

$$B_N = jt\rho c x^N \left(\sum_{i=1}^{\lfloor (N-1)/2 \rfloor} \left(\sum_{\substack{1 \leq j_1 < \dots < j_i \\ \dots < i_1 < \dots < i_i \leq N}} \frac{a_{i_1} \dots a_{i_i}}{a_{j_1} \dots a_{j_i}} \right) (-t^2)^i \right), \quad (A5b)$$

$$C_N = \frac{jt x^N}{\rho c} \left(\sum_{i=1}^{\lfloor (N-1)/2 \rfloor} \left(\sum_{\substack{1 \leq j_1 < \dots < j_i \\ \dots < i_1 < \dots < i_i \leq N}} \frac{a_{j_1} \dots a_{j_i}}{a_{i_1} \dots a_{i_i}} \right) (-t^2)^i \right), \quad (A5c)$$

$$D_N = x^N \left(1 + \sum_{i=1}^{\lfloor N/2 \rfloor} \left(\sum_{\substack{1 \leq j_1 < \dots < j_i \\ \dots < i_1 < \dots < i_i \leq N}} \frac{a_{i_1} \dots a_{i_i}}{a_{j_1} \dots a_{j_i}} \right) (-t^2)^i \right). \quad (A5d)$$

It is noticeable that coefficients A_N and D_N (resp. C_N and B_N) have the same form but with inverted cross-section ratios. B_N is homogeneous to an impedance $\rho c/a$ while C_N is homogeneous to an admittance $a/\rho c$.

Each sum of A_N and D_N contains $\binom{N}{2i}$ terms ($\binom{n}{p}$ is the binomial coefficient) and each sum of B_N and C_N contains $\binom{N}{2i+1}$ terms. So, the second coefficient of D_N , $\sum_{1 \leq i < j \leq N} a_i/a_j$, contains $N(N-1)/2$ terms and the last contains a single term, $\prod_{i=1}^{N/2} a_{2i-1}/a_{2i}$, when N is even and N terms when it is odd.

A2. Eigenfrequencies

The eigenfrequencies of a given stepped guide can be obtained writing

$$C_N = 0, \quad \text{when the guide is closed at each end} \\ (U_{in} = U_{out} = 0), \quad (A6a)$$

$$B_N = 0, \quad \text{when the guide is open at each end} \\ (P_{in} = P_{out} = 0), \quad (A6b)$$

$$A_N = 0, \quad \text{when the guide is open at the input} \\ \text{and closed at the output}, \quad (A6c)$$

$$D_N = 0, \quad \text{when the guide is closed at the input} \\ \text{and open at the output}. \quad (A6d)$$

As the tangent is a periodic and odd function, if f_i is an eigenfrequency such that $0 < f_i < c/4L$ and $i \leq N$, then all frequencies $f_{2Nm \pm i} = m(c/2L) \pm f_i$ with m natural integer are eigenfrequencies.

When $t = 0$,

$$[M_N] = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix},$$

which implies that $f_m = m c/(2L)$ are eigenfrequencies for homogeneous boundary conditions.

When $x = 0$, two cases have to be considered:

1. If N is even,

$$[M_N] = \begin{pmatrix} \pm \prod_{i=1}^{N/2} \frac{a_{2i-1}}{a_{2i}} & 0 \\ 0 & \pm \prod_{i=1}^{N/2} \frac{a_{2i}}{a_{2i-1}} \end{pmatrix},$$

which implies that $f_m = (2m + 1) c/4L$ are eigenfrequencies for mixed boundary conditions.

2. If N is odd,

$$[M_N] = \begin{pmatrix} 0 & \pm j \frac{\rho c}{a_1} \prod_{i=1}^{N/2} \frac{a_{2i}}{a_{2i-1}} \\ \pm j \frac{\rho c}{a_1} \prod_{i=1}^{N/2} \frac{a_{2i}}{a_{2i-1}} & 0 \end{pmatrix},$$

which implies that $f_m = (2m + 1) c/4L$ are eigenfrequencies for homogeneous boundary conditions.

Finally eigenfrequencies are deduced from the roots of a polynomial in t^2 and from $x = 0$ when N is odd.

