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Radiation Characteristics of Two Semi-Infinite Overlapped Cylindrical Waveguides with Impedance Loading

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Summary
The radiation of plane waves by two coaxial overlapping pipes is investigated in the case where the walls of the pipes lying in the overlap region are impedance boundaries. By using the eigenfunction expansion in the overlap region and using the Fourier transform technique elsewhere we obtain directly two uncoupled modified Wiener-Hopf equations of the second kind whose solutions involve two set of infinitely many unknown expansion coefficients satisfying two systems of linear algebraic equations. Numerical solution of these systems are obtained for various values of the parameters of the problem whereby are studied the effects of these parameters on the radiation phenomenon.

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1. Introduction

In the present work we reconsider the radiation of acoustic waves from two coaxial opposite semi-infinite circular cylinders in the case where the walls of the ducts in the overlapping region are impedance boundaries. This configuration may be used for the purposes of electromagnetic shielding, the determination of the sound from a hole in an exhaust system. The case where the walls of the pipes lying in the overlap region is perfectly rigid was treated by Lawrie, Abrahams and Linton [1]. By using the Fourier transform technique, they formulated the related boundary value problem as a matrix Wiener-Hopf equation and then solved by using the weak factorization method [2] [3] [4].

The main objective of this paper is to reveal the influence of the impedance loading on the radiated field and to present an alternative method of formulation which enable to skip the most challenging step of matrix factorization. The method adopted in this paper is similar to that employed previously in [5] and consists of expanding the field in the overlap region into a series of complete set of orthogonal eigenfunctions and using the Fourier transform technique elsewhere. The problem is then reduced directly into a pair of uncoupled modified Wiener-Hopf equations of the second kind whose solutions involve two set of infinitely many unknown expansion coefficients satisfying two infinite systems of linear algebraic equations. Numerical solution to these systems are obtained for various values of the parameters of the problem such as the radii of the semi infinite waveguides, the overlap length and the impedance loading whereby the effects of these parameters on the radiation phenomenon are presented graphically. One should also note that the method used in [1] is not suitable to extend the analysis to the case where an impedance loading in the overlap region is present.

The time dependence is assumed to be $\exp(-i\omega t)$ with ω being the angular frequency and suppressed throughout.

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2. Formulation of the problem

Consider two opposite semi-infinite circular cylindrical waveguides of different radii with common longitudinal axis, say z, in a cylindrical polar coordinate system (ρ, ϕ, z). They occupy the regions $\rho = a,\ z < l$ and $\rho = b < a,\ z > 0$, respectively, where l represents the overlap length. The parts of the surfaces $r = a+0$ and $\rho = b-0$ lying in the overlap region $0 < z < l$ of the waveguides are assumed to be treated by acoustically absorbing linings which are characterized by constant but different surface impedances, say $Z_1$and Z_2, respectively; while the remaining parts are perfectly rigid (see Figure 1). The waveguides are immersed in the inviscid and compressible stationary fluid of density $\tilde{\rho}_0$ and sound speed c. A plane sound wave is incident in the positive z-direction, through the waveguide of radius $\rho = a$. From the symmetry of the geometry of the problem and the incident field the acoustic field everywhere will be independent of the ϕ coordinate. We shall therefore introduce a scalar potential $u(\rho, z)$ which defines the acoustic pressure and velocity by $p = i\omega\tilde{\rho}_0 u$ and $\mathbf{v} = \text{grad}\, u$, respectively.

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Let the incident field be given by

$$u^i = \exp\left(\mathrm{i}kz\right), \tag{1}$$

where $k = \omega/c$ denotes the wave number. For the sake of analytical convenience we will assume that the surrounding medium is slightly lossy and k has a small positive imaginary part. The lossless case can be obtained by letting $\mathrm{Im}\, k \to 0$ at the end of the analysis.

The total field $u^T(\rho, z)$ can be written as

$$u^T = \begin{cases} u_1(\rho, z), & \rho > b, z \in (-\infty, \infty), \\ u_2^{(1)}(\rho, z), & \rho \in (a, b), z \in (-\infty, 0), \\ u_2^{(2)}(\rho, z), & \rho \in (a, b), z \in (0, l), \\ u_2^{(3)}(\rho, z), & \rho \in (a, b), z \in (l, \infty), \\ u_3(\rho, z) + u^i, & \rho < a, z \in (-\infty, \infty). \end{cases} \tag{2}$$

$u_1(\rho, z), u_2^{(j)}(\rho, z)$ and $u_3(\rho, z)$ $(j = 1, 2, 3)$ denote the scattered fields which satisfy the Helmholtz equation,

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) + \frac{\partial^2}{\partial z^2} + k^2\right] \begin{bmatrix} u_1(\rho, z) \\ u_2^{(j)}(\rho, z) \\ u_3(\rho, z) \end{bmatrix} = 0, \tag{3}$$

are to be determined with the help of the following boundary and continuity relations:

$$\frac{\partial}{\partial\rho}u_1(b, z) = 0, \quad z > 0, \tag{4a}$$

$$\frac{\partial}{\partial\rho}u_2^{(1)}(a, z) = 0, \quad z < 0, \tag{4b}$$

$$\left[\mathrm{i}k\eta_1 + \frac{\partial}{\partial\rho}\right] u_2^{(2)}(a, z) = 0, \quad 0 < z < l, \tag{4c}$$

$$\left[\mathrm{i}k\eta_2 - \frac{\partial}{\partial\rho}\right] u_2^{(2)}(b, z) = 0, \quad 0 < z < l, \tag{4d}$$

$$\frac{\partial}{\partial\rho}u_2^{(3)}(b, z) = 0, \quad z > l, \tag{4e}$$

$$\frac{\partial}{\partial\rho}u_3(a, z) = 0, \quad z < l, \tag{4f}$$

$$u_1(b, z) - u_2^{(1)}(b, z) = 0, \quad z < 0, \tag{4g}$$

$$\frac{\partial}{\partial\rho}u_1(b, z) - \frac{\partial}{\partial\rho}u_2^{(1)}(b, z) = 0, \quad z < 0, \tag{4h}$$

$$u_2^{(1)}(\rho, 0) - u_2^{(2)}(\rho, 0) = 0, \quad a < \rho < b, \tag{4i}$$

$$\frac{\partial}{\partial z}u_2^{(1)}(\rho, 0) - \frac{\partial}{\partial z}u_2^{(2)}(\rho, 0) = 0, \quad a < \rho < b, \tag{4j}$$

$$u_2^{(2)}(\rho, l) - u_2^{(3)}(\rho, l) = 0, \quad a < \rho < b, \tag{4k}$$

$$\frac{\partial}{\partial z}u_2^{(2)}(\rho, l) - \frac{\partial}{\partial z}u_2^{(3)}(\rho, l) = 0, \quad a < \rho < b, \tag{4l}$$

$$u_3(a, z) + u^i(a, z) - u_2^{(3)}(a, z) = 0, \quad z > l, \tag{4m}$$

$$\frac{\partial}{\partial\rho}u_3(a, z) + \frac{\partial}{\partial\rho}u^i(a, z) - \frac{\partial}{\partial\rho}u_2^{(3)}(a, z) = 0, \quad z > l, \tag{4n}$$

where in (4c,d) η_1, η_2 are admittances of the acoustic linings which are related by surface impedances as $\eta_1 = 1/Z_1, \eta_2 = 1/Z_2$. To ensure the uniqueness of the mixed boundary-value problem stated by (3) and (4a-n), one has to take into account the following radiation and edge conditions:

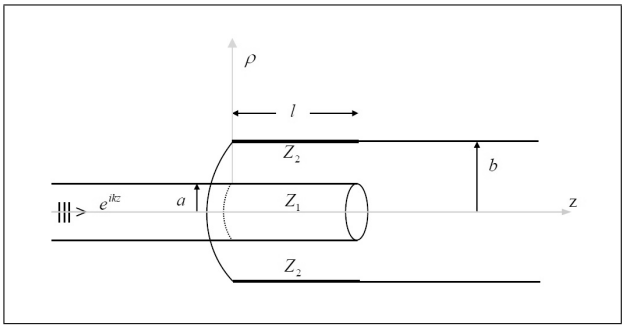


Figure 1. Geometry of the problem.

$$u \sim \frac{\mathrm{e}^{\mathrm{i}kr}}{r}, \quad r = \sqrt{\rho^2 + z^2} \to \infty, \tag{5}$$

$$u^T(\rho, z) = O(1), \quad z \to 0, \tag{6a}$$

$$\frac{\partial}{\partial\rho}u^T(\rho, z) = O(z^{-1/2}), \quad z \to 0, \tag{6b}$$

$$u^T(\rho, z) = O(1), \quad z \to l, \tag{6c}$$

$$\frac{\partial}{\partial\rho}u^T(\rho, z) = O\left((z - l)^{-1/2}\right), \quad z \to l. \tag{6d}$$

2.1. The Wiener-Hopf Equations

Let us consider the Fourier transform of the Helmholtz equation which the field $u_1(\rho, z)$ satisfies in the region $\rho > b$ for $z \in (-\infty, \infty)$.

