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Secondary Bjerknes Force and Its Influence on the Radiation Impedance of Coupled Bubbles

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Summary

Based on Lagrange's equation, the linear second-order differential equations of motion for two coupled bubbles system are obtained. The secondary Bjerknes force and the radiation impedance of two coupled bubbles in a weak acoustic field are investigated. The results show that the secondary Bjerknes force between two bubbles is in agreement with the well-known linear theory. For two coupled bubbles, the radiation damping is related to the driving frequency and radii of two bubbles. In particular, if the driving frequency lies between the linear resonance frequencies of two large bubbles, the radiation damping of coupled bubbles increases obviously. The linear resonance frequencies of two coupled bubbles are almost not changed by the mutual interaction between bubbles.

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1. Introduction

Pulsating bubbles expend a portion of their energy in the form of spherical sound waves. Radiation damping is just this loss of energy. Devin discussed the fundamental processes by which a pulsating gas bubble in liquids dissipated its energy. The total damping was explained by losses originating from three parts: thermal damping, sound radiation damping and viscous damping. His survey showed that the radiation damping of a single bubble was independent of frequency. Moreover, the damping at resonance was due to thermal and viscous, and the radiation damping was very small compared to other damping [1]. But experimental results verified that the damping at resonance was primarily due to thermal and radiation [2]. The experimental results showed that the radiation damping at resonance may play a big role in the damping. The discrepancy between the theory and experimental results may be due to the coupled pulsation of bubbles and the mutual interaction between bubbles. In other words, the mutual interaction between bubbles may cause a big change of radiation damping of a bubble.

The mutual interaction between bubbles in an acoustic field is a famous acoustical phenomenon, which has a significant effect on coalescence and movement of bubbles, such as multi-bubbles sonoluminescence and targeted drug delivery [3, 4, 5, 6]. The force between two bubbles due to the "secondary" sound field emitted by other bubbles is called the secondary Bjerknes force [7]. After the discovery of the secondary Bjerknes [8], the force was investigated experimentally and theoretically by many authors [9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19], but the influence on the radiation damping was a blank.

In this paper, we address to investigate the secondary Bjerknes force between two bubbles in linear pulsations and the radiation damping of two coupled bubbles. We also expect to give a theoretical explanation for the experimental results.

2. The volume pulsation equations of bubbles

A bubble, which is in a uniform but alternating pressure field, can't be in equilibrium with this oscillating pressure unless the bubble itself is pulsating. Devin derived the volume pulsation equation of a single bubble in an adiabatic pulsation through Lagrange's equation in 1959 [1],

$$\frac{\rho}{4\pi R_0}\ddot{v} + b\dot{v} + \frac{\gamma P_0}{V_0}v = -P_A e^{j\omega t}, \tag{1}$$

where v is the change in volume from the equilibrium bubble volume, and R_0 is the equilibrium radius of the bubble. P_A is the amplitude of driving sound pressure, and ρ is the density of liquid. ω is the angular frequency of driving sound, and V_0 is the volume of the bubble without sound. γ is the ratio of specific heats, and P_0 is the static pressure at which the bubble has the radius R_0. b is the dissipation coefficient, which can be expressed as

$$b = \frac{\delta\sqrt{3\gamma\rho P_0}}{4\pi R_0^2}, \tag{2}$$

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where δ is a damping constant. Equation (1) is the fundamental theory of a single bubble, and many important conclusions can be obtained from equation (1).

Let us consider two gas bubbles in a liquid driven by a stationary sound field. We suppose that the bubbles are far enough apart that their pressure emissions have the effect of an additional external driving of their neighbor without distorting each other's sphericity. If the volume amplitude of a single bubble is small, we can use a linear second-order differential equation to describe the motion of the bubble system. If the liquid is incompressible, Lagrange's equation of the system can be expressed as

$$\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)\Big(\frac{\partial L}{\partial \dot{v}}\Big) - \frac{\partial L}{\partial v} + +b\dot{v} = C, \tag{3}$$

where L is the Lagrangian function defined as the kinetic energy minus the potential energy of the system, and C is a generalized driving function. For the liquid containing two bubbles, the potential energy of the system can be expressed as

$$E_p = -\int_0 v_1 \Delta P_1 \,\mathrm{d}v_1 - \int_0 v_2 \Delta P_2 \,\mathrm{d}v_2. \tag{4}$$

