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# Acoustic Pulses in Media with Power Law Hysteretic Nonlinearity

Andrey Radostin[1,2)]
[1)] Institute of Applied Physics, Russian Academy of Science, Nizhny Novgorod, Russia. radostin@ipfran.ru
[2)] Department of Applied Mathematics, Nizhny Novgorod State Technical University, n.a. R.E. Alekseev, Nizhny Novgorod, Russia

**Summary**
This paper presents analytical results on propagation of acoustic unipolar pulses in media with hysteretic nonlinearity characterized by a power law with an arbitrary exponent. Exact solutions for pulse profiles are derived for 1D propagation in the case of integer power-law exponents. For fractional ones, numeric and approximate analytical solutions are obtained. These results are used for interpretation of experimental data from the study by Y. Yasumoto, *et al.* [Acta Acustica united with Acustica 30(5) (1974) 260–267], where propagation of acoustic pulses in aluminum samples with different degree of annealing was observed. It is found that the dependence of the received pulse amplitude on the excitation amplitude corresponds to exponent $n = 4$ in the hysteretic equation of state, the nonlinear parameters of which increase with increasing temperature annealing.

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## 1. Introduction

Experimental data on nonlinear acoustic phenomena in many metals, rocks and some artificial materials with microstructural damage are not consistent with quadratic or cubic nonlinearity adopted in the classical theory of elasticity [1, 2]. Among main indications of this inconsistency is the fact that the complementary variations in the elastic modulus and damping coefficient demonstrate power-law dependences on the acoustic perturbation amplitude with the same exponent.

To describe the nonlinear phenomena of this type, equations of state with hysteresis are usually applied [1, 2, 3]. Many of these materials (such as copper, zinc, granite, sandstones, etc.) are characterized by quadratic hysteretic nonlinearity, for which nonlinear wave transformations, including propagation of unipolar pulses, have been studied thoroughly [2, 4, 5, 6, 7, 8, 9]. Nevertheless, there are also many experimental demonstrations, where a power exponent of the hysteretic nonlinearity is not equal to 2. For example, S. Takahashi [10] demonstrated that in samples of Cu–Zn alloys the power exponent changes from 2.1 to 4.2 with growth of zinc concentration from 0% to 3%; for Cu–P alloys the power exponent changes from 2 to 3.4 with growth of phosphor concentration from 0.002% to 0.6%, and for Cu–Al alloy the power exponent is 3.8. V. Nazarov and co-authors [2, 11] observed the values of the power exponent 3 and 5/3 in different ranges of the strain in the limestone sample. P. Johnson and co-authors [12] observed the values of the power exponent from 1.76 to 2.47 in several rock samples from dependence of resonant frequency shift on the strain amplitude. Figure 1 displays a variety of the power exponent values compiled from literature [1, 2, 3, 10, 11, 12, 13, 14, 15]. It is worth noting that this list is far from complete.

As clearly seen from this histogram the power exponent $n$ equals 2 or 3 for most materials, a considerable quantity of materials has fractional values of $n$ in the range from 2 to 3. There are five examples with $n$ in the range from 1 to 2. In addition we note that the values of $n$ in the range from 1 to 2 were also observed by O. Vinogradov and I. Pivovarov in the experiments with stranded cable [16]. Certainly, power-law exponents evaluated from experimental measurements may have some inaccuracy and for theoretical description, the use of integer $n$ is of special interest.

In what follows, we restrict our consideration to longitudinal waves, however, noting that quadratic hysteretic nonlinearity has also been observed for flexural vibrations in the concrete samples [17] and torsion waves [18]. Also V. Gusev and co-authors developed a theory for propagation of flexural wave in the plates of continuously varying thickness with hysteretic nonlinearity of an arbitrary power [19].

All these factors stipulate the interest to the study of evolution of unipolar pulses in media with hysteretic nonlinearity of arbitrary power ($n > 1$), when perturbations of a medium are described by different dependences of stress

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on strain. As pointed by V. Nazarov [20], consideration of this problem is exceptionally interesting, because shock front does not form, but pulse amplitude decreases because of nonlinear losses equal to the hysteresis loop area; and it duration grows due to difference in velocities of the ascending and descending parts of a pulse.