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + K^2(\alpha)\right] F(\rho, \alpha) = 0, \tag{7}$$

where $F(\rho, \alpha)$ is the Fourier transform of the field $u_1(\rho, z)$ defined by

$$F(\rho, \alpha) = \int_{-\infty}^{\infty} u_1(\rho, z)\, \mathrm{e}^{\mathrm{i}\alpha z}\, \mathrm{d}z = F_+(\rho, \alpha) + F_-(\rho, \alpha), \tag{8a}$$

with

$$F_\pm(\rho, \alpha) = \pm\int_0^{\pm\infty} u_1(\rho, z)\, \mathrm{e}^{\mathrm{i}\alpha z}\, \mathrm{d}z. \tag{8b}$$

Owing to the analytical properties of Fourier integrals, $F_+(\rho, \alpha)$ and $F_-(\rho, \alpha)$ are regular functions in the upper half-plane $\mathrm{Im}\,\alpha > \mathrm{Im}(-k)$ and in the lower half-plane $\mathrm{Im}\,\alpha < \mathrm{Im}\, k$, respectively. The solution of (7) satisfying the radiation condition for $\rho > b$ reads

$$F(\rho, \alpha) = -A(\alpha)\frac{H_0^{(1)}(K\rho)}{K(\alpha)H_1^{(1)}(Kb)}, \tag{9}$$

where $A(\alpha)$ is a spectral coefficient to be determined and $K(\alpha)$ is the square-root function

$$K(\alpha) = \sqrt{k^2 - \alpha^2}, \tag{10a}$$

which is defined in the complex α-plane such that $K(0) = k$. In (9), $H_n^{(1)}$ stands for the Hankel function of the first kind and n-th order, given by

$$H_n^{(1)} = J_n + \mathrm{i}Y_n. \tag{10b}$$

Consider now the Fourier transform of (4a), namely

$$\dot{F}_+(b, \alpha) = 0, \tag{11}$$

where the dot specifies the derivative with respect to ρ. The differentiation of (9) with respect to ρ gives

$$\dot{F}_-(b, \alpha) = A(\alpha). \tag{12}$$

substituting (12) into (9) yields

$$F_-(\rho, \alpha) = -\frac{\dot{F}_-(b, \alpha)}{K(\alpha)} \frac{H_0^{(1)}(K\rho)}{H_1^{(1)}(Kb)} - F_+(\rho, \alpha). \tag{13}$$

In the region $a < \rho < b$ the field $u_2^{(1)}(\rho, z)$ satisfies the Helmholtz equation for $z \in (-\infty, 0)$ as denoted in (3). The Fourier transform of this equation for the region in question is

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + K^2(\alpha)\right] G_-(\rho, \alpha) = \mathrm{i}\alpha f(\rho) - g(\rho), \tag{14}$$

where

$$f(\rho) = u_2^{(1)}(\rho, 0) \tag{15a}$$

and

$$g(\rho) = \frac{\partial}{\partial\rho} u_2^{(1)}(\rho, 0). \tag{15b}$$

In (14), $G_-(\rho, \alpha)$ is a regular function in the lower half of the α-plane which is defined as

$$G_-(\rho, \alpha) = \int_{-\infty}^{0} u_2^{(1)}(\rho, z)\mathrm{e}^{\mathrm{i}\alpha z}\,\mathrm{d}z. \tag{16}$$

Particular solutions to (14) can be found easily by using the Green's function technique which satisfies the Helmholtz equation,

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + K^2\right] \mathcal{G}(\rho, t, \alpha) = 0, \quad \rho \neq t, \quad \rho, t \in (a, b), \tag{17}$$

with the conditions

$$\mathcal{G}(t + 0, t, \alpha) = \mathcal{G}(t - 0, t, \alpha) \tag{18a}$$

$$\frac{\partial}{\partial\rho}\mathcal{G}(t + 0, t, \alpha) - \frac{\partial}{\partial\rho}\mathcal{G}(t - 0, t, \alpha) = \frac{1}{t}, \tag{18b}$$

$$\frac{\partial}{\partial\rho}\mathcal{G}(b, t, \alpha) = 0 \tag{18c}$$

$$\frac{\partial}{\partial\rho}\mathcal{G}(a, t, \alpha) = 0. \tag{18d}$$

The solution is

$$\mathcal{G}(\rho, t, \alpha) = \frac{1}{K^2(\alpha)M(\alpha)} Q(\rho, t, \alpha), \tag{19}$$

with

$$Q(\rho, t, \alpha) = \frac{\pi}{2} \cdot \begin{cases} \left[J_0(K\rho)KY_1(Ka) - KJ_1(Ka)Y_0(K\rho)\right] \\ \left[J_0(Kt)KY_1(Kb) - KJ_1(Kb)Y_0(Kt)\right], & a \le \rho \le t, \\ \left[J_0(K\rho)KY_1(Kb) - KJ_1(Kb)Y_0(K\rho)\right] \\ \left[J_0(Kt)KY_1(Ka) - KJ_1(Ka)Y_0(Kt)\right], & t \le \rho \le b \end{cases} \tag{20a}$$

and

$$M(\alpha) = \left[J_1(Ka)Y_1(Kb) - J_1(Kb)Y_1(Ka)\right]. \tag{20b}$$

When the boundary condition (4b) is taken into account the solution of (14) can now be written as

$$G_-(\rho, \alpha) = \frac{1}{K^2(\alpha)M(\alpha)} \cdot \left\{ C(\alpha)\left[J_0(K\rho)KY_1(Ka) - Y_0(K\rho)KJ_1(Ka)\right] + \int_a^b \left[\mathrm{i}\alpha f(t) - g(t)\right] Q(t, \rho, \alpha)\, t\,\mathrm{d}t \right\}. \tag{21}$$

Here, $C(\alpha)$ is a spectral coefficient to be determined.

The continuity relation in (4h) requires

$$\dot{G}_-(b, \alpha) = \dot{F}_-(b, \alpha). \tag{22}$$

$C(\alpha)$ can be solved uniquely from (22) as

$$C(\alpha) = \dot{F}_-(b, \alpha). \tag{23}$$

The substitution of (23) into (21) gives

$$G_-(\rho, \alpha) = \frac{1}{K^2(\alpha)M(\alpha)} \cdot \left\{ \dot{F}_-(b, \alpha)\left[J_0(K\rho)KY_1(Ka) - Y_0(K\rho)KJ_1(Ka)\right] + \int_a^b \left[\mathrm{i}\alpha f(t) - g(t)\right] Q(t, \rho, \alpha)\, t\,\mathrm{d}t \right\}. \tag{24}$$

Although the left-hand side of (24) is regular in the half-plane $\mathrm{Im}(\alpha) < \mathrm{Im}(k)$, the regularity of the right-hand side is violated by the presence of simple poles lying at the lower half plane, namely at $\alpha = -\alpha_m$ ($\mathrm{Im}\,\alpha_m > \mathrm{Im}\,k$) with

$$K_m^2\left[J_1(K_m a)Y_1(K_m b) - J_1(K_m b)Y_1(K_m a)\right] = 0, \quad K_m = K(\alpha_m), \quad m = 0, 1, 2, \ldots \tag{25}$$

These poles can be isolated in order to provide regularity in the lower half of the α-plane for the right hand side of (24) with

$$\dot{F}_-(b,-k) = -\frac{1}{b}\int_a^b \left[g(t)+\mathrm{i}kf(t)\right] t\,\mathrm{d}t, \tag{26a}$$

$$\dot{F}_-(b,-\alpha_m) = \frac{\pi}{2}\frac{J_1(K_mb)}{J_1(K_ma)}\int_a^b \Big\{ \left[g(t)+i\alpha_m f(t)\right] \cdot K_m\left[J_0(K_mt)Y_1(K_ma)-Y_0(K_mt)J_1(K_ma)\right] t\,\mathrm{d}t \Big\} \tag{26b}$$

for $m = 0, 1, 2, \ldots$ Here, f_m and g_m's are the expansion coefficients of the functions $f(\rho)$ and $g(\rho)$ which may be represented with the following complete sets of biorthogonal functions.

$$\begin{bmatrix} f(\rho) \\ g(\rho) \end{bmatrix} = \frac{2}{\pi a}\begin{bmatrix} f_0 \\ g_0 \end{bmatrix} + \sum_{m=1}^{\infty}\begin{bmatrix} f_m \\ g_m \end{bmatrix} \cdot K_m\left[J_0(K_m\rho)Y_1(K_ma)-Y_0(K_m\rho)J_1(K_ma)\right]. \tag{27a}$$

Note that

$$\int_a^b \Big\{ \left[g(t)+\mathrm{i}\alpha_m f(t)\right] \cdot K_m\left[J_0(K_mt)Y_1(K_ma)-Y_0(K_mt)J_1(K_ma)\right] t\,\mathrm{d}t \Big\} = \left[g_m+\mathrm{i}\alpha_m f_m\right]\frac{2}{\pi^2}\frac{J_1^2(K_ma)-J_1^2(K_mb)}{J_1^2(K_mb)}, \tag{27b}$$

which allows (26a,b) to be

$$\dot{F}_-(b,-k) = \frac{1}{\pi}\left[g_0+\mathrm{i}kf_0\right]\frac{a^2-b^2}{ab} \tag{27c}$$

$$\dot{F}_-(b,-\alpha_m) = \frac{1}{\pi}\left[g_m+\mathrm{i}\alpha_m f_m\right]\frac{J_1^2(K_ma)-J_1^2(K_mb)}{J_1(K_ma)J_1(K_mb)}. \tag{27d}$$