Based on Devin's methods [1], the potential energy of the system can be written as

$$\begin{aligned} E_p = {} & \frac{\gamma P_0}{2V_{10}}\Big(1 + \frac{2\sigma}{P_0 R_{10}} - \frac{2\sigma}{3\gamma P_0 R_{10}}\Big)v_1^2 \\ & + \frac{\gamma P_0}{2V_{20}}\Big(1 + \frac{2\sigma}{P_0 R_{20}} - \frac{2\sigma}{3\gamma P_0 R_{20}}\Big)v_2^2, \end{aligned} \tag{5}$$

where γ is the ratio of specific heats. The velocity of a liquid particle at a distance r, due to a simple source, in a liquid at rest at infinity, is [1]

$$u_i = \frac{\dot{v}_i}{4\pi r^2} \quad (i = 1, 2). \tag{6}$$

When only two bubbles are in the liquid, the kinetic energy of the additional mass of the liquid can be expressed as

$$\begin{aligned} E_k &= \frac{1}{2}\rho \int u_1^2 4\pi r_1^2 \,\mathrm{d}r_1 + \frac{1}{2}\rho \int u_2^2 4\pi r_2^2 \,\mathrm{d}r_2 \\ &= \frac{\rho}{8\pi R_{10}}\dot{v}_1^2 + \frac{\rho}{8\pi R_{20}}\dot{v}_2^2. \end{aligned} \tag{7}$$

For the first (second) bubble, the generalized driving function C is that the pressure P_2 (P_1) generated by the second (first) bubble adds to the external driving pressure P_A. To calculate the respective pressure field of a single bubble, Euler equation is used,

$$\frac{\partial \dot{r}}{\partial t} + \big(\dot{r}\cdot\nabla\big)\dot{r} = -\frac{1}{\rho}\mathrm{grad}\, R_{\mathrm{rad}}. \tag{8}$$

Substituting equation (6) into equation (8), and integration, the pressure of a single bubble can be expressed as

$$p_{\mathrm{rad}i} = \frac{\rho \ddot{v}_i}{4\pi r} \quad (i = 1, 2). \tag{9}$$

Taking expressions (9) and (7) into equation (3) and making corresponding differentiations, we can get the dynamical equations of two coupled bubbles in a liquid,

$$\begin{aligned} &\frac{\rho}{4\pi R_{10}}\ddot{v}_1 + b_1\dot{v}_1 + \frac{\gamma P_0}{V_{10}}\Big(1 + \frac{2\sigma}{P_0 R_{10}} - \frac{2\sigma}{3\gamma P_0 R_{10}}\Big)v_1 \\ &\qquad = -P_A \mathrm{e}^{\mathrm{j}\omega t} + \frac{\rho \ddot{v}_2}{4\pi d}, \end{aligned} \tag{10}$$

$$\begin{aligned} &\frac{\rho}{4\pi R_{20}}\ddot{v}_2 + b_2\dot{v}_2 + \frac{\gamma P_0}{V_{20}}\Big(1 + \frac{2\sigma}{P_0 R_{20}} - \frac{2\sigma}{3\gamma P_0 R_{20}}\Big)v_2 \\ &\qquad = -P_A \mathrm{e}^{\mathrm{j}\omega t} + \frac{\rho \ddot{v}_1}{4\pi d'}, \end{aligned} \tag{11}$$

where d is the distance between bubbles. If the distance between bubbles is so large, the mutual interaction is negligible, and equations (10) and (11) will be reduced to the equation of a single bubble. The analytical solutions of equations (10) and (11) are

$$\begin{aligned} v_1 = {} & \frac{P_A}{\sqrt{\omega^2 b_1^2 + \big(\rho/(4\pi R_{10})\big)^2\big(\omega_{10}^2 - \omega^2\big)^2}} \\ & \cdot \left(1 + \frac{\rho\omega^2 \mathrm{e}^{-\varphi_2 \mathrm{j}}}{4\pi d\sqrt{\omega^2 b_2^2 + \big(\rho/(4\pi R_{20})\big)^2\big(\omega_{20}^2 - \omega^2\big)^2}}\right) \\ & \cdot \mathrm{e}^{\mathrm{j}(\omega t - \varphi_1}, \end{aligned} \tag{12}$$

$$\begin{aligned} v_2 = {} & \frac{P_A}{\sqrt{\omega^2 b_2^2 + \big(\rho/(4\pi R_{20})\big)^2\big(\omega_{20}^2 - \omega^2\big)^2}} \\ & \cdot \left(1 + \frac{\rho\omega^2 \mathrm{e}^{-\varphi_1 \mathrm{j}}}{4\pi d\sqrt{\omega^2 b_1^2 + \big(\rho/(4\pi R_{10})\big)^2\big(\omega_{10}^2 - \omega^2\big)^2}}\right) \\ & \cdot \mathrm{e}^{\mathrm{j}(\omega t - \varphi_2}, \end{aligned} \tag{13}$$

where φ_1 and φ_2 are pulsation phases of bubble 1 and bubble 2 respectively, and the expressions of φ_1 and φ_2 can be written as