Noting that similar problem was considered in [20, 21], but hysteretic function in equation of state had terms with maximal power only, which is not quite correct from the point of view of the theory of hysteretic models construction [22, 23].

This paper deals with analytical study of plane acoustic unipolar pulses propagation in media with power hysteretic nonlinearity in the framework of Davidenkov's model [3, 16, 24]. The exact solutions, describing pulse profile and parameters, are derived for the case of integer values of the power exponent $n$, whereas numeric and approximate solutions are proposed for the case of fractional $n$. These results are used for qualitative description of experimental results from the paper by Y. Yasumoto *et al.* [25], where propagation of acoustic pulses in aluminum samples was observed. It is demonstrated that dependence of received pulse amplitude on pulse amplitude at source corresponds to hysteretic equation of state with power exponent equal to 4, and values of nonlinear parameters increase with growth of annealing temperature.

## 2. Basic equations

In the low-frequency range, at sufficiently large strain amplitudes, when the linear dissipation can be neglected, the equation of state of a medium with hysteretic nonlinearity, i.e., the dependence $\sigma = \sigma(\varepsilon, \mathrm{sgn}\dot{\varepsilon})$, can be represented in the form [2]

$$\sigma(\varepsilon, \mathrm{sgn}\dot{\varepsilon}) = E\left[\varepsilon - f(\varepsilon, \mathrm{sgn}\dot{\varepsilon})\right], \tag{1}$$

where $\sigma$, $\varepsilon$, and $\dot{\varepsilon}$ are the stress, strain, and strain rate, respectively; $E$ is the elastic modulus; $f(\varepsilon, \mathrm{sgn}\dot{\varepsilon})$ is the hysteresis function, and $|f_\varepsilon \sigma(\varepsilon, \mathrm{sgn}\dot{\varepsilon})| \ll 1$. For Davidenkov's model of hysteresis [24], the function $f(\varepsilon, \mathrm{sgn}\dot{\varepsilon})$ has the form

$$f(\varepsilon, \mathrm{sgn}\dot{\varepsilon}) = \frac{\gamma}{n}\left\{\begin{array}{rl} \varepsilon^n, & \dot{\varepsilon} > 0, \\ -(\varepsilon_m - \varepsilon)^n + \varepsilon_m^n, & \dot{\varepsilon} < 0, \end{array}\right. \tag{2}$$

where $\gamma$ is parameter of hysteretic nonlinearity $\gamma > 0$, $|\gamma\varepsilon_m^{n-1}| \ll 1$, $n > 1$.

As is clearly seen, the lower branch of hysteretic function contains all terms with power exponents of $\varepsilon$ from 0 to $n$ (in the case of integer values of $n$), whereas in above mentioned papers [20, 21] equation with 1st and $n$th – power terms was only used.

In deriving of Equation (2) N. Davidenkov used the power dependence of elastic modulus deviation on strain, hence this equation evidently has no direct relation to the microstructure of the medium. Nevertheless the equation of this type can be considered as an approximation of dislocation hysteresis proposed by A. Granato and K. Lücke

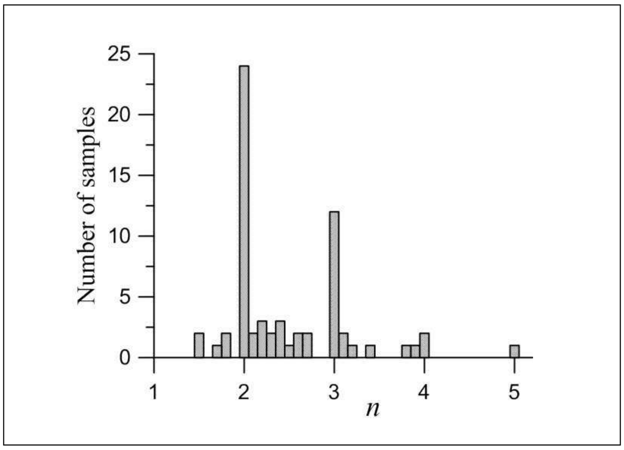


Figure 1. A histogram for the power exponent $n$ for 65 samples [1, 2, 3, 10, 11, 12, 13, 14, 15].

with nonlinear restoring force [3, 26]. V. Gusev demonstrated that quadratic hysteretic equation of state can be derived as cumulative response of ensemble of mechanical elements of friction type with homogeneous statistical distribution over their basic parameters [7], and change in distribution [27, 28] or consideration of the elements of different type (for example accounting of adhesion [29, 30]) caused change in power of nonlinearity.