Using the continuity relation (4g) together with (24) and taking into account (13) with the relation

$$\frac{1}{K^2(\alpha)M(\alpha)}\int_a^b \left[\mathrm{i}\alpha f(t)-g(t)\right]Q(t,b,\alpha)\,t\,\mathrm{d}t = \frac{2}{\pi a}\frac{1}{K^2(\alpha)}\left[\mathrm{i}\alpha f_0-g_0\right] - \frac{2}{\pi b}\sum_{m=1}^{\infty}\left[\mathrm{i}\alpha f_m-g_m\right]\frac{J_1(K_ma)}{J_1(K_mb)}\frac{1}{K^2-K_m^2} \tag{28a}$$

gives

$$\frac{b}{2}F_+(b,\alpha)-\frac{L(\alpha)}{K^2(\alpha)}\dot{F}_-(b,\alpha) = -\frac{b}{\pi a}\frac{\left[\mathrm{i}\alpha f_0-g_0\right]}{K^2(\alpha)}+\frac{1}{\pi}\sum_{m=1}^{\infty}\frac{\left[\mathrm{i}\alpha f_m-g_m\right]}{\alpha_m^2-\alpha^2}\frac{J_1(K_ma)}{J_1(K_mb)}, \tag{28b}$$

which is nothing but the first decoupled Wiener-Hopf equation to be solved through classical procedures. Here, $L(\alpha)$ stands for

$$L(\alpha) = \frac{H_1^{(1)}(Ka)}{\pi M(\alpha)H_1^{(1)}(Kb)}. \tag{29}$$

The final solution of the above equation is determined to be

$$\dot{F}_-(b,\alpha) = \frac{b}{\pi a}\frac{\left[\mathrm{i}kf_0-g_0\right]}{L_-(\alpha)L_+(k)}-\frac{(k-\alpha)}{\pi L_-(\alpha)} \cdot \sum_{m=1}^{\infty}\frac{\left[\mathrm{i}\alpha_m f_m-g_m\right]}{2\alpha_m(\alpha_m-\alpha)}\frac{J_1(K_ma)}{J_1(K_mb)}\frac{(k+\alpha_m)}{L_+(\alpha_m)}, \tag{30}$$

where $L_\pm(\alpha)$ are the split functions

$$L(\alpha) = L_+(\alpha)L_-(\alpha), \tag{31a}$$

$$L_-(\alpha) = L_+(-\alpha). \tag{31b}$$

The explicit expression for $L_+(\alpha)$ can be obtained by using the results of [6] and [7] as

$$L_+(\alpha) = \sqrt{\frac{H_1^{(1)}(ka)}{\pi H_1^{(1)}(kb)\left[J_1(ka)Y_1(kb)-J_1(kb)Y_1(ka)\right]}} \cdot \exp\left\{q_2(\alpha)-q_1(\alpha)\right\}\prod_{m=1}^{\infty}\frac{1}{(1+\frac{\alpha}{\alpha_m})\mathrm{e}^{-\alpha/\alpha_m}} \cdot \exp\left[\frac{\mathrm{i}k(b-a)}{2}+\frac{K(\alpha)(a-b)}{\pi}\ln\frac{\alpha+\mathrm{i}K(\alpha)}{k}\right] \cdot \exp\left\{\frac{\alpha}{\pi i}(b-a)\left[1-C+\ln\left(\frac{2\pi i}{k(b-a)}\right)\right]\right\}, \tag{32a}$$

where C is the Euler's constant given by $C = 0.57721\ldots$ and $q_{1,2}(\alpha)$ stands for

$$q_1(\alpha) = \frac{1}{\pi}\mathcal{P}\int_0^{\infty}\left[1-\frac{2}{\pi x}\frac{1}{J_1^2(x)+Y_1^2(x)}\right] \cdot \ln\left(1+\frac{\alpha b}{\left[(kb)^2-x^2\right]^{1/2}}\right)\mathrm{d}x \tag{32b}$$

$$q_2(\alpha) = \frac{1}{\pi}\mathcal{P}\int_0^{\infty}\left[1-\frac{2}{\pi x}\frac{1}{J_1^2(x)+Y_1^2(x)}\right] \cdot \ln\left(1+\frac{\alpha a}{\left[(ka)^2-x^2\right]^{1/2}}\right)\mathrm{d}x. \tag{32c}$$

In (32b,c), the letter $\mathcal{P}$ denotes the Cauchy principle value at the singularity $x = ka$ and $x = kb$. Note that when we let $|\alpha| \to \infty$ in their respective regions of regularity, we have

$$L^{\pm}(\alpha) = (\pm\alpha)^{1/2}. \tag{32d}$$

Let us now consider the Fourier transform of the Helmholtz equation which the field $u_3(\rho, z)$ satisfies in the region $\rho < a$ for $z \in (-\infty, \infty)$, namely,

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+K^2(\alpha)\right]H(\rho,\alpha) = 0, \tag{33}$$

where $H(\rho, \alpha)$ is the Fourier transform of the field $u_3(\rho, z)$ defined to be

$$H(\rho,\alpha) = \mathrm{e}^{\mathrm{i}\alpha l}\left[H_+(\rho,\alpha)+H_-(\rho,\alpha)\right] \tag{34a}$$

with

$$H_-(\rho,\alpha) = \int_{-\infty}^{l} u_3(\rho,z)\mathrm{e}^{\mathrm{i}\alpha(z-l)}\,\mathrm{d}z, \tag{34b}$$

$$H_+(\rho,\alpha) = \int_{l}^{\infty} u_3(\rho,z)\mathrm{e}^{\mathrm{i}\alpha(z-l)}\,\mathrm{d}z. \tag{34c}$$

Owing to the analytical properties of Fourier integrals, $H_+(\rho,\alpha)$ and $H_-(\rho,\alpha)$ are regular functions in the upper half-plane $\mathrm{Im}\,\alpha > \mathrm{Im}(-k)$ and in the lower half-plane $\mathrm{Im}\,\alpha < \mathrm{Im}\,k$, respectively. The solution of (33) reads

$$H(\rho,\alpha) = -B(\alpha)\frac{J_0(K\rho)}{K(\alpha)J_1(Ka)}, \tag{35}$$

where $B(\alpha)$ is a spectral coefficient to be determined. Consider now the Fourier transform of (4f), namely

$$\dot{H}_-(a,\alpha) = 0. \tag{36}$$

The differentiation of (35) with respect to ρ and then substituting $\rho = a$ gives

$$\mathrm{e}^{\mathrm{i}\alpha l}\dot{H}_+(a,\alpha) = B(\alpha). \tag{37}$$

Substituting (37) into (35) yields

$$H_+(\rho,\alpha) = -\dot{H}_+(a,\alpha)\frac{J_0(K\rho)}{K(\alpha)J_1(Ka)} - H_-(\rho,\alpha). \tag{38}$$

In the region $a < \rho < b$ the field $u_2^{(3)}(\rho,z)$ satisfies the Helmholtz equation for $z \in (l,\infty)$ as denoted in (3). The Fourier transform of this equation for the region in question is

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\frac{\partial}{\partial\rho}) + K^2(\alpha)\right] G_+(\rho,\alpha) = p(\rho) - \mathrm{i}\alpha q(\rho) \tag{39}$$

where

$$p(\rho) = \frac{\partial}{\partial\rho}u_2^{(3)}(\rho,l) \tag{40a}$$

and

$$q(\rho) = u_2^{(3)}(\rho,l). \tag{40b}$$

In (39), $G_+(\rho,\alpha)$ is a regular function in the upper half of the complex α-plane which is defined as

$$G_+(\rho,\alpha) = \int_{l}^{\infty} u_2^{(3)}(\rho,z)\mathrm{e}^{\mathrm{i}\alpha(z-l)}\,\mathrm{d}z. \tag{41}$$

Particular solutions to (39) can be found by using the Green's function given in (19). When the boundary condition (4e) is taken into account the solution can be written as

$$G_+(\rho,\alpha) = \frac{1}{K^2(\alpha)M(\alpha)} \cdot \left\{ D(\alpha)\left[J_0(K\rho)KY_1(Kb) - Y_0(K\rho)KJ_1(Kb)\right] + \int_a^b \left[p(t) - \mathrm{i}\alpha q(t)\right]Q(t,\rho,\alpha)\,t\,\mathrm{d}t \right\}. \tag{42}$$

Here, $D(\alpha)$ is a spectral coefficient to be determined.