$$\begin{aligned} \tan\varphi_1 &= \frac{\omega b_1}{\big(\rho/(4\pi R_{10})\big)\big(\omega_{10}^2 - \omega^2\big)^2}, \\ \tan\varphi_2 &= \frac{\omega b_2}{\big(\rho/(4\pi R_{20})\big)\big(\omega_{10}^2 - \omega^2\big)^2}, \\ \omega_{10}^2 &= \frac{\frac{\gamma P_0}{V_{10}}\Big(1 + \frac{2\sigma}{P_0 R_{10}} - \frac{2\sigma}{3\gamma P_0 R_{10}}\Big)}{\big(\rho/(4\pi R_{10})\big)}, \\ \omega_{20}^2 &= \frac{\frac{\gamma P_0}{V_{20}}\Big(1 + \frac{2\sigma}{P_0 R_{20}} - \frac{2\sigma}{3\gamma P_0 R_{20}}\Big)}{\big(\rho/(4\pi R_{20})\big)}. \end{aligned} \tag{14}$$

The calculations were carried out with atmospheric static pressure $P_0 = 1.01\,\mathrm{bar}$, the amplitude of driving pressure $P_A = 0.95\,\mathrm{bar}$ and driving frequency $f = 200\,\mathrm{kHz}$. Other parameters were set to $\rho = 1000\,\mathrm{kg/m^3}$, $\delta = 0.1$, $\sigma = 0.0725\,\mathrm{N/m}$, $\gamma = 1.4$ The influence of the mutual interaction between two coupled bubbles on the volume pulsation is illustrated in Figure 1. The curve I is the volume change of the first bubble for the coupled case ($R_{10} = 3\mu\mathrm{m}$, $R_{20} = 3\mu\mathrm{m}$ and $d = 200\mu\mathrm{m}$,) and the curve

of II is volume change of a single bubble for the uncoupled case ($R_{10} = 3\mu m$). The contrastive results show that the mutual interaction between two bubbles can change the volume of a single bubble, and influence factors are radii of bubbles, driving frequency and distance between two bubbles.

3. The secondary Bjerknes force between bubbles

When two bubbles are pulsating in an acoustic field, each one can be seen as a second sound source. The net radiation force acting on a neighboring spherical bubble in the acoustic field is called the secondary Bjerknes force. The known formula for the secondary Bjerknes force is [10]

$$\begin{aligned} F_B &= -\frac{\rho}{4\pi d^2}\left\langle \dot{V}_1 \dot{V}_2 \right\rangle \\ &= -\frac{\rho}{4\pi d^2}\left\langle \dot{v}_1 \dot{v}_2 \right\rangle \end{aligned} \tag{15}$$

where $\langle ... \rangle$ denotes the time average. V_1 and V_2 are instantaneous volume of the first bubble and the secondary bubble respectively, and dot denotes the time derivative. Substituting equations (12) and (13) into equation (15), the secondary Bjerknes force can be expressed as

$$F_B = -\frac{\rho\omega^2}{4\pi d^2} v_1' v_2' \cos\left(\varphi_1 - \varphi_2\right), \tag{16}$$

where

$$\begin{aligned} v_1' = &\frac{P_A}{\sqrt{\omega^2 b_1^2 + \left(\rho/(4\pi R_{10})\right)^2\left(\omega_{10}^2 - \omega^2\right)^2}} \\ &\left(1 + \frac{\rho\omega^2 \mathrm{e}^{-\varphi_2 \mathrm{j}}}{4\pi d\sqrt{\omega^2 b_2^2 + \left(\rho/(4\pi R_{20})\right)^2\left(\omega_{20}^2 - \omega^2\right)^2}}\right), \end{aligned} \tag{17}$$

$$\begin{aligned} v_2' = &\frac{P_A}{\sqrt{\omega^2 b_2^2 + \left(\rho/(4\pi R_{20})\right)^2\left(\omega_{20}^2 - \omega^2\right)^2}} \\ &\left(1 + \frac{\rho\omega^2 \mathrm{e}^{-\varphi_1 \mathrm{j}}}{4\pi d\sqrt{\omega^2 b_1^2 + \left(\rho/(4\pi R_{10})\right)^2\left(\omega_{10}^2 - \omega^2\right)^2}}\right). \end{aligned} \tag{18}$$

For presentation of the results, the secondary Bjerknes force cofficient f_B is used, which is defined as [12]

$$f_B = -\frac{\rho}{4\pi}\left\langle \dot{v}_1 \dot{v}_2 \right\rangle. \tag{19}$$

The sign of f_B indicates attraction ($f_B > 0$) or repulsion ($f_B < 0$) of the bubbles.