Equation of state (1) together with the equation of motion $\rho U''_{tt} = \sigma_x(\varepsilon, \mathrm{sgn}\dot{\varepsilon})$ determine wave processes in hysteretic media ($U$ is displacement, $\varepsilon = U_x$, $\rho$ is density). According to common practice, we solve Equations (1)–(2) without taking into account the geometric nonlinearity of the equations of motion (compared to the strong hysteretic nonlinearity of the medium) by assuming that $\gamma \gg 1$. Then, we study the propagation of unipolar disturbances by assuming (for definiteness) that they are of positive polarity ($\varepsilon > 0$). In this case, no shock fronts (or discontinuities) are formed in the pulses, because small perturbations ($\varepsilon \ll \varepsilon_m$) of identical amplitudes at the ascending ($\dot{\varepsilon} > 0$) and descending ($\dot{\varepsilon} < 0$) parts of a pulse move with different velocities $C_1(\dot{\varepsilon} > 0) = C = \sqrt{E/\rho}$ and $C_2(\dot{\varepsilon} < 0) = C[1 - \gamma\varepsilon_m^{n-1}]$, where $C_1 > C_2$. As a result, in the course of the pulse propagation, the duration of the disturbance increases while its amplitude and energy decrease due to hysteretic loss. In solving Equations (1)–(2), we use the method of "sewing together" simple waves [2, 7, 20, 21] corresponding to each of the hysteresis branches. Substituting Equations (1)– (3) into the equation of motion, differentiating it with respect to $x$, and changing to the variables $\tau = x - x/C$, $x' = x$, we obtain single-wave equations for simple strain waves corresponding to the ascending $\varepsilon_1(x, \tau)$ and descending $\varepsilon_2(x, \tau)$ parts of the disturbance,

$$\frac{\partial\varepsilon_1}{\partial x} = -\frac{1}{2C}\gamma\varepsilon_1^{n-1}\frac{\partial\varepsilon_1}{\partial\tau}, \qquad \frac{\partial\varepsilon_1}{\partial\tau} > 0, \tag{3}$$

$$\frac{\partial\varepsilon_2}{\partial x} = -\frac{1}{2C}\gamma\left[(\varepsilon_m - \varepsilon_2)^{n-1}\right]\frac{\partial\varepsilon_2}{\partial\tau}, \qquad \frac{\partial\varepsilon_2}{\partial\tau} < 0. \tag{4}$$

The solution to these equations for a pulse with initially triangular shape

$$\varepsilon(0,\tau) = \frac{2\varepsilon_0}{T}\begin{cases} \tau, & 0 \le \tau \le T/2, \\ T-\tau, & T/2 \le \tau \le T \end{cases}$$

has the form

$$\theta = \begin{cases} \dfrac{1}{2}\epsilon(z,\theta)^{n-1}z + \dfrac{1}{2}\epsilon(z,\theta), & \dot\epsilon > 0, \\ \dfrac{1}{2}\displaystyle\int_0^z \left(\epsilon_m(z') - \epsilon(z',\theta)\right)^{n-1}\mathrm{d}z' \\ \qquad +1 - \epsilon(z,\theta)/2, & \dot\epsilon < 0, \end{cases} \tag{5}$$

where $\theta = \tau/T$, $z = \gamma x \varepsilon_0^{n-1}/(cT)$, $\epsilon = \varepsilon/\varepsilon_0$, $\epsilon_m = \varepsilon_m/\varepsilon_0$, $\epsilon_m(0) = 1$.