The continuity relation in (4n) requires

$$\dot{G}_+(a,\alpha) = \dot{H}_+(a,\alpha). \tag{43}$$

$D(\alpha)$ can be solved uniquely from (43) as

$$D(\alpha) = -\dot{H}_+(a,\alpha). \tag{44}$$

The substitution of (44) into (42) gives

$$G_+(\rho,\alpha) = \frac{1}{K^2(\alpha)M(\alpha)} \cdot \left\{ -\dot{H}_+(a,\alpha)\left[J_0(K\rho)KY_1(Kb) - Y_0(K\rho)KJ_1(Kb)\right] + \int_a^b \left[p(t) - \mathrm{i}\alpha q(t)\right]Q(t,\rho,\alpha)\,t\,\mathrm{d}t \right\}. \tag{45}$$

Although the left-hand side of (45) is regular in the half-plane $\mathrm{Im}(\alpha) > \mathrm{Im}(-k)$, the regularity of the right-hand side is violated by the presence of simple poles lying at the upper half plane, namely at $\alpha = \alpha_m$ ($\mathrm{Im}(\alpha_m) > \mathrm{Im}(k)$) with

$$K_m^2\left[J_1(K_m a)Y_1(K_m b) - J_1(K_m b)Y_1(K_m a)\right] = 0, \qquad K_m = K(\alpha_m), \qquad m = 0,1,2,\ldots \tag{46}$$

In order to provide regularity of the right hand side of (45) in the upper half of the α-plane, these poles must be eliminated by imposing that their residues are zero. This gives

$$\dot{H}_+(a,k) = \frac{1}{\pi}\left[p_0 - \mathrm{i}kq_0\right]\frac{a^2-b^2}{ab} \tag{47a}$$

$$\dot{H}_+(a,\alpha_m) = \frac{1}{\pi}\left[p_m - \mathrm{i}\alpha_m q_m\right]\frac{J_1^2(K_m a) - J_1^2(K_m b)}{J_1(K_m a)J_1(K_m b)}. \tag{47b}$$

for $m = 0,1,2,\ldots$ Here, p_m and q_m's are the expansion coefficients of the functions $p(\rho)$ and $q(\rho)$, respectively which may be represented trough the following complete sets of orthogonal functions.

$$\begin{bmatrix} p(\rho) \\ q(\rho) \end{bmatrix} = \frac{2}{\pi b}\begin{bmatrix} p_0 \\ q_0 \end{bmatrix} + \sum_{m=1}^{\infty}\begin{bmatrix} p_m \\ q_m \end{bmatrix} \cdot K_m\left[J_0(K_m\rho)Y_1(K_m b) - Y_0(K_m\rho)J_1(K_m b)\right]. \tag{48}$$

Using the continuity relation (4m) together with (45) and taking into account (38) gives

$$\frac{\dot{H}_+(a,\alpha)}{K^2(\alpha)}N(\alpha) + \frac{a}{2}H_-(a,\alpha) = -\frac{a}{\pi b}\frac{[p_0 - \mathrm{i}\alpha q_0]}{K^2(\alpha)} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{[p_m - \mathrm{i}\alpha q_m]}{\alpha_m^2 - \alpha^2}\frac{J_1(K_m b)}{J_1(K_m a)} - \frac{a}{2}\frac{\mathrm{e}^{\mathrm{i}kl}}{\mathrm{i}(\alpha+k)} \tag{49}$$

the second decoupled Wiener-Hopf equation to be solved through classical procedures. Here, $N(\alpha)$ stands for

$$N(\alpha) = \frac{J_1(Kb)}{\pi M(\alpha)J_1(Ka)}. \tag{50}$$

The final solution of the above equation is determined to be

$$\frac{\dot{H}_+(a,\alpha)}{(k+\alpha)}N_+(\alpha) = -\frac{a}{\pi b}\frac{[p_0+\mathrm{i}kq_0]}{N_+(k)(k+\alpha)} + \frac{1}{\pi}\sum_{m=1}^{\infty}\frac{[p_m+\mathrm{i}\alpha_m q_m}{2\alpha_m(\alpha_m+\alpha)}\frac{J_1(K_m b)}{J_1(K_m a)}\frac{k+\alpha_m}{N_+(\alpha_m)} - \frac{ka\mathrm{e}^{\mathrm{i}kl}}{\mathrm{i}(k+\alpha)N_+(k)}, \tag{51}$$

where $N_\pm(\alpha)$ are the split functions resulting from the Wiener-Hopf factorization of $N(\alpha)$ as

$$N(\alpha) = N_+(\alpha)\,N_-(\alpha). \tag{52}$$

Their explicit expressions are given in [6] as

$$N_+(\alpha) = \sqrt{\pi\frac{J_1(ka)}{J_1(kb)}\left[J_1(ka)Y_1(kb) - J_1(kb)Y_1(ka)\right]} \cdot \mathrm{e}^{-\alpha\chi}\prod_{n=0}^{\infty}\frac{\left(1+\alpha/\sqrt{k^2-\left(\frac{j_n}{b}\right)^2}\right)}{\left(1+\alpha/\sqrt{k^2-\left(\frac{j_n}{a}\right)^2}\right)\left(1+\alpha/\alpha_n\right)}, \tag{53a}$$

where j_n's are the roots of the Bessel function of the first kind,

$$J_1(j_n) = 0, \quad n = 0, 1, \ldots$$

and

$$\chi = \frac{\mathrm{i}}{\pi}\left[b\ln b - a\ln a - (b-a)\ln(b-a)\right]$$

with

$$N_-(\alpha) = N_+(-\alpha). \tag{53b}$$

2.2. Determination of the Unknown Coefficients

The field in the region $a < \rho < b$, $z \in (0, l)$ can be expressed is terms of the waveguide normal modes as

$$u_2^{(2)}(\rho,z) = \sum_{n=0}^{\infty}\left[a_n\mathrm{e}^{\mathrm{i}\beta_n z} + b_n\mathrm{e}^{-\mathrm{i}\beta_n z}\right] \cdot \left[J_0(\xi_n\rho) - R_nY_0(\xi_n\rho)\right] \tag{54}$$

with

$$R_n = \frac{\mathrm{i}k\eta_1 J_0(\xi_n a) - \xi_n J_1(\xi_n a)}{\mathrm{i}k\eta_1 Y_0(\xi_n a) - \xi_n Y_1(\xi_n a)} = \frac{\mathrm{i}k\eta_2 J_0(\xi_n b) + \xi_n J_1(\xi_n b)}{\mathrm{i}k\eta_2 Y_0(\xi_n b) + \xi_n Y_1(\xi_n b)}. \tag{55}$$

In (54), ξ_n's are the roots of the equation

$$\frac{\mathrm{i}ka\eta_1 J_0(\xi_n a) - \xi_n a J_1(\xi_n a)}{\mathrm{i}ka\eta_1 Y_0(\xi_n a) - \xi_n a Y_1(\xi_n a)} - \frac{\mathrm{i}kb\eta_2 J_0(\xi_n b) + \xi_n b J_1(\xi_n b)}{\mathrm{i}kb\eta_2 Y_0(\xi_n b) + \xi_n b Y_1(\xi_n b)} = 0, \tag{56}$$

while β_n's are defined as

$$\beta_n = \sqrt{k^2 - \xi_n^2}. \tag{57}$$

Taking into account (27a) and (54), the continuity relation (4i) and (4j) can be written in the form

$$\frac{2}{\pi a}\left[g_0 + \mathrm{i}\alpha f_0\right] + \sum_{m=1}^{\infty}\left[g_m + \mathrm{i}\alpha f_m\right]K_m \cdot \left[J_0(K_m\rho)Y_1(K_m a) - Y_0(K_m\rho)J_1(K_m b)\right] = \mathrm{i}\sum_{n=0}^{\infty}\left[a_n(\alpha+\beta_n) + b_n(\alpha-\beta_n)\right]\left[J_0(\xi_n\rho) - R_nY_0(\xi_n\rho)\right]. \tag{58}$$

Multiplying equation (58) by

$$K_m\left[J_0(K_m\rho)Y_1(K_m a) - Y_0(K_m\rho)J_1(K_m a)\right]\rho$$

and integrating over ρ from a to b reads

$$\left[g_0 + \mathrm{i}\alpha f_0\right] = \frac{\mathrm{i}}{S_0}\sum_{n=0}^{\infty}\left[a_n(\alpha+\beta_n) + b_n(\alpha-\beta_n)\right]\triangle_{0n}^{(1)} \tag{59a}$$

$$\left[g_m + \mathrm{i}\alpha f_m\right] = \frac{\mathrm{i}}{S_m}\sum_{n=0}^{\infty}\left[a_n(\alpha+\beta_n) + b_n(\alpha-\beta_n)\right]\triangle_{mn}^{(1)} \tag{59b}$$

with

$$\triangle_{0n}^{(1)} = \frac{2}{\pi\xi_n}\left\{\left[J_1(\xi_n a) - R_nY_1(\xi_n a)\right] - \frac{b}{a}\left[J_1(\xi_n b) - R_nY_1(\xi_n b)\right]\right\}, \tag{60a}$$

$$\triangle_{mn}^{(1)} = \frac{2}{\pi}\frac{\xi_n}{\xi_n^2 - K_m^2}\left\{\left[J_1(\xi_n a) - R_nY_1(\xi_n a)\right] - \frac{J_1(K_m a)}{J_1(K_m b)}\left[J_1(\xi_n b) - R_nY_1(\xi_n b)\right]\right\}, \tag{60b}$$

where S_m stands for

$$S_0 = \frac{2}{\pi^2}\frac{a^2-b^2}{a^2}, \tag{60c}$$

$$S_m = \frac{2}{\pi^2}\frac{J_1^2(K_m a) - J_1^2(K_m b)}{J_1^2(K_m b)}. \tag{60d}$$