The influence of the radii of two bubbles on the direction of the secondary Bjerknes force is illustrated in Figure 2. For two coupled bubbles of size $R_{10} = 3\mu m$ and $R_{20} = 5\mu m$, the function $\dot{v}_1\dot{v}_2$ is altered according to Figure 2a, and averaging results in a negative $\langle \dot{v}_1 \dot{v}_2 \rangle = 4.7030\dot{1}0^{-10}\,\mathrm{mm^2/s^2}$. In this case, the secondary Bjerknes between two bubbles is repulsive. For two coupled bubbles of size $R_{10} = 5\mu m$ and $R_{20} = 20\mu m$, the function $\dot{v}_1\dot{v}_2$ is

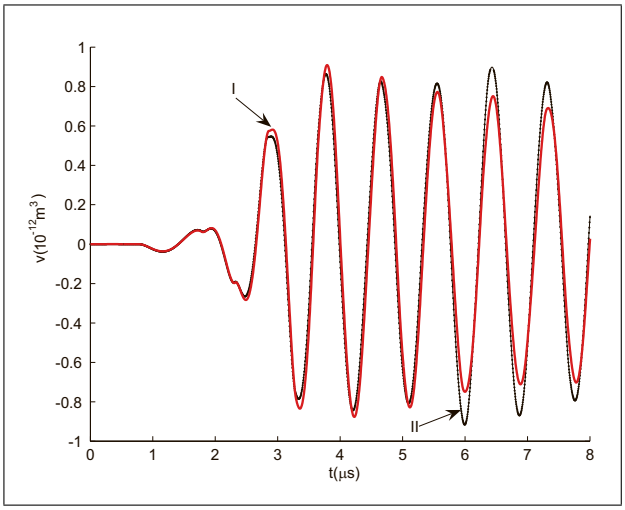


Figure 1. The change of volume of a single bubble ($P_A = 0.95\,\mathrm{bar}$ and $f = 200\,\mathrm{kHz}$). Curves I correspond to v1 for the coupled case ($R_{10} = 3\,\mu m$, $R_{20} = 5\,\mu m$ and $d = 200\,\mu m$). Curve II correspond tov1 for uncoupled case($R_{10} = 3\,\mu m$).

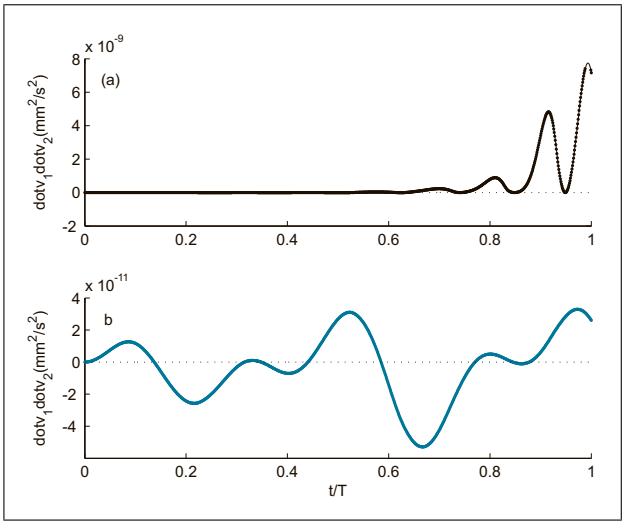


Figure 2. $\dot{v}_1\dot{v}_2$ and normalized time t/T during one driving period ($P_A = 0.95\,\mathrm{bar}$, $f = 200\,\mathrm{kHz}$ and $d = 200\,\mu m$) (a). $R_{10} = 3\,\mu m$ and $R_{20} = 5\,\mu m$, (b). $R_{10} = 5\,\mu m$ and $R_{20} = 20\,\mu m$.

altered according to Figure 2b, and averaging results in a negative $\langle \dot{v}_1 \dot{v}_2 \rangle = -2.3572\dot{1}0^{-12}\,\mathrm{mm^2/s^2}$. In this case, the secondary Bjerknes between two bubbles is repulsive. This phenomenon implies that for weaker driving ($P_A < 1$ bar) if the bubbles both have radii larger (or smaller) than the resonance size, then they attract each other. If the resonance radius lies between the two bubble radii, then repulsion takes place.

For the considered parameter values a linear response of a single bubble occurs, which is shown in Figure 3. If the driving frequency is smaller (or larger) than the resonance frequencies of two coupled bubbles, then they attract each other. If the driving frequency lies between the resonance frequencies of two bubbles, then repulsion takes place. This result is consistent with the well-known linear resonance of the bubble.