The sewing condition $\theta(\epsilon_m, \dot\epsilon > 0) = \theta(\epsilon_m, \dot\epsilon < 0) = \theta_m(z)$ yields the equation determining dependence of pulse amplitude on propagation distance

$$\frac{1}{2}\epsilon_m^{n-1}(z)z - \frac{1}{2}\int_0^z \left[\epsilon_m(z') - \epsilon_m(z)\right]^{n-1}\mathrm{d}z' = 1 - \epsilon_m(z). \tag{6}$$

This nonlinear integral equation with respect to $\epsilon_m(z)$ can be simply solved in the case of integer values of $n$. Equation (6) can be transformed into canonical form of Volterra integral equation of the second kind by using partial integration and substitution $\epsilon_m(z) = 1 - y$,

$$z(y) = \frac{n-1}{(1-y)^{n-1}}\int_0^y z(t)\left[y-t\right]^{n-2}\mathrm{d}t + \frac{2y}{(1-y)^{n-1}}. \tag{7}$$

The latter equation can be transformed into differential one in quite simple form,

$$\frac{\mathrm{d}}{\mathrm{d}y}\left[z(y)\left(1-y\right)\right] = z(y) + 2, \text{ for } n = 2,$$

and

$$\frac{\mathrm{d}^{n-1}}{\mathrm{d}y^{n-1}}\left[z(y)\left(1-y\right)^{n-1}\right] = (n-1)!\, z(y), \text{ for } n > 2.$$

These equations should be supplemented with initial conditions at $y = 0$, which are written as

$$\frac{\mathrm{d}^k z(y)}{\mathrm{d}y^k}\Big|_{y=0} = \frac{2k(n-3+k)!}{(n-2)!},$$

where $k = 0, 1, \ldots, n-2$ is the derivative order. For instance, $\mathrm{d}z/\mathrm{d}y\big|_{y=0} = 2$, $\mathrm{d}^2z/\mathrm{d}y^2\big|_{y=0} = 4(n-1)$, $\mathrm{d}^3z/\mathrm{d}y^3\big|_{y=0} = 6(n-1)n$, etc.

## 3. Solutions in the case of integer values of n

The solution in the case of $n = 2$ corresponds to the well-known one in the form of triangle pulse derived by V. Gusev [7],

$$\theta = \begin{cases} \dfrac{\epsilon z}{2} + \dfrac{\epsilon}{2}, & \dot\epsilon > 0, \\ \sqrt{1+z} - \dfrac{\epsilon z}{2} - \dfrac{\epsilon}{2}, & \dot\epsilon < 0. \end{cases}$$

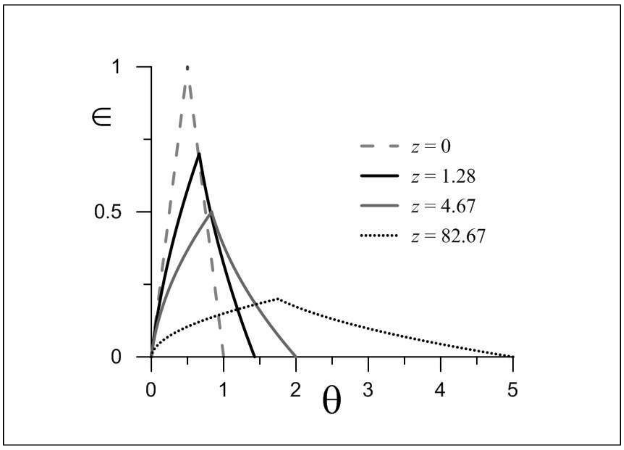


Figure 2. Evolution of a pulse in a cubic hysteretic medium.

In this case pulse amplitude and duration are written as $\epsilon_m = 1/\sqrt{1+z}$, $\theta^* = \theta(0) = \sqrt{1+z}$. Note that actually $\epsilon$ is the function of two arguments, $\epsilon(z,\theta)$ as in Equation (5), however we omit the arguments here and further in purposes of compactness.

In the case of $n = 3$ we derive

$$\theta = \begin{cases} \dfrac{\epsilon}{2} + \dfrac{\epsilon^2 z}{2}, & \dot\epsilon > 0, \\ \sqrt[3]{1+\dfrac{3}{2}z} - \dfrac{\epsilon}{\epsilon_m^2} + \dfrac{\epsilon}{2} + \dfrac{\epsilon^2 z}{2}, & \dot\epsilon < 0, \end{cases}$$

$$\epsilon_m = \frac{1}{\sqrt[3]{1+\frac{3}{2}z}}, \qquad \theta^* = \sqrt[3]{1+\frac{3}{2}z}.$$

Figure 2 displays the wave profiles for $n = 3$ at various propagation distances $z$.