A3. Dual resonators

Consider a resonator for which the series of the cross section areas is $[1 a_1/a_2 \dots a_1/a_N]$. For example, if one consider the stepped waveguide $[1 2 3]$, its dual is $[1 1/2 1/3]$ or, when reversed, $[1/3 1/2 1]$ which equivalent to $[1 3/2 3]$. So, it can be noted that a resonator and its reversed dual have the same a_N/a_1 ratio. As noticed before D_N (resp. C_N) can be deduced from A_N (resp. B_N) by replacing the cross-sections ratios, each by its invert. As a consequence, when a discrete resonator, open (resp. closed) at both ends, has given eigenfrequencies, its dual will have the same eigenfrequencies when closed (resp. open) at both ends. When mixed boundary conditions are considered the same reasoning, considering B_N and C_N , allows to say that two dual resonators have the same eigenfrequencies when the first is open at the input and closed at the end while the second is closed at the input and open at the end. So reversing one of the two resonators, both have the same eigenfrequencies when closed (open) at the input and open (closed) at the output. For example, resonators $[1 2 3]$ and $[1 3/2 3]$ have the same eigenfrequencies in mixed boundary conditions.

Some resonators are their own dual. For example, the dual of $[1 2 4]$ is $[1 1/2 1/4]$ which reverted is $[1/4 1/2 1]$ which is equivalent to $[1 2 4]$. A stepped guide is its own dual (autodual) when $a_{i+1}/a_i = a_{N-i+1}/a_{N-i}$. Consequently any two stepped wave guide is its own dual. A stepped exponential horn is autodual, as in that case $a_{i+1}/a_i = Cte$.

A4. Inverse problem

Aside from the derivation of the characteristic polynomials, given the geometry $[a_1 a_2 a_3 a_4 \dots]$, another question is the inverse problem of finding the geometry $[a_1 a_2 a_3 a_4 \dots]$ from the knowledge of the eigenfrequencies. To this end, given the roots t_i^2 , one has to solve $A_N = 0$, $B_N = 0$, $C_N = 0$ or $D_N = 0$, as a system of equations in $[a_1 a_2 a_3 a_4 \dots]$.

As noticed before, if f_i is an eigenfrequency, then all frequencies $f_{2Nm \pm i} = m(c/2L) \pm f_i$ are eigenfrequencies. As a consequence, when looking for a resonator made of N cylinders with arbitrary chosen eigenfrequencies with mixed boundary conditions, only $\lfloor N/2 \rfloor$ independent equations $A_N = 0$ or $D_N = 0$ are obtained. This is therefore a problem with $\lfloor N/2 \rfloor$ constraints and N unknown cross sections a_i . So there will be $\lfloor (N + 1)/2 \rfloor$ degrees of freedom (see also previous work [9] about this question of

non uniqueness). However, all problems will not have feasible solutions because the solution space is bounded and cross-section areas must be positive. In other words, only a subset of the free $[(N + 1)/2]$ parameters will give feasible discrete resonators. Different ways can be considered. The first one is a direct evaluation of $D_N = 0$ (closed-open situation) at the arbitrary chosen roots t_i^2 . Notice that considering $A_N = 0$ (open-closed situation) leads to the same solutions inverted. This approach was used on a polynomial in x [8]. It allows finding easily solutions for $N = 2$,

$$a_2 = a_1 t^2, \quad (A7)$$

and for $N = 3$:

$$t^2 = \left(\frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_2}{a_3} \right)^{-1},$$

which leads to

$$a_3 = a_2 \frac{(a_1 + a_2)t^2}{a_2 - a_1 t^2}, \quad (A8)$$

a_1 and a_2 being arbitrarily chosen with the restriction $a_2/a_1 > t^2$.