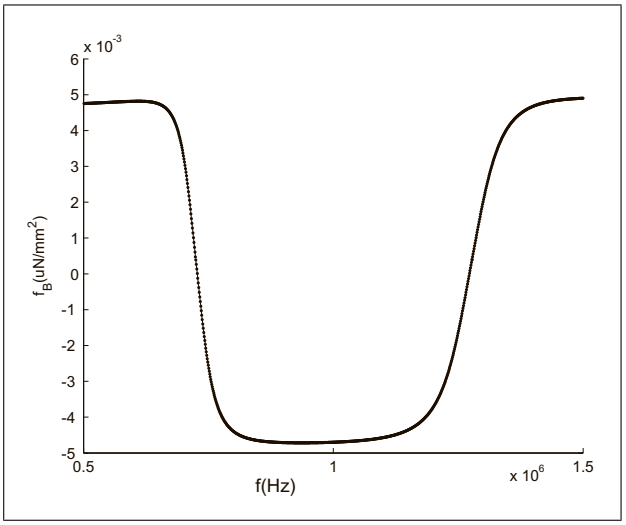


Figure 3. The secondary Bjerknes force cofficient f_B and driving frequency ($P_A = 0.95$ bar, $f = 200$ kHz, $d = 200$ µm, $R_{10} = 3$ µm and $R_{20} = 5$ µm).

4. Influence of the secondary Bjerknes force on radiation impedance of bubbles

In an incompressible liquid, a bubble excited into volume pulsation expends a portion of its energy by radiating spherical sound waves. The bubble is considered as a simple sound source. The bubble is also driven by the sound fields emitted by other bubbles at the same time, so the radiation energy and pulsation mass of this bubble can be influenced by other bubbles. In order to research the influence of the mutual interaction on the radiation damping and pulsation mass, we derive the expression for the mutual radiation impedance of coupled bubbles.

As the secondary sound fields, two bubbles emitted the spherical sound wave in liquids. Considering time delay effects, the radiation pressure of the second bubble can be expressed as

$$\begin{aligned} P_{\text{rad2}} &= -\frac{\rho P_A \omega^2}{4\pi r\sqrt{\omega^2 b_2^2 + \left(\rho/(4\pi R_{20})\right)^2\left(\omega_{20}^2-\omega^2\right)^2}} \\ &\quad\left(1+\frac{\rho\omega^2 \mathrm{e}^{-\varphi_1 \mathrm{j}}}{4\pi d\sqrt{\omega^2 b_1^2 + \left(\rho/(4\pi R_{10})\right)^2\left(\omega_{10}^2-\omega^2\right)^2}}\right) \\ &\quad\cdot \mathrm{e}^{\mathrm{j}[\omega(t-r/c)-\varphi_2]}. \end{aligned} \tag{20}$$

From the calculation above, the secondary Bjerknes force is not negligible at a small distance between bubbles, and this force has a considerable importance in radiation impedance of a bubble. The radiation resistance, radiation energy and pulsation mass of the bubble can be changed by the mutual interaction force. If the amplitude of the volume pulsation is small?the instantaneous force acting from the secondary bubble on the first one can be expressed as

$$\begin{aligned} F_{21} &= -V_1 \nabla P_{\text{rad2}} \\ &= -\frac{\rho\omega^2 V_{10} P_A}{4\pi d^2\sqrt{\omega^2 b_2^2 + \left(\rho/(4\pi R_{20})\right)^2\left(\omega_{20}^2-\omega^2\right)^2}} \\ &\quad\cdot\left(1+\frac{\rho\omega^2 \mathrm{e}^{-\mathrm{j}\varphi_1}}{4\pi d\sqrt{\omega^2 b_1^2 + \left(\rho/(4\pi R_{10})\right)^2\left(\omega_{10}^2-\omega^2\right)^2}}\right) \\ &\quad\cdot\left(1+\mathrm{j}\omega d/c\right)\mathrm{e}^{\mathrm{j}[\omega(t-d/c)-\varphi_2]}. \end{aligned} \tag{21}$$

The mutual radiation impedance due to the secondary Bjerknes force is

$$Z_{21} = -F_{21}/\dot{u}_1, \tag{22}$$

and the radial velocity of the first bubble is

$$\begin{aligned} \dot{u}_1 &= \frac{P_A\omega \mathrm{j}}{4\pi R_{10}^2\sqrt{\omega^2 b_1^2 + \left(\rho/(4\pi R_{10})\right)^2\left(\omega_{10}^2-\omega^2\right)^2}} \\ &\quad\left(1+\frac{\rho\omega^2 \mathrm{e}^{-\mathrm{j}\varphi_2}}{4\pi d\sqrt{\omega^2 b_2^2 + \left(\rho/(4\pi R_{20})\right)^2\left(\omega_{20}^2-\omega^2\right)^2}}\right) \\ &\quad\cdot \mathrm{e}^{\mathrm{j}(\omega(t-R_0/c)-\varphi_1}. \end{aligned} \tag{23}$$