In the case of $n = 4$ one can obtain

$$\theta = \begin{cases} \dfrac{\epsilon}{2} + \dfrac{\epsilon^3 z}{2}, & \dot\epsilon > 0, \\ \dfrac{(\epsilon_m-\epsilon)^3 z}{2} + \dfrac{\epsilon}{2} - \dfrac{9}{11}\Bigg[-\dfrac{1}{\epsilon_m} + \dfrac{\epsilon}{\epsilon_m^2} + \dfrac{\epsilon^2}{3\epsilon_m^3} \\ +\dfrac{\sqrt{2}\sin(\sqrt{2}\ln\epsilon_m)}{18}\left(-5\epsilon_m^2 + 16\epsilon_m\epsilon - 9\epsilon^2\right) \\ +\dfrac{\cos(\sqrt{2}\ln\epsilon_m)}{9}\left(-2\epsilon_m^2 + 2\epsilon_m\epsilon + 3\epsilon^2\right)\Bigg] \\ & \dot\epsilon < 0, \end{cases}$$

$$z = \frac{6}{11}\left[\frac{1}{\epsilon_m^4} - \frac{\sqrt{2}}{3\epsilon_m}\sin\left(\sqrt{2}\ln\epsilon_m\right) - \frac{1}{\epsilon_m}\cos\left(\sqrt{2}\ln\epsilon_m\right)\right],$$

$$\theta^* = \frac{\epsilon_m^2}{11}\left[\frac{12}{\epsilon_m^3} + \frac{3\sqrt{2}\sin(\sqrt{2}\ln\epsilon_m)}{2} - \cos\left(\sqrt{2}\ln\epsilon_m\right)\right].$$

By reason of extremely cumbersomeness of expressions we give only dependence of the pulse amplitude on propagation distance in the case of $n = 5$,

$$z = \frac{12}{25}\left[\frac{1}{\epsilon_m^5} - \frac{1}{6} - \frac{\sqrt{15}}{6\epsilon_m^{5/2}}\sin\left(\frac{\sqrt{15}}{2}\ln\epsilon_m\right) - \frac{5}{6\epsilon_m^{5/2}}\cos\left(\frac{\sqrt{15}}{2}\ln\epsilon_m\right)\right].$$

## 4. Approximate solution for an arbitrary power

In the case of fractional power the attempts to obtain the exact solution have not met with success, nevertheless, one can solve this problem numerically replacing the integral in Equation (6) by the left Riemann sum, approximating the function by its value at the left-end point, since its value at the right-end point is unknown. It should also be noted that Equation (7) can be considered as fractional differential one and corresponding approaches can be used to obtain its solutions [31].

Figure 3 displays wave amplitudes versus propagating distance for various values of n. The dependences are calculated in according with exact solutions for integer values $n$ from 2 to 5 (solid lines), in the cases of fractional values 3/2, 5/3, 5/2 the curves correspond to numerical solutions of Equation (7) (solid lines). In addition this figure presents approximate dependences (dashed lines), except cases of 2 and 3, which can be written as

$$\epsilon_m = \frac{1}{\sqrt[n]{1 + az}}. \tag{8}$$

The values of the parameter a in these cases are 0.64 for 3/2, 0.76 for 5/3, 1.3 for 5/2, and 1.9 for both 4 and 5. As is clearly seen the approximate dependences agree well with numeric and exact ones.

Hence, the approximate solution for arbitrary power exponent $n$ has the form

$$\theta = \begin{cases} \dfrac{\epsilon(z,\theta)}{2} + \dfrac{\epsilon(z,\theta)^{n-1} z}{2}, & \dot{\epsilon} > 0, \\ \dfrac{1}{2a\epsilon}\Big[\big(1 - \epsilon(z,\theta)\big)^n & \\ -\Big(1 - \frac{\epsilon(z,\theta)}{\epsilon_m}\Big)^n\Big] + 1 - \frac{\epsilon(z,\theta)}{2}, & \dot{\epsilon} < 0. \end{cases} \tag{9}$$

Note, that there is no singularity at $\epsilon(z,\theta) = 0$,, for particular values of $n$ it can be eliminated by using standard technique. The expression for pulse duration has the form

$$\theta^* = 1 + \frac{n(1 - \epsilon_m)}{2a\epsilon_m}, \tag{10}$$

asymptotically, when $\epsilon_m \to 0$, the duration is in inverse proportion to the pulse amplitude as in the cases of integer values of $n$.

