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Lumped Parameter Description of Thermoviscous Acoustics in Tubes with Equilateral Triangular Cross-Sections

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Summary
An accurate acoustic model of a tube in the millimeter or sub-millimeter size should include the losses that occur due to viscosity and thermal conduction, collectively termed thermoviscous losses. These losses are prominent in a boundary layer near the tube walls. Under the assumption that the boundary layer is thick compared to a characteristic geometry length, such as the radius for a circular cross-section, and fills out the entire cross-section, the tube can be described via a lumped parameter model. The lumped parameters have been known for several decades for a tube with circular cross-section, but the equilateral triangular tube has only been investigated in terms of transmission line parameters. The lumped parameters have here been established for the equilateral triangular tube as a truncated series solution of the transmission line parameters. One result observed is that the lumped parameter model for a triangular tube has 38% more viscous loss than a circular tube for a given area.

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1. Introduction

When modeling small tubes in the millimeter or sub-millimeter size one has to take into account the so-called thermoviscous losses. These losses occur because the fluid has a small, but not negligible, viscosity, and because of heat exchange between the tube wall and the fluid. The viscosity leads to drag so that the air particles near the boundary of the tube have a lower velocity than those in the bulk of the fluid. Hence, acoustic energy is converted into heat. In most cases the fluid has a low heat capacity and conductivity compared to the tube walls, and the temperature variation is zero near the boundary, which also results in energy being taken out of the acoustic wave. The effects are important in a boundary layer near the tube walls. The boundary layer thickness is approximately $2/\sqrt{f}$ millimeters [1], where f is the frequency, and so for geometries small enough and/or frequencies low enough, the boundary layer thickness will be of the same size as the geometry. In this case, as will be shown later, a lumped parameter model can be established for certain regular cross-sectional shapes, where the inertial, viscous, compressional and thermal effects each are modeled by a single component. The component values will depend on the dimensions of the tube and on the shape of the cross-section. The equilateral triangular shaped cross-section is shown in Figure 1.

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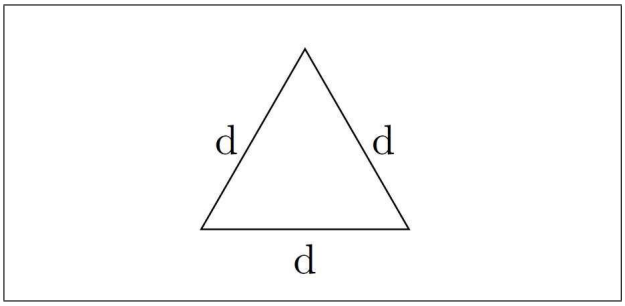

Figure 1. The equilateral triangular cross-section with side length d.

The acoustics in such a tube can be described via a transmission line model, where the relationship between the input and output variables, in this case the sound pressure and volume velocity, is described by a transmission matrix. The transmission line parameters needed to describe the acoustic behaviour of a tube are well-known for a circular cylindrical tube [1, 2, 3] and also for the triangular tube these parameters have been found [4]. For narrow tubes a lumped parameter model will suffice in describing the thermoviscous acoustic behavior. These lumped parameters have been found for the circular geometry [1, 2, 5], and for slits [6]. In the following the lumped parameters for the equilateral triangular cross-section case are established via the corresponding transmission line description. The lumped parameters for the circular and the triangular tubes have been compared to each other, and especially the differences in the viscous and thermal losses are discussed.

2. Transmission Line Model

The so-called transmission line model provides a one-dimensional representation of the sound field in a tube. Strictly speaking, the sound field in a tube is not one-dimensional in the case of thermoviscous wave propagation, since the velocity in the axial direction and the temperature vary in the cross-section. However, the pressure is practially constant [7] in the cross-section, and the model accurately describes the acoustic behavior in an average sense. The input and output variables are related via a transfer matrix

$$\begin{pmatrix} p_i \\ q_i \end{pmatrix} = \begin{pmatrix} \cosh(\Gamma l) & Z_c \sinh(\Gamma l) \\ Z_c^{-1} \sinh(\Gamma l) & \cosh(\Gamma l) \end{pmatrix} \begin{pmatrix} p_o \\ q_o \end{pmatrix}, \tag{1}$$

where p_i and q_i are the input pressure and volume velocity, p_o and q_o are the output pressure and volume velocity, l is the tube length, Z_c is the characteristic impedance and Γ is the propagation wavenumber. The characteristic impedance Z_c and the propagation wavenumber Γ can both be expressed through a so-called series impedance per unit length Z' and a shunt admittance per unit length Y' in that

$$Z_c = \sqrt{\frac{Z'}{Y'}} \tag{2}$$

and

$$\Gamma = \sqrt{Z'Y'}. \tag{3}$$

The series impedance Z' and the shunt admittance Y' are generally complex and frequency dependent, and they both depend on the cross- sectional shape. For example, the series impedance per unit length for a circular cylindrical tube (subscript $\circ$), with

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a radius of a and assuming rigid walls and a no-slip condition of the tangential velocity at the wall, is [1, 2]

$$Z'_\circ = -\frac{\mathrm{i}\omega\rho_0}{\pi a^2}\frac{J_0\left(a\sqrt{-\mathrm{i}\omega\rho_0/\mu}\right)}{J_2\left(a\sqrt{-\mathrm{i}\omega\rho_0/\mu}\right)}, \tag{4}$$

where ω is the angular frequency, ρ_0 is the density, μ is the viscosity, and J_0 and J_2 are Bessel functions of zeroth and second order, respectively. The boundary layer thickness [1] can be defined as $\sqrt{\mu/(\rho_0\omega)}$, so the argument in the Bessel functions is seen to be proportional to the ratio between the radius and the boundary layer thickness.