Substituting expressions (21) and (23) into expression (22), the mutual radiation impedance can be expressed as

$$\begin{aligned} Z_{21} &= R_{21} + X_{21} \\ &= \frac{\rho\omega^2 R_{10}^2 V_{10}}{d^2 \mathrm{j}}\frac{\sqrt{\omega^2 b_1^2 + \left(\rho/(4\pi R_{10})\right)^2\left(\omega_{10}^2-\omega^2\right)^2}}{\sqrt{\omega^2 b_2^2 + \left(\rho/(4\pi R_{20})\right)^2\left(\omega_{20}^2-\omega^2\right)^2}} \\ &\quad\frac{\left(1+\frac{\rho\omega^2 \mathrm{e}^{-\mathrm{j}\varphi_1}}{4\pi d\sqrt{\omega^2 b_1^2+\left(\rho/(4\pi R_{10})\right)^2\left(\omega_{10}^2-\omega^2\right)^2}}\right)}{\left(1+\frac{\rho\omega^2 \mathrm{e}^{-\mathrm{j}\varphi_2}}{4\pi d\sqrt{\omega^2 b_2^2+\left(\rho/(4\pi R_{20})\right)^2\left(\omega_{20}^2-\omega^2\right)^2}}\right)} \\ &\quad\left(1+\mathrm{j}\omega d/c\right)\mathrm{e}^{\mathrm{j}[\varphi_1-\varphi_2+\omega/c(R_0-d)]}. \end{aligned} \tag{24}$$

From expression (24), the mutual radiation impedance divides two parts. The real part is resistance of the mutual radiation impedance, which shows the change of the radiation damping of the first bubble caused by the mutual interaction between two bubbles. If the resistance is positive, the mutual interaction between two coupled bubbles will hinder the pulsation of the first bubble, causing the radiation damping of the first bubble to increase, otherwise the radiation damping decreases. The imaginary part X_{21} is reactance of mutual radiation impedance, which shows the change of radiation mass of the first bubble caused by the mutual interaction between two bubbles. If the reactance is positive, the mutual interaction between two bubbles will have an inertial effect on the first bubble, causing the pulsation mass of the first bubble to increase. Otherwise, the pulsation mass of the first bubble decreases. According to a similar calculation?the radiation impedance of the first bubble itself can be expressed as

$$Z_{11} = R_{11} + X_{11}\mathrm{j} = \frac{\rho\omega^2 V_{10} R_{10}}{c} - \rho\omega V_{10}\mathrm{j}, \tag{25}$$

where R_{11} is the resistance of radiation impedance of the first bubble, which shows the change of the radiation

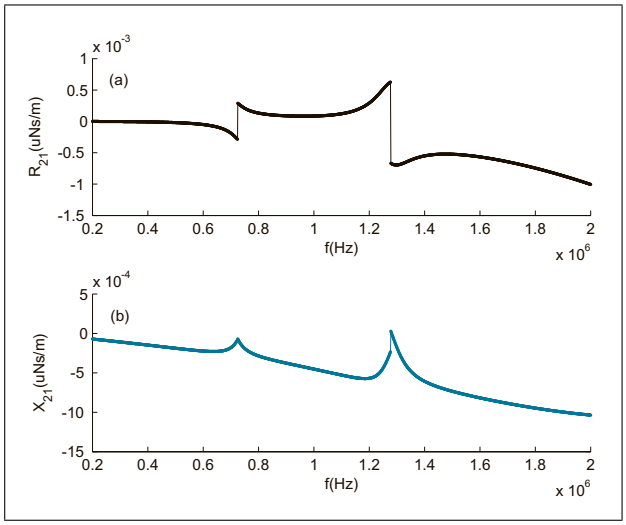


Figure 4. The mutual radiation impendences Z21 and driving frequency ($P_A = 0.95$ bar, $f = 200$ kHz, $d = 200$ µm, $R_{10} = 3$ µm and $R_{20} = 5$ µm).