Figure 4 displays profiles of the pulse for the case $n = 5/3$. Descending edge of each pulse calculated numerically are shown by solid lines, whereas dashed lines correspond to approximate solution (Equation 9) when $q = 0.76$. As clearly seen from this figure the approximate solution is very close to numerical one. Comparing Figure 2 and Figure 4, it is worth noting the marked distinction in form for the cases of $n > 2$ and $n < 2$; in the first case a ascending part of a pulse is concave and descending one is salient, and vice versa in the second case. Note that in [21] both edges of a pulse were concave for $n > 2$ and were salient for $n < 2$.

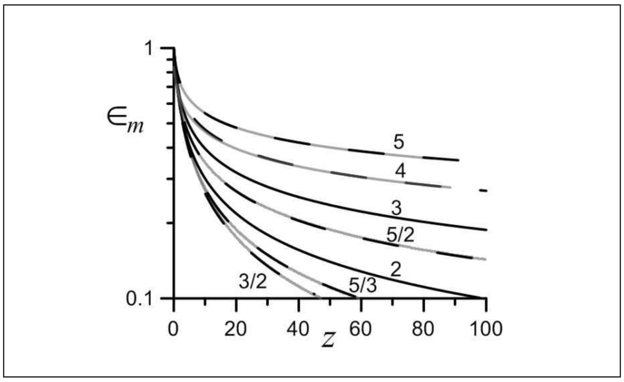


Figure 3. Pulse amplitude versus propagation distance $z$ for various values of $n$.

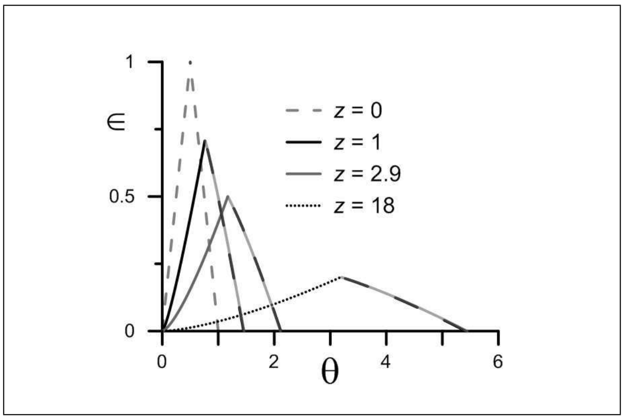


Figure 4. Evolution of a pulse in a hysteretic medium with $n = 5/3$.

## 5. Qualitative description of pulse evolution

As mentioned above, in the most cases hysteretic nonlinearity manifests itself indirectly via similar dependences of changes in elastic modulus and attenuation on wave amplitude that can be easily observed in rod resonators [1, 2]. Significantly greater level of strain is required in the case of propagating waves. To the best of our knowledge the only observation of unipolar pulses propagation and nonlinear distortion without shock front formation was described by Y. Yasumoto and co-authors in [25]. In this work unipolar pulses were excited in aluminum rods subjected to annealing at various temperatures (up to 500 °C). The received pulse amplitude was observed to decrease with growth of annealing temperature as compared to linear propagation in sample without annealing, whereas pulse duration has noticeable increase. Since the data are not calibrated we will try to describe observed phenomena qualitatively using change in pulse amplitude and duration related to linear parameters corresponding to samples without annealing and annealing at temperature 250 °C. It can be shown (see Appendix) that the power exponent $n$ that well describes pulse amplitude decrease is equal to 4. In addition we estimated the ratios of nonlinear parameters corresponding to sample annealed at temperatures 350 °C, 450 °C, and 500 °C. Figures 5 display the profiles of pulses in the cases of 64% (a) and 100% (b) of maximal input pulse amplitude. Comparing these figures with Fig. 14 in [25] demonstrates that proposed theoretical description agrees quite well with experimental results.

Supposing, that pulse duration in the case of 500 °C annealed sample has almost two times increase relative to linear case, allows estimating of absolute value for nonlinear parameter. Assuming $\theta^* = 2$, $n = 4$, $a = 1.9$ in Equation (10) yields $\epsilon_m = 0.51$. Then, using Equation (8) and notations after Equation (5), as well as parameters of the experiment ($x = 0.25$ m is sample's length, $c = 5240$ m/s is wave velocity in aluminum rod, $T = 18 \cdot 10^{-6}$ s, and $\varepsilon_0 = 10^{-4}$), one can obtain $\gamma \approx 2.67 \cdot 10^{12}$. The values of nonlinear parameters for the cases of samples annealed at 350 °C and 450 °C are correspondently 15 and 5 times lower (see Appendix).