Performing the same procedure for the continuity relations (4k) and (4l), together with the equations (48) and (54) gives

$$\frac{2}{\pi b}\left[p_0 + \mathrm{i}\alpha q_0\right] + \sum_{m=1}^{\infty}\left[p_m + i\alpha q_m\right]K_m \cdot \left[J_0(K_m\rho)Y_1(K_m b) - Y_0(K_m\rho)J_1(K_m b)\right] = \mathrm{i}\sum_{n=0}^{\infty}\left\{\left[a_n(\alpha+\beta_n)\mathrm{e}^{\mathrm{i}\beta_n l} + b_n(\alpha-\beta_n)\mathrm{e}^{-\mathrm{i}\beta_n l}\right] \cdot \left[J_0(\xi_n\rho) - R_nY_0(\xi_n\rho)\right]\right\} \tag{61a}$$

and

$$\left[p_0 + \mathrm{i}\alpha q_0\right] = \frac{\mathrm{i}}{P_0}\sum_{n=0}^{\infty}\left[a_n(\alpha+\beta_n)\mathrm{e}^{\mathrm{i}\beta_n l} + b_n(\alpha-\beta_n)\mathrm{e}^{-\mathrm{i}\beta_n l}\right]\triangle_{0n}^{(2)}, \tag{61b}$$

$$[p_m + i\alpha q_m] = \frac{\mathrm{i}}{P_m}\sum_{n=0}^{\infty}\left[a_n(\alpha+\beta_n)\mathrm{e}^{\mathrm{i}\beta_n l} + b_n(\alpha-\beta_n)\mathrm{e}^{-\mathrm{i}\beta_n l}\right]\triangle_{mn}^{(2)}, \tag{61c}$$

with

$$\triangle_{0n}^{(2)} = \frac{a}{b}\triangle_{0n}^{(1)}, \tag{62a}$$

$$\triangle_{mn}^{(2)} = \frac{J_1(K_m b)}{J_1(K_m a)}\triangle_{mn}^{(1)}, \tag{62b}$$

where P_m stands for

$$P_0 = \frac{2}{\pi^2}\frac{a^2-b^2}{b^2} = \frac{a^2}{b^2}S_0, \tag{63a}$$

$$P_m = \frac{2}{\pi^2}\frac{J_1^2(K_m a) - J_1^2(K_m b)}{J_1^2(K_m a)} = \frac{J_1^2(K_m b)}{J_1^2(K_m a)}S_m. \tag{63b}$$

With the help of (26),(30),(47),(51),(59) and (61) we obtain the following systems of algebraic equations to determine a_n, b_n, f_m, g_m, p_m and q_m as:

$$\frac{\pi^2}{2}\frac{a}{b}L_+(k)\sum_{n=0}^{\infty}\left[a_n(k+\beta_n) + b_n(k-\beta_n)\right]\triangle_{0n}^{(1)} = \frac{b}{a}\frac{1}{L_+(k)}\frac{1}{S_0}\sum_{n=0}^{\infty}\left[a_n(k-\beta_n) + b_n(k+\beta_n)\right]\triangle_{0n}^{(1)} - k\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\left[a_n(\alpha_m-\beta_n) + b_n(\alpha_m+\beta_n)\right] \cdot \frac{\triangle_{mn}^{(1)}}{\alpha_m L_+(\alpha_m)}\frac{J_1(K_m a)}{S_m J_1(K_m b)}\right\}, \tag{64a}$$

$$\frac{\pi^2}{2}L_+(\alpha_r)\frac{J_1(K_r b)}{J_1(K_r a)}\sum_{n=0}^{\infty}\left[a_n(\alpha_r+\beta_n) + b_n(\alpha_r-\beta_n)\right]\triangle_{rn}^{(1)} = \frac{b}{a}\frac{1}{L_+(k)}\frac{1}{S_0}\sum_{n=0}^{\infty}\left[a_n(k-\beta_n) + b_n(k+\beta_n)\right]\triangle_{0n}^{(1)} - \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\left[a_n(\alpha_m-\beta_n) + b_n(\alpha_m+\beta_n)\right] \cdot \frac{(k+\alpha_r)\triangle_{mn}^{(1)}}{2\alpha_m(\alpha_m+\alpha_r)}\frac{J_1(K_m a)}{S_m J_1(K_m b)}\frac{(k+\alpha_m)}{L_+(\alpha_m)}\right\} \tag{64b}$$

$$\frac{\pi^2}{2}N_+(k)\sum_{n=0}^{\infty}\left[a_n(k-\beta_n)\mathrm{e}^{\mathrm{i}\beta_n l} + b_n(k+\beta_n)\mathrm{e}^{-\mathrm{i}\beta_n l}\right]\triangle_{0n}^{(1)} = \frac{1}{S_0 N_+(k)}\sum_{n=0}^{\infty}\left[a_n(k+\beta_n)\mathrm{e}^{\mathrm{i}\beta_n l} + b_n(k-\beta_n)\mathrm{e}^{-\mathrm{i}\beta_n l}\right]\triangle_{0n}^{(1)} - k\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\left[a_n(\alpha_m+\beta_n)\mathrm{e}^{\mathrm{i}\beta_n l} + b_n(\alpha_m-\beta_n)\mathrm{e}^{-\mathrm{i}\beta_n l}\right] \cdot \frac{\triangle_{mn}^{(1)}}{\alpha_m N_+(\alpha_m)}\frac{1}{S_m}\right\} - \frac{\pi k a \mathrm{e}^{\mathrm{i}kl}}{N_+(k)}. \tag{64c}$$

$$\frac{\pi^2}{2}N_+(\alpha_r)\sum_{n=0}^{\infty}\left[a_n(\alpha_r-\beta_n)\mathrm{e}^{\mathrm{i}\beta_n l} + b_n(\alpha_r+\beta_n)\mathrm{e}^{-i\beta_n l}\right]\triangle_{rn}^{(1)} = \frac{1}{S_0 N_+(k)}\sum_{n=0}^{\infty}\left[a_n(k+\beta_n)\mathrm{e}^{\mathrm{i}\beta_n l} + b_n(k-\beta_n)\mathrm{e}^{-\mathrm{i}\beta_n l}\right]\triangle_{0n}^{(1)} - \sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\left\{\left[a_n(\alpha_m+\beta_n)\mathrm{e}^{\mathrm{i}\beta_n l} + b_n(\alpha_m-\beta_n)\mathrm{e}^{-\mathrm{i}\beta_n l}\right] \cdot \frac{(k+\alpha_r)\triangle_{mn}^{(1)}}{2\alpha_m(\alpha_m+\alpha_r)}\frac{1}{S_m}\frac{k+\alpha_m}{N_+(\alpha_m)}\right\} - \frac{\pi k a \mathrm{e}^{\mathrm{i}kl}}{N_+(k)} \tag{64d}$$

3. Analysis of the Fields

3.1. The radiated field

The radiated field $u_1(\rho, z)$ can be obtained by taking the inverse Fourier transform of $F(\rho, \alpha)$. From (9) and (12), we get

$$u_1(\rho, z) = -\frac{1}{2\pi}\int_{\mathcal{L}}\dot{F}_-(b,\alpha)\frac{H_0^{(1)}(K\rho)}{K(\alpha)H_1^{(1)}(Kb)}\mathrm{e}^{-\mathrm{i}\alpha z}\,\mathrm{d}\alpha, \tag{65}$$

where $\mathcal{L}$ is a straight line parallel to the $\mathrm{Re}(\alpha)$ axis, lying in the strip $\mathrm{Im}(-k) < \mathrm{Im}(\alpha) < \mathrm{Im}(k)$. Utilizing the asymptotic expansion of $H_0^{(1)}(K\rho)$ as $k\rho \to \infty$,

$$H_0^{(1)}(K\rho) = \sqrt{\frac{2}{\pi K\rho}}\,\mathrm{e}^{\mathrm{i}(K\rho-\pi/4)}, \tag{66}$$

the asymptotic evaluation of the integral in (61) using the saddle point technique, yields for the radiated field,

$$u_1(r,\theta) = \mathcal{F}(\theta)\frac{\mathrm{e}^{\mathrm{i}kr}}{kr}, \tag{67a}$$

with

$$\mathcal{F}(\theta) = \frac{\mathrm{i}}{\pi^2}\frac{1}{L_+(k\cos\theta)\sin\theta H_1^{(1)}(kb\sin\theta)} \cdot \left\{-\frac{b}{a}\frac{[g_0 - \mathrm{i}kf_0]}{L_+(k)} + (k + k\cos\theta) \cdot \sum_{m=1}^{\infty}\frac{[g_m - \mathrm{i}\alpha_m f_m]}{2\alpha_m(\alpha_m + k\cos\theta)}\frac{J_1(K_m a)}{J_1(K_m b)}\frac{(k+\alpha_m)}{L_+(\alpha_m)}\right\}. \tag{67b}$$

Here r and θ are the usual spherical coordinates defined by $\rho = r\sin\theta$, $z = r\cos\theta$.

3.2. Reflected and transmitted fields

According to (34a), the scattered field in the region $0 < \rho < a$, i.e., $u_3(\rho, z)$ can be obtained by taking the inverse Fourier transform of $H(\rho, \alpha)$. .By considering (35) and (37) we write

$$u_3(\rho, z) = -\frac{1}{2\pi}\int_{\mathcal{L}} \dot{H}_+(a, \alpha)\frac{J_0(K\rho)}{K(\alpha)J_1(Ka)} \cdot \mathrm{e}^{-\mathrm{i}\alpha(z-l)}\,\mathrm{d}\alpha. \tag{68}$$

The evaluation of this integral for $z < l$ and $z > l$, will give us the reflected wave propagating backward in the inner cylinder and the transmitted wave, respectively.