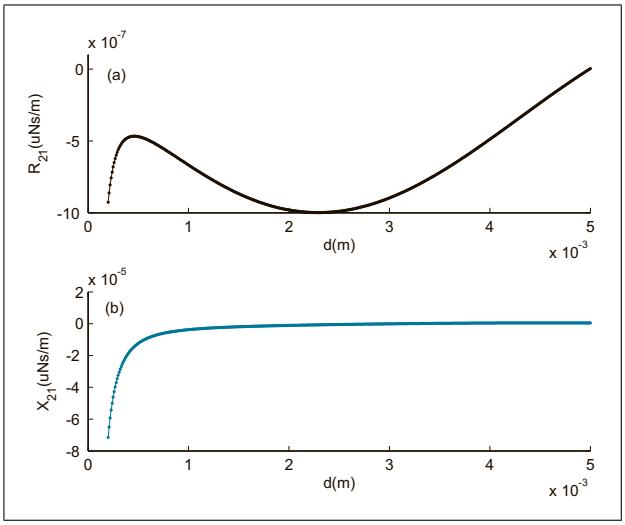


Figure 5. The mutual radiation impedance Z_{21} and distance between bubbles ($P_A = 0.95$ bar, $f = 200$ kHz, $d = 200$ µm $R_{10} = 3$ µm and $R_{20} = 5$ µm.

damping caused by the first bubble itself. X_{11} is the reactance of radiation impedance of the first bubble, which shows the change of radiation mass caused by the first bubble itself.

For two coupled bubbles, if the resonance frequencies of two are smaller (or larger) than the driving frequency, the radiation damping will decrease. If the driving frequency lies between the resonance frequencies of two bubbles, the radiation damping will increase (Figure 4a). This result is consistent with well-known linear resonance of the bubble. This implies that the change of the radiation damping of a single bubble is related to direction of the secondary Bjerknes force. If two coupled bubbles reject each other, the radiation damping of a single bubble will increase. If two coupled bubbles attract each other, the radiation damping of a single bubble will decrease. We found that the influence of the mutual interaction on the pulsation mass of a bubble is small, where is shown in Figure 4b. For two bubbles of size $R_{10} = 3$ µm and $R_{20} = 5$ µm, X_{21} is negative with the increasing driving frequency. This implies that the mutual interaction between bubbles is an elastic effect on the pulsation of a single bubble, resulting in the decrease of the pulsation mass.

In addition, the mutual radiation impedance is related to the distance between two bubbles and the radii of two bubbles. R_{21} and X_{21} tend to 0 with the increasing distance between bubbles, which are shown in Figure 5a and Figure 5b. If the bubbles are far apart, pulsations are uncoupled and the mutual interaction between bubbles is negligible.

The radii of two coupled bubbles have a significant effect on the mutual radiation impedance. The curves of the mutual radiation impedance are plotted in Figure 6a and Figure 6b. If the radius of the first bubble is 2 µm, R_{21} is negative and decreases with the increasing radius of the second bubble. At about 17 µm a resonance-like maximum appears. R_{21} enlarges at a linear resonance which is shown

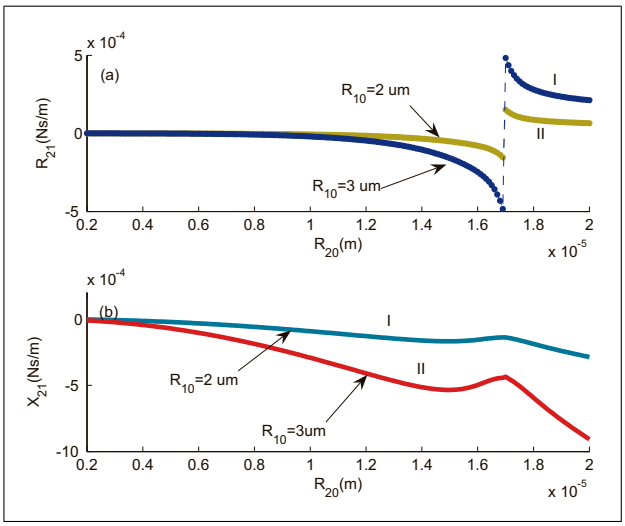


Figure 6. The mutual radiation impedance and the radius of the second bubble ($P_A = 0.95$ bar, $f = 200$ kHz and $d = 200$ µm).

as curve II in Figure 6a. After the linear resonance, R_{21} is positive and decreases with the increasing radius of the second bubble. For X_{21}, the similar results are shown in Figure 6b. Furthermore, the magnitude of the mutual radiation impedance for two large coupled bubbles is bigger than that of two small coupled bubbles, which are shown as curve I and II in Figure 7a and Figure 7b.