Note that effect of annealing on internal friction in aluminum was considered by V. Betekhtin and co-authors in [32]. They proposed that annealing can result in a significant decrease in the internal stress fields because of a decrease in the dislocation density, and an increase in the amplitude-dependent decrement is in turn stipulated by significant dislocation displacements from equilibrium positions under vibration stresses without limitations associated with interaction of dislocation.

To the best of our knowledge a value of the parameter of hysteretic nonlinearity with power exponent 4 was only estimated for a sample of Koelga white marble [33], and it achieved a value of $6.8 \cdot 10^{14}$ that two orders greater than our estimations for aluminum sample, nevertheless as demonstrated in [14], even in the case of quadratic hysteretic nonlinearity the parameters of nonlinearity for rocks are 10–100 times greater than ones for metals.

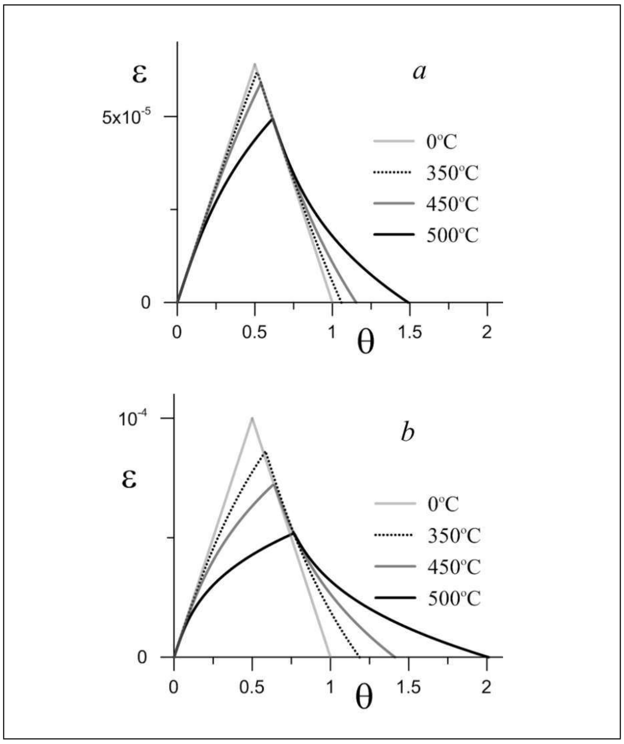


Figure 5. Qualitative views of pulse profiles in the cases of 64% (a) and 100% (b) of maximal input pulse amplitude.

## 6. Conclusion

This paper presents analytical and numeric studies of propagation of plane acoustic pulses in the media with hysteretic nonlinearity characterized by a power law with an arbitrary exponent. The exact solutions are derived for the cases of integer values of power exponent. For fractional ones, numeric and approximate analytical solutions are obtained. A comparison of the derived solutions with experimental results from the paper by Y. Yasumoto and co-authors [25] demonstrates that dependence of received pulse amplitude on pulse amplitude at source corresponds to exponent $n = 4$ in the hysteretic equation of state, and values of nonlinear parameters increase with growth of annealing temperature.

These results are of interest to the development of theory of nonlinear wave processes in media with hysteretic nonlinearity.

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## Appendix

Determination of the power exponent and relative nonlinear parameters for experimental data from [25]

Since the form of pulses for various input amplitudes was analyzed in the experiment, at first we obtain an expression for dependence of received pulse amplitude and duration on electrostatic energy charged up in the condenser. From Equation (7) and (8) follows that

$$\varepsilon_m = \frac{kE_0}{\sqrt[n]{1 + a\gamma(kE_0)^{n-1}\big(x/(cT)\big)}} \tag{A1}$$

$$\theta^* = 1 + \frac{n\left[\sqrt[n]{1 + a\gamma(kE_0)^{n-1}\big(x/(cT)\big)} - 1\right]}{2a}, \tag{A2}$$

where $k$ is coefficient, accounting the linear relations between input amplitude and electrostatic energy as well as output strain amplitude and the peak value of output electric signal, $x, c, T$ are the mentioned above parameters of the experiment. As follows from Equation (A1) in the range of the small initial amplitudes $\varepsilon_m^* \approx kE_0^*$, with growth of the amplitude Equation (A1) can be written as

$$\varepsilon_m \approx \frac{kE_0}{1 + a\gamma(kE_0)^{n-1}\big(x/(cT)\big)}. \tag{A3}$$