Benade [1] also gives a full expression for shunt admittance per unit length for a circular tube, assuming isothermal conditions at the tube wall as

$$Y'_\circ = \frac{\mathrm{i}\omega\pi a^2}{\rho_0 c_0^2}\left(\gamma + (\gamma-1)\frac{J_2\left(a\sqrt{-\mathrm{i}\omega\rho_0 C_p/\lambda}\right)}{J_0\left(a\sqrt{-\mathrm{i}\omega\rho_0 C_p/\lambda}\right)}\right), \tag{5}$$

with the thermal conductivity λ, the specific heat at constant pressure C_p, the adiabatic sound speed c_0 and the ratio of specific heats γ.

Similarly, Stinson and Champoux [4] give the general series impedance per unit length for a tube with equilateral triangular cross-section with side length d (subscript $\triangle$) as

$$Z'_\triangle = \mathrm{i}\left(\frac{\omega\rho_0}{S}\right)\left(\frac{\epsilon_v^2}{\epsilon_v^2 - 3\epsilon_v\coth\epsilon_v + 3}\right), \tag{6}$$

with $\epsilon_v = d\sqrt{i3\omega\rho_0/16\mu}$, and the shunt admittance per unit length as

$$Y'_\triangle = \mathrm{i}\omega\frac{S}{\rho_0 c_0^2}\left(\gamma - (\gamma-1)\left(\frac{\epsilon_t^2 - 3\epsilon_t\coth\epsilon_t + 3}{\epsilon_t^2}\right)\right), \tag{7}$$

with $\epsilon_t = \sigma\epsilon_v = d\sqrt{\mathrm{i}3\omega\rho_0 C_p/16\lambda}$ and $\sigma = \sqrt{\mu C_p/\lambda}$ being the square root of the Prandtl number.

3. Lumped Parameter Model

When the so-called shear wavenumber [1] $r_v = a\sqrt{\rho_0\omega/\mu}$ is smaller than one, the boundary layer fills out the entire tube, and the viscous effects dominate over the inertia effects. In this low shear wavenumber region the series impedance and shunt admittance can be described via lumped parameters. With the lumped parameter modelling method a complex system can be reduced to a system of simpler components, and transfer functions can be established by applying methodology from electronics engineering. These expressions for the circular tube can be found in the literature, see e.g. [1].

$$Z'_\circ = R'_\circ + \mathrm{i}\omega L'_\circ = \frac{8\mu}{\pi a^4} + \mathrm{i}\omega\frac{4\rho_0}{3\pi a^2}, \tag{8}$$

where R' is the resistance per unit length and L' is the inertance per unit length. Alternatively, with the area $S = \pi a^2$ the series impedance per unit length can be written as

$$Z'_\circ = R'_\circ + \mathrm{i}\omega L'_\circ = \frac{8\pi\mu}{S^2} + \mathrm{i}\omega\frac{4\rho_0}{3S}. \tag{9}$$

To find the equivalent series impedance per unit length for the equilateral triangular tube the transmission line expresion in Equation (6) is modified by assuming that ϵ_v is small, i.e. $\epsilon_v \ll 1$, which is equivalent to a low shear wavenumber. Then the hyperbolic cotangent function can be written as a series solution [8, page 85],

$$\coth\epsilon_v = \frac{1}{\epsilon_v} + \frac{\epsilon_v}{3} - \frac{\epsilon_v^3}{45} + \frac{2\epsilon_v^5}{945} - O(\epsilon_v^7). \tag{10}$$

Inserting terms up to the fifth order gives

$$Z'_\triangle \approx \mathrm{i}\left(\frac{\omega\rho_0}{S}\right)\left(\frac{15}{\epsilon_v^2}\left(1 - \frac{\epsilon_v^2}{10.5}\right)^{-1}\right). \tag{11}$$

The series impedance can be rewritten, by applying the Taylor expansion [8] $(1-x)^{-1} \approx 1 + x$, which holds for small values of x. Here, $x = \epsilon_v^2/10.5$, and since $\epsilon_v \ll 1$ then $\epsilon_v^2/10.5 \ll 1$, and so

$$Z'_\triangle \approx \mathrm{i}\left(\frac{\omega\rho_0}{S}\right)\left(\frac{15}{\epsilon_v^2}\left(1 + \frac{\epsilon_v^2}{10.5}\right)\right) \tag{12}$$

and by collecting terms

$$Z'_\triangle \approx \mathrm{i}\left(\frac{\omega\rho_0}{S}\right)\left(\frac{15}{\epsilon_v^2} + \frac{30}{21}\right). \tag{13}$$