Another approach consists in using the elementary symmetric polynomials σ of the roots t_i^2 and their well-known relationship to the coefficients γ_n of the polynomials $p_N(t^2)$ of order $[N/2]$ in t^2 :

$$\sigma_j = (-1)^j \frac{\gamma_{[N/2]-j}}{\gamma_{[N/2]}} \quad (A9)$$

and

$$\sigma_j = \sum_{\substack{1 \leq i_1 < \dots < i_j \\ \leq [N/2]}} t_{i_1}^2 t_{i_2}^2 \dots t_{i_j}^2.$$

Notice that $\sigma_1 = \sum_1^{[N/2]} t_i^2$ and that $\sigma_{[N/2]} = t_1^2 t_2^2 \dots t_{[N/2]}^2$. In our case, if N is even, this last equation leads to the remarkable result

$$\prod_{i=1}^{N/2} \frac{a_{2i}}{a_{2i-1}} = \prod_{i=1}^{N/2} t_i^2. \quad (A10)$$

These relations allow finding the solutions for $N = 4$,

$$t_1^2 + t_2^2 = \left(\frac{a_1}{a_2} + \frac{a_1}{a_3} + \frac{a_1}{a_4} + \frac{a_2}{a_3} + \frac{a_2}{a_4} + \frac{a_3}{a_4} \right) \frac{a_2 a_4}{a_1 a_3}$$

and

$$t_1^2 t_2^2 = \frac{a_2 a_4}{a_1 a_3}. \quad (A11)$$

This leads to

$$\frac{a_4}{a_3} = \frac{a_1}{a_2} t_1^2 t_2^2$$

and

$$\frac{a_2}{a_3} = \frac{t_1^2 + t_2^2 - \frac{a_2}{a_1} - \frac{a_4}{a_3}}{1 + \frac{a_4}{a_3} + \frac{a_2}{a_1} + t_1^2 t_2^2}, \quad (A12)$$

which give a_3 and a_4 knowing a_1 and a_2 with the restriction conjectured by Le Vey [8]:

$$t_i^2 < a_2/a_1 < t_2^2. \quad (A13)$$

This last formula is useful to find the solutions as the solutions space can be rather small especially when the roots t_1^2 and t_2^2 are close to each other. For example if $k_1 L = \pi/30$ and $k_2 L = \pi/15$, there will be solutions only for $0.011 < a_2/a_1 < 0.045$.

A5. Specific relationships for N-natural series

The interest being focused on resonators with harmonic resonances, in the case of the stepped cone family found by Dalmont and Kergomard [6], the eigenvalues $t_i^2 = \tan^2(i\pi/(N + 1))$ are solutions of the polynomial [18]

$$p_N = 1 + \frac{1}{N + 1} \sum_{i=1}^{[N/2]} \binom{N + 1}{2i + 1} (-t^2)^i. \quad (A14)$$

For a resonator made with N cylinders for which eigenfrequencies correspond to the N -natural series, t_i are also roots of the polynomial D_N/x^N . So, since the first coefficient is taken as unit in both cases, polynomial D_N/x^N and p_N are identical. It is then possible to write, for any N and any i , the following identity between the polynomials coefficients:

$$\sum_{\substack{1 \leq i_1 < j_1 < \dots < i_n < j_n \leq N}} \frac{a_{i_1} \dots a_{i_n}}{a_{j_1} \dots a_{j_n}} = \frac{1}{N + 1} \binom{N + 1}{2i + 1}. \quad (A15)$$

This last result is interesting because, in that case, the roots of the polynomials have not to be explicitly known. Moreover, it shows that cross section areas ratios can be chosen rational as for stepped cones. The first coefficient ($i = 0$) equals one and the second one ($i = 1$) is

$$\sum_{1 \leq i < j \leq N} \frac{a_i}{a_j} = \frac{N(N - 1)}{6} \quad (A16)$$

and contains $N(N - 1)/2$ terms. This means that the mean value of a_i/a_j with $j > i$ is $1/3$. The last coefficient ($i = [N/2]$) is

$$\prod_{j=1}^{N/2} \frac{a_{2j-1}}{a_{2j}} = \frac{1}{N + 1} \quad (A17)$$

when N is even and

$$\sum_{\substack{1 \leq i_1 < j_1 < \dots < i_n < j_n \leq N}} \frac{a_{i_1} \dots a_{i_n}}{a_{j_1} \dots a_{j_n}} = 1, \quad (A18)$$

which contain N terms when N is odd.

Considering the stepped cone for which $a_{2j-1} = j(2j - 1)a_1$ and $a_{2j} = j(2j + 1)a_1$ equation (A17) is easily verified. Thus, equation (A17) as well as equation (A18) indicates that there is an average global growing of the cross sections analogous to that of the stepped cones.

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