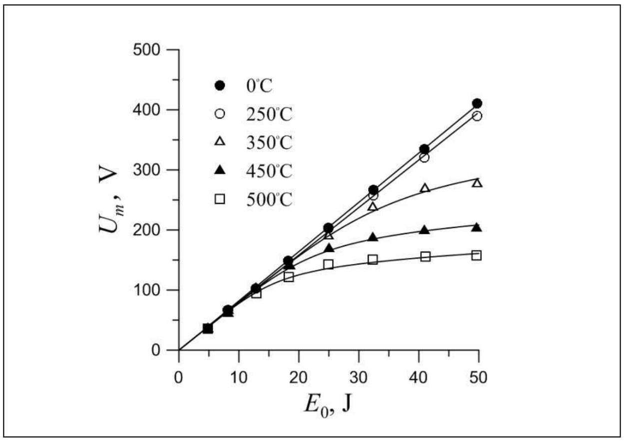


Figure A1. Relationship between the peak value of output signal in Volts and the electrostatic energy charged up in the condenser from [25] (symbols). The lines correspond to Equation (A1).

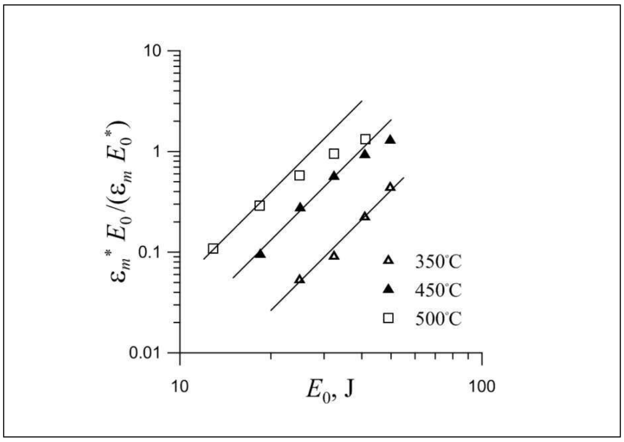


Figure A2. Experimental data from [25] (symbols) and the dependences corresponding to the Equation (A-4), the slope of straight lines is 3.

Dividing $\varepsilon_m^* \approx k E_0^*$ by this equation and taking logarithm yields

$$\ln\left(\frac{\varepsilon_m^*}{\varepsilon_m}\frac{E_0}{E_0^*} - 1\right) \approx \quad (A4)$$
$$\left(n-1\right)\ln E_0 + \ln\left(a\gamma x k^{n-1}/(ncT)\right),$$

using this equation and experimental data allows determining the value of the power exponent $n$. Figure A1 displays the original dependences (symbols) of the received pulse amplitude on the electrostatic energy and dependences corresponding to the Equation (A1) (lines). Figure A2 displays data and the dependences corresponding to the Equation (A4), the slope of straight lines is 3. We also can estimate relative values of the parameter $\gamma$. Since the other parameters in the second term of the right hand of Equation (A4) are equal, the parameter $\gamma$ is only sensitive to the change of sample annealing temperature. Hence, if one sets $\gamma = \gamma_0$ for the case of 350 °C then $\gamma = 5\gamma_0$ and $\gamma = 15\gamma_0$ are obtained for the cases of 450 °C and 500 °C, correspondently. The smallness of nonlinearity for the case of 350 °C annealing results in close fit of the straight line and experimental data in the whole range as also clearly seen from Figure (A2). The growth of the annealing temperature causes an increase in nonlinearity and in turn results in a noticeable difference between Equation (A1) and (A2) in the range of great input pulse amplitudes. It is worth noting that the similar behavior of nonlinear parameter was observed by V. Nazarov in copper [34], where the value of nonlinear parameter of the sample annealed at 800 °C was 70 and 7 times greater than ones for sample annealed at 600 °C and non-annealed sample, correspondently.

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