Inserting ϵ_v explicitly gives the series impedance per unit length as

$$Z'_\triangle = R'_\triangle + \mathrm{i}\omega L'_\triangle = \frac{80\mu}{Sd^2} + \mathrm{i}\omega\frac{30\rho_0}{21S} \tag{14}$$

or alternatively with the area of the triangular cross-section $S = d^2\sqrt{3/16}$ the final result is

$$Z'_\triangle = R'_\triangle + \mathrm{i}\omega L'_\triangle = \frac{20\sqrt{3}\mu}{S^2} + \mathrm{i}\omega\frac{30\rho_0}{21S}. \tag{15}$$

By comparing the series impedance per unit length for circular and triangular tubes, respectively, it can be seen that whereas the inertance is only slightly affected by the shape of the tube, the resistance - related to the viscous loss - is 37.8% higher for an equilateral triangular cross-section than for a circular one for a given area at low shear wavenumbers. From the transmission line parameters, it can be calculated that for high shear wave numbers the viscous loss in the triangular tube is 28.6% higher than for the circular tube, for a given area. This latter result is exactly what was predicted by Pierce [9] who used a variational technique to find that the damping in a wide tube is proportional to circumference of the cross-section divided by the cross-sectional area. And the circumference of the equilateral triangle is 28.6% larger than that of the circle for a given area.

In the limit of a narrow circular tube the lumped parameter terms for the shunt admittance per unit length for the circular tube are given as [1]

$$Y'_\circ = G'_\circ + \mathrm{i}\omega C'_\circ = \frac{\omega^2 S^2 C_p}{8\pi\lambda c_0^2}(\gamma-1) + \mathrm{i}\omega\frac{\gamma S}{\rho_0 c_0^2}, \tag{16}$$

where G' is the conductance per unit length, and C' is the compliance per unit lenght. To arrive at the equivalent lumped parameter shunt admittance for the equilateral triangular tube the starting point is the transmission line expression in Equation (7). Replacing the hyperbolic cotangent function with the series solution in Equation (10) gives

$$Y'_\triangle \approx \mathrm{i}\omega\frac{S}{\rho_0 c_0^2}\left(\gamma - (\gamma-1)\left(\frac{\epsilon_t^2}{15}\left(1 - \frac{\epsilon_t^2}{10.5}\right)\right)\right). \tag{17}$$

Inserting ϵ_t and removing a higher order term gives

$$Y'_{\triangle} = G'_{\triangle} + \mathrm{i}\omega C'_{\triangle} = \frac{\omega^2 S d^2 C_p}{80\lambda c_0^2}(\gamma - 1) + \mathrm{i}\omega\frac{\gamma S}{\rho_0 c_0^2}, \tag{18}$$

or written in terms of area the final expression is

$$Y'_{\triangle} = G'_{\triangle} + \mathrm{i}\omega C'_{\triangle} = \frac{\omega^2 S^2 C_p}{20\sqrt{3}\lambda c_0^2}(\gamma - 1) + \mathrm{i}\omega\frac{\gamma S}{\rho_0 c_0^2}. \tag{19}$$

It is seen the conductance - its inverse being related to the thermal loss - is about 27% lower for the triangular case than for the circular, whereas there is no difference in compliance. It should also be noted that the lumped conductance value itself is frequency-dependent, as opposed to the resistance, inertance, and compliance, which all have frequency independent values. The means that the lumped conductance does not have a true counterpart in the electronics domain.

The lumped expressions are valid for low shear wavenumbers, where the tube is so narrow that boundary layers take up the entire bulk. If at the same time the tube is short compared to the acoustic wavelength ($|\Gamma l| \ll 1$), the impedance for a tube open in both ends can be found as

$$Z_a = Z' l, \tag{20}$$

with viscous and inertial effects being the most important ones. On the other hand, if the geometry is a closed volume, then the compressional effects along with the associated thermal effects will be important, in which case only the shunt admittance needs to be considered as

$$Y_a = Y' l. \tag{21}$$

Longer tubes can be represented as a series of shorter tubed described via lumped parameters. If each sub-length is short compared to the wavelength, the series coupling will be equivalent to the transmission line description by definition.

4. Conclusion

New expressions, Equation (15) and Equation (19), respectively, have been presented for the lumped parameter acoustic modelling of tubes with equilateral triangular shaped cross-sections. The lumped parameters include the thermoviscous effects, which are of particular importance, when the tubes are narrow in cross-section, i.e. millimetre scale or smaller. The lumped parameters supplement those already known for tubes with circular cross-section and slit structures, and are accurate under the same assumption of a low shear wavenumber. The series impedance and the shunt admittance for the equilateral triangular tube case differ from that of the circular tube, which demonstrates that when thermoviscous losses are included, the cross-sectional shape of a tube will greatly affect its acoustic properties. One important observation from the lumped parameter expressions is that the resistance due to viscous drag is 38% higher for the equilateral triangular tube than for a tube with circular cross-section for a given area.

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