For $z < l$, the integral is calculated by closing the contour in the upper half plane and evaluating the residues contributions from the simple poles occuring at the zeros of $J_1(Ka)$ lying in the upper $\alpha-$half-plane, namely at $Ka = j_n$. The reflection coefficient $\mathcal{R}$ of the fundamental mode is defined as the complex coefficient multiplying the travelling wave term $\exp(-\mathrm{i}kz)$ and is computed from the contribution of the first pole at $\alpha = k$. The result is

$$\mathcal{R} = -\frac{\mathrm{e}^{\mathrm{i}2kl}}{[N_+(k)]^2} - \frac{\mathrm{i}}{\pi kb}\frac{[p_0 + \mathrm{i}kq_0]}{[N_+(k)]^2}\mathrm{e}^{\mathrm{i}kl} + \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}kl}}{\pi a N_+(k)}\sum_{m=1}^{\infty}\frac{[p_m + \mathrm{i}\alpha_m q_m]}{\alpha_m N_+(\alpha_m)}\frac{J_1(K_m b)}{J_1(K_m a)}. \tag{69}$$

The first term is the reflection coefficient related to the case where a semi infinite rigid duct is inserted axially into a larger rigid tube of infinite length [6] whereas the rest is the correction term involving the effect of the impedances of the annular region and the overlap length.

Similarly, the transmission coefficient $\mathcal{T}$ of the fundamental mode which is defined as to be the complex coefficient of $\exp(\mathrm{i}kz)$, is obtained by evaluating the integral in (66) for $z > l$. This integral is now computed by closing the contour in the lower half of the complex α-plane. The pole of interest is at $\alpha = -k$ whose contribution gives

$$-\frac{\mathrm{i}}{\pi kb}\left(1 - \frac{a^2}{b^2}\right)\left[p_0 + \mathrm{i}kq_0\right]\mathrm{e}^{-\mathrm{i}kl} + \frac{a^2}{b^2} - 1, \tag{70a}$$

with

$$\mathcal{T} = \frac{a^2}{b^2} - \frac{i}{\pi kb}\left(1 - \frac{a^2}{b^2}\right)\left[p_0 + \mathrm{i}kq_0\right]\mathrm{e}^{-\mathrm{i}kl}. \tag{70b}$$

The last term in (70a) cancel out the incident wave in the region $\rho < a,\ z > l$, while the rest is the transmission coefficient of the fundamental mode.

4. Computational Results

Since the solution includes unknown expansion coefficients they have to be determined from the solution of the obtained algebraic system. The algebraic system in (64a-d) is consistent and can be solved numerically to any required accuracy by truncation. The truncation number depends on the cut-on mode number and consequently on waveguide radii. It means greater truncation numbers N are needed for greater b/a values. In all examples the real part of the lining impedances are taken as $Re(Z_1) = Re(Z_2) = 1$ except for the rigid case in which $\eta_1 = 1/Z_1 = 0$ and $\eta_2 = 1/Z_2 = 0$.

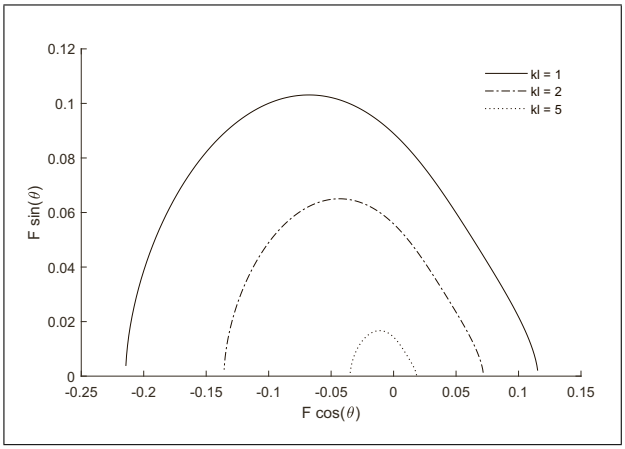


Figure 2. Polar plot of the far field amplitude function $\mathcal{F}(\theta)$ for $ka = 1, kb = 2, Z_1 = 1 - \mathrm{i}, Z_2 = 1 + \mathrm{i}$.

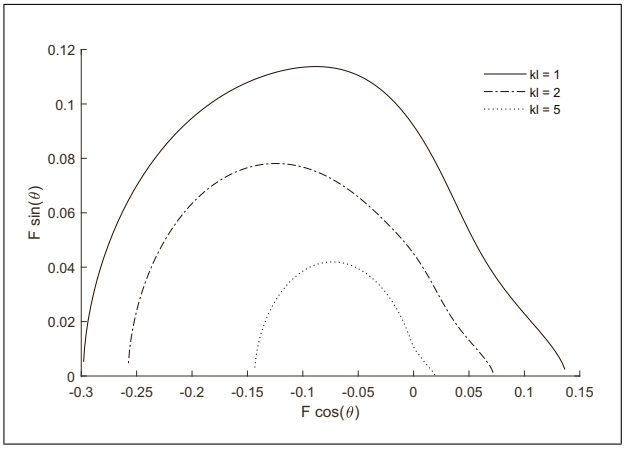


Figure 3. Polar plot of the far field amplitude function $\mathcal{F}(\theta)$ for $ka = 1, kb = 4, Z_1 = 1 - \mathrm{i}0.5, Z_2 = 1 - \mathrm{i}0.5$.

In Figure 2 and Figure 3 the variation of the far field amplitude in polar coordinates for different values of dimensionless overlap length kl is depicted. For increasing kl values it is seen that the far field amplitude is decreased. This result can be confirmed analytically by letting $l \to \infty$ in the algebraic system (64a-d). By doing so, we see that the far field approaches to zero for increasing values of kl.

For the figures (Figures 4–9) the far field amplitude is plotted in polar coordinates to see the effect of the lining impedances Z_1 and Z_2 in radiation phenomenon. In these far field graphs radiated field is pictured for different imaginary values of Z_1 and Z_2 with increasing ratio b/a. It is observed that the most attenuation in the radiated field occured at the values $Z_1 = Z_2 = 1 - \mathrm{i}0.5$ for the ratio $b = 2a$. When we inreased the ratio b/a it is seen that $Z_1 = 1 - \mathrm{i}0.5$ still satisfies the most attenuation, but on the other side $|Z_2| \to \infty$ keeps the shape of the radiation pattern by providing some attenuation on the radiated field

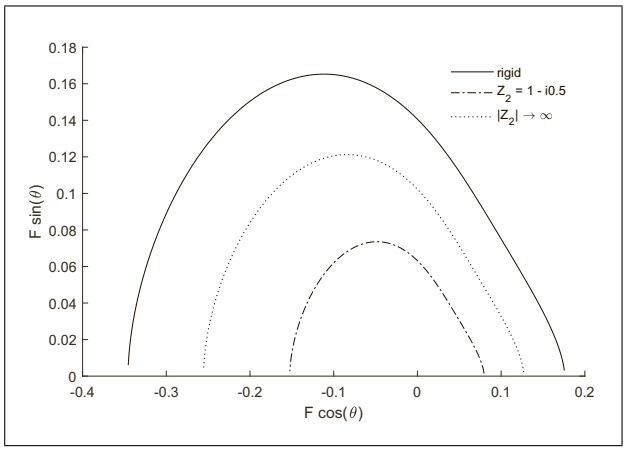


Figure 4. Polar plot of the far field amplitude function $\mathcal{F}(\theta)$ for $ka = 1, kb = 2, kl = 1, Z_1 = 1 - \mathrm{i}0.5$.

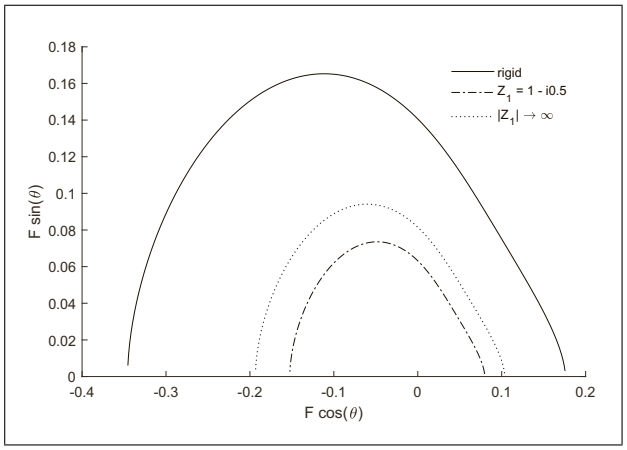


Figure 5. Polar plot of the far field amplitude function $\mathcal{F}(\theta)$ for $ka = 1, kb = 2, kl = 1, Z_2 = 1 - \mathrm{i}0.5$.