To research the influence of mechanics characteristics of a bubble caused by the mutual interaction between two coupled bubbles, we calculated R_{21}/R_{11} and X_{21}/X_{11} for the considered parameter. It is an important result of Figures 7a and 7b that the influence of mutual interaction on radiation damping of the first bubble is much more than that of pulsation mass, which is two orders of magnitude in the considered range of driving frequency. Moreover, the influence of the mutual force between two large bubbles on the radiation damping is much more than that of

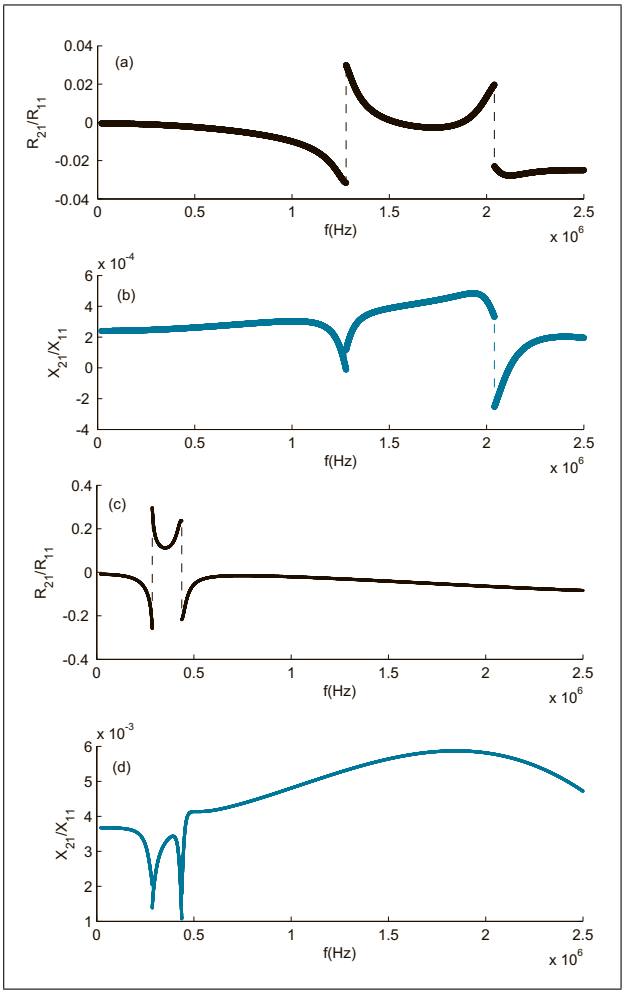


Figure 7. Z_{21}/Z_{11} and the driving frequency ($P_A = 0.95$ bar, $f = 200$ kHz, $d = 200\,\mu$m). (a) and (b): $R_{10} = 2\,\mu$m, $R_{20} = 3\,\mu$m, (c) and (d): $R_{10} = 8\,\mu$m, $R_{20} = 12\,\mu$m.

two small bubbles (Figure 7). R_{21}/R_{11} of two big bubbles ($R_{10} = 8\,\mu$m, $R_{20} = 12\,\mu$m) is ten times the magnitude of that of two small bubbles ($R_{10} = 3\,\mu$m, $R_{20} = 5\,\mu$m) at resonance. In particular, for two large coupled bubbles ($R_{10} = 8\,\mu$m, $R_{20} = 12\,\mu$m), the increased radiation damping of the first bubble becomes comparable in magnitude to the radiation damping of the bubble itself, which is illustrated in Figure 7c.

For a single bubble, the radiation damping is independent of driving frequency, but this result is inapplicable for the coupled bubbles. Radiation damping of coupled bubbles is dependent of driving frequency and the radii of bubbles. In particular, the radiation damping of large bubbles increases obviously. This implies that we should reconsider the radiation damping of the bubble clouds. The influence of the mutual interaction force between bubbles on the pulsation mass of a single bubble is very small, which almost not cause the change of the resonance frequency of the bubble.

5. Conclusion

The secondary Bjerknes force and the radiation impedance of two coupled bubbles in a weak acoustic field ($P_A <$ 1 bar) have been investigated assuming that the spacing between the bubbles is large enough that the bubbles remain spherical for all times. The calculations have been carried out using the linear second-order differential equations of motion for two coupled bubbles system.

For the considered parameter values a linear response of a bubble occurs. If the bubbles both have radii smaller (or larger) than the resonance radius, then they attract each other. If the resonance radius lies between the two bubble radii, then repulsion takes place.

If the driving frequency lies between the linear resonance frequencies of two bubbles, the radiation damping of a single bubble will increase; otherwise the radiation damping decreases. This result may be explained using the well-known linear resonance of the bubble as an analogy. For two coupled bubbles, the result show that the radiation damping is related to the driving frequency and radii of two bubbles. The result also shows that the mutual force between bubbles almost not change the resonance frequency of a single bubble, which is consistent with [20].

Our finding is important for static clusters of large bubbles in weak sound fields [21], as a single bubble modeling may lead to severe underestimation of the radiation damping. Radiation damping of a bubble in bubble clouds is bigger than that of a single bubble due to the mutual interaction between bubbles. The radiation damping at resonance may play a big role in the damping for static clusters of large bubbles. Future investigations will address to radiation damping of clusters of bubbles in strong acoustic fields, for instance the coupled pulsation of multi-bubbles and chaotic phenomena.

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