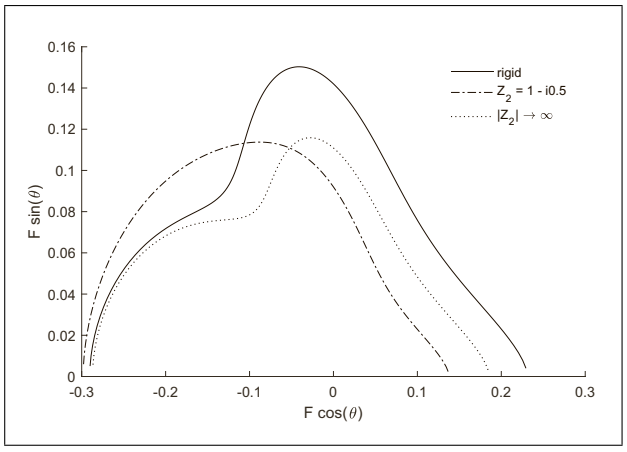


Figure 6. Polar plot of the far field amplitude function $\mathcal{F}(\theta)$ for $ka = 1, kb = 4, kl = 1, Z_1 = 1 - \mathrm{i}0.5$.

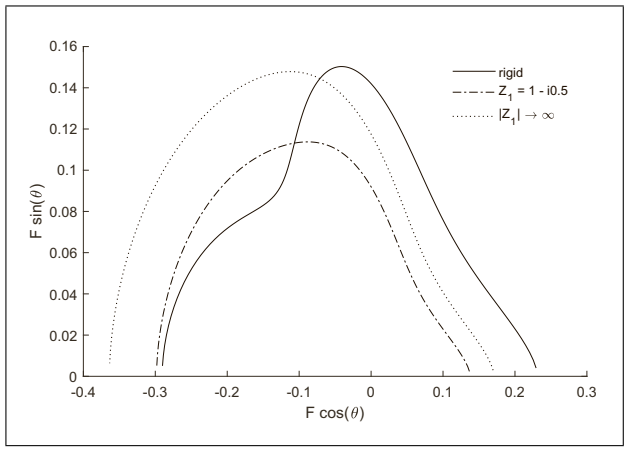


Figure 7. Polar plot of the far field amplitude function $\mathcal{F}(\theta)$ for $ka = 1, kb = 4, kl = 1, Z_2 = 1 - \mathrm{i}0.5$.

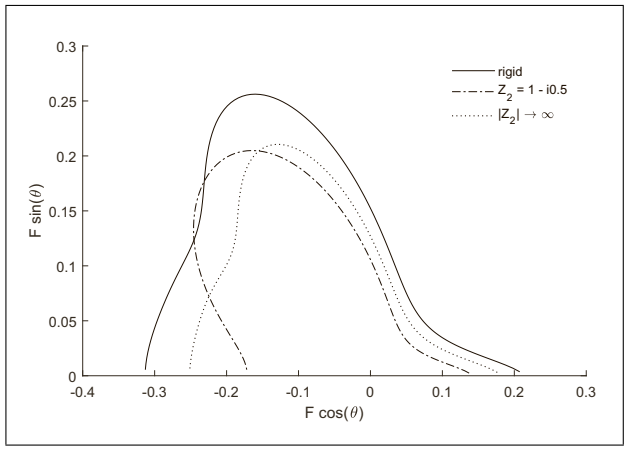


Figure 8. Polar plot of the far field amplitude function $\mathcal{F}(\theta)$ for $ka = 1, kb = 8, kl = 1, Z_1 = 1 - \mathrm{i}0.5$.

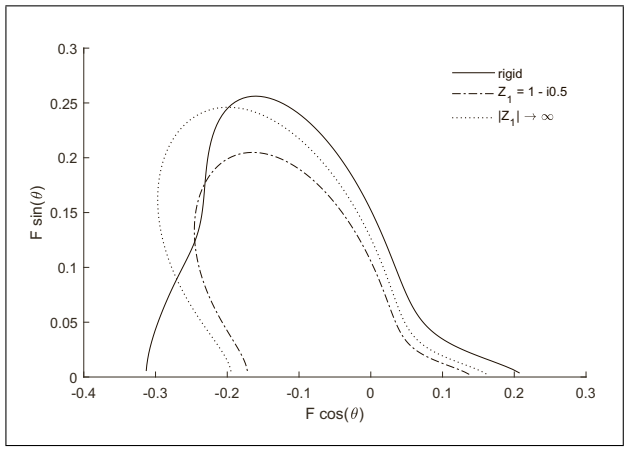


Figure 9. Polar plot of the far field amplitude function $\mathcal{F}(\theta)$ for $ka = 1, kb = 8, kl = 1, Z_2 = 1 - \mathrm{i}0.5$.

at each direction while $Z_2 = 1 - \mathrm{i}\,0.5$ reduces the radiated field by some oscillation.

Figure 10 and Figure 11 show the effect of the ratio b/a and lining impedances on the radiated field amplitude. The main lobe steers up and noisy region narrows with increasing ratio b/a. Oscillations appear for larger values of b/a and magnitude of these oscillations decrease with increasing lining impedance $|Z_2| \to \infty$.

To support the observations from Figure 4, Figure 12 and Figure 13 are depicted. As we mentioned before, when choosing $Z_1 = Z_2 = 1 - i0.5$ the most attenuation in the radiated field is obtained for the ratio $b = 2a$

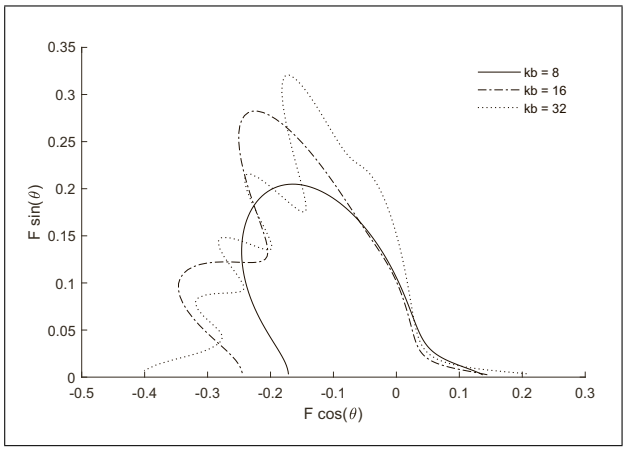


Figure 10. Polar plot of the far field amplitude function $\mathcal{F}(\theta)$ for $ka = 1, kl = 1, Z_1 = 1 - \mathrm{i}\,0.5, Z_2 = 1 - \mathrm{i}\,0.5$.

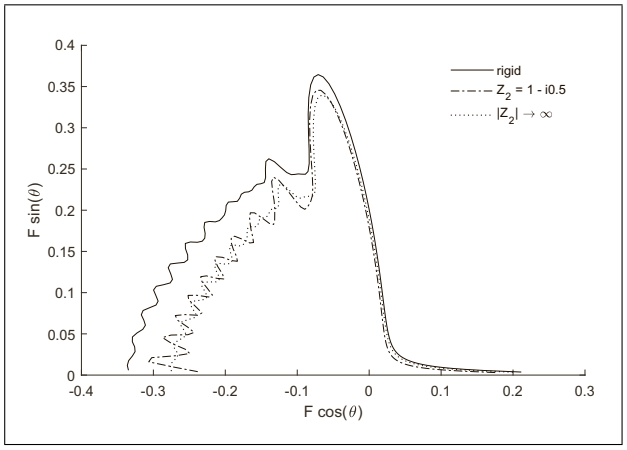


Figure 11. Polar plot of the far field amplitude function $\mathcal{F}(\theta)$ for $ka = 1, kb = 64, kl = 1, Z_1 = 1 - \mathrm{i}0.5$.

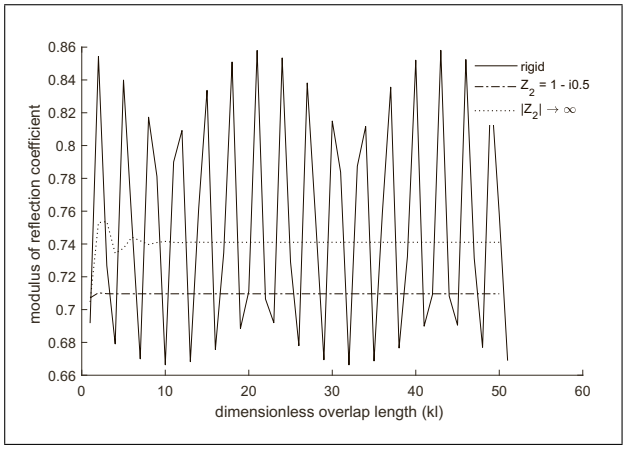


Figure 12. Modulus of reflection coefficient $|\mathcal{R}|$ vs. overlap length kl for $ka = 1, kb = 2, Z_1 = 1 - \mathrm{i}\,0.5$.

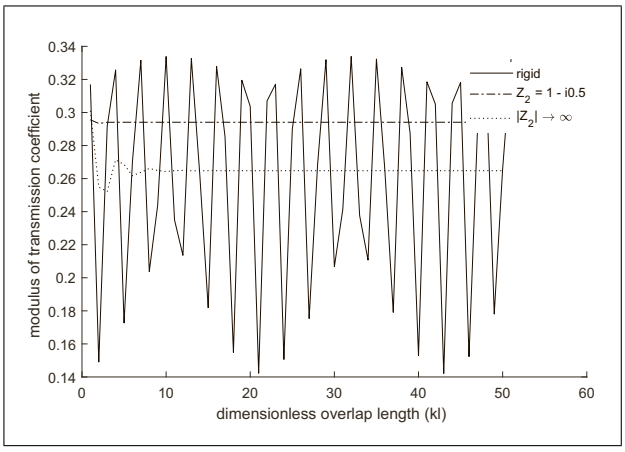


Figure 13. Modulus of transmission coefficient $|\mathcal{T}|$ vs. overlap length kl for $ka = 1, kb = 2, Z_1 = 1 - \mathrm{i}\,0.5$.

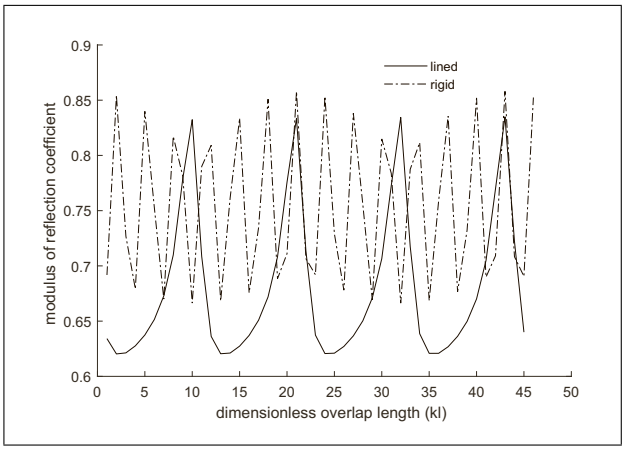


Figure 14. Modulus of reflection coefficient $|\mathcal{R}|$ vs. overlap length kl for $ka = 1, kb = 2, Z_1 = Z_2 = -2\mathrm{i}$.

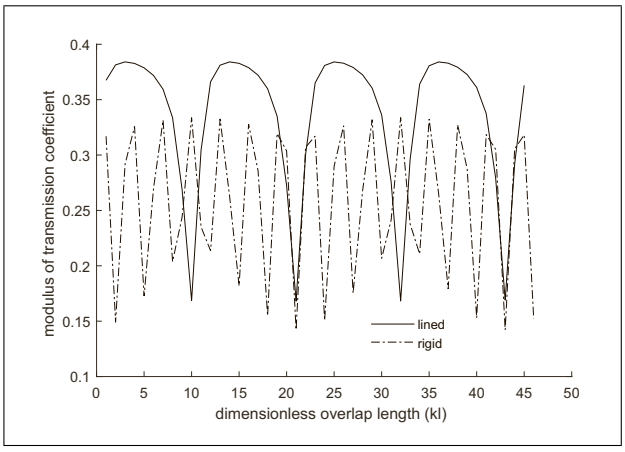


Figure 15. Modulus of transmission coefficient $|\mathcal{T}|$ vs. overlap length kl for $ka = 1, kb = 2, Z_1 = Z_2 = -2\mathrm{i}$.

while we get less reflection and more transmission in the waveguide as can be seen from Figure 12 and Figure 13. It is also observed that the oscillatory behaviour of reflection and transmission coefficients is damped with increasing kl when the lining exist. Additionally, if the lining impedances are purely imaginary such that $Z_1 = \mathrm{i}X_1, Z_2 = \mathrm{i}X_2$ the reflection and transmission coefficients still continue their oscillatory behaviour (see Figures 14, 15)

Finally in Figures 16, 17, 18, the radiated total field 40 meters far from the origin is calculated in dB scale for the configuration $a = 0.02\,\mathrm{m}$, $l = a$, $Z_1 = Z_2 = 1 - \mathrm{i}\,0.5$

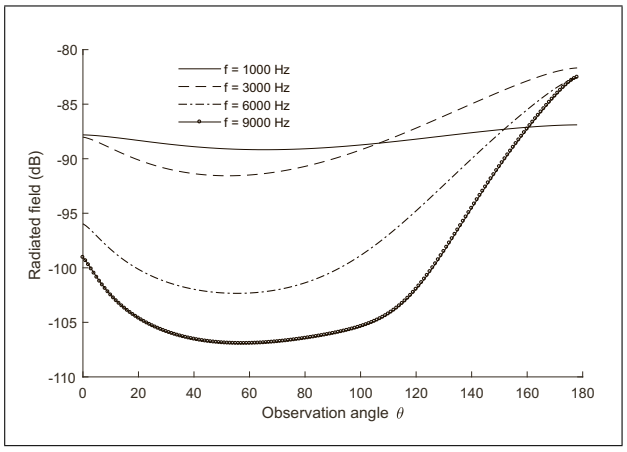


Figure 16. $20\log|u_1|$ vs. observation angle θ for $b = 2a$.

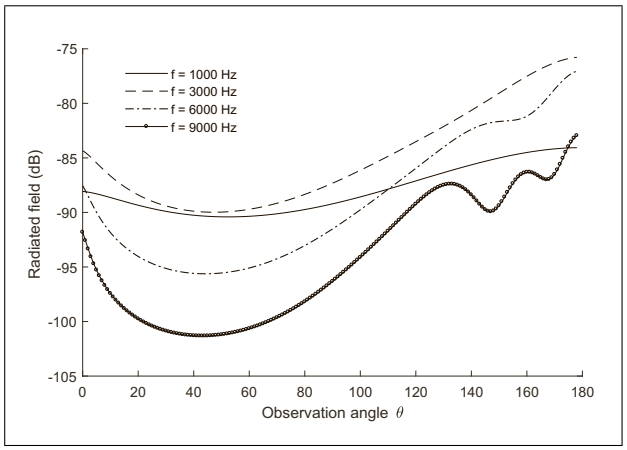


Figure 17. $20\log|u_1|$ vs. observation angle θ for $b = 4a$.

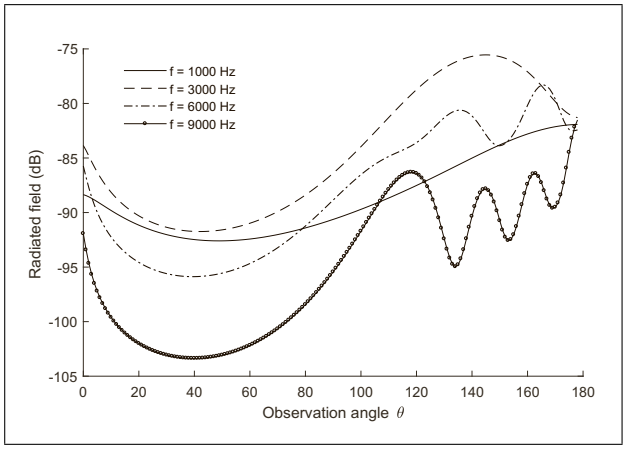


Figure 18. $20\log|u_1|$ vs. observation angle θ for $b = 8a$.

at different frequencies. Sound speed in the medium is taken as 346 m/s and frequencies are chosen below 10 kHz which is the cut-off frequency of the fundamental mode depend on the radius of the central duct and sound speed. We observe in these figures that the radiated total field starts to oscillate with increasing frequency and ratio b/a at high elevation angles as we see in previous polar plots of the far field amplitude.

5. Concluding Remarks

The radiation of sound from two semi-infinite circular cylinders having a common axis but different radii is investigated. Two cylinders overlap in a finite region and in this region they have different linings on their walls. Using the mode matching method in overlap region and Fourier transform elswhere the well-known Wiener-Hopf technique successfully applied to the problem. The problem is first reduced to two Wiener Hopf equations and then solved following usual factorization and decompozition procedures. The solution involves two systems of linear algebraic equations involving two sets of infinitely many unknown expansion coefficients. Numerical solution to these systems is obtained for various values of the problem parameters such as overlap length l, lining impedances $Z_{1,2}$. In the case where the lining impedances approaches to infinity (rigid case), the results obtained in this paper are compared with the results of [1] and the agreement is perfect. Furthermore, it is shown numerically that choosing impedance values appropriately, it is possible to attenuate radiation and to increase transmission, as intended.

References

[1] J. B. Lawrie, I. D. Abrahams, C. M. Linton: Acoustic radiation from two opposed seni-infinite coaxial cylindrical waveguides. I: Overlapping edges. Wave Motion **18** (1993) 121–142.

[2] M. İdemen: A new method to obtain exact solutions of vector Wiener-Hopf equations. Z. Angew. Math. Mech. **59** (1979) 656–658.

[3] I. D. Abrahams: Scattering of sound by two parallel semi-infinite screens. Wave Motion **9** (1987) 289–300.

[4] I. D. Abrahams, G. R. Wickham: The scattering of water waves by two semi-infinite opposed verical walls. Wave Motion **14** (1991) 145–168.

[5] A. Büyükaksoy, G. Uzgören, F. Birbir: The scattering of a plane wave by two parallel semi-infinite overlapping screens with dielectric loading. Wave Motion **34** (2001) 375–389.

[6] A. D. Rawlins: A bifurcated circular waveguide problem. IMA J. Appl. Math. **54** (1995) 59–81.

[7] S.-W. Lee, V. Jamnejad, R. Mittra: Near field of scattering by a hollow semi-infinite cylinder and its application to sensor booms. IEEE Trans Antennas and Propagat AP-**21** (1973) 182–188.