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Measuring Frequency Dependent Material Properties by Use of N-Parameter Models, C. Kling and M. Schmelzer, Acta Acustica **vol. 94** (Number 4), 2008, pp. 568-579

DOI
https://doi.org/10.3813/AAA.918066

ACTA ACUSTICA UNITED WITH ACUSTICA
Vol. 94 (2008) 568 – 579

DOI 10.3813/AAA.918066

Measuring Frequency Dependent Material Properties by Use of N-Parameter Models

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Summary
The measurement of material properties is an important task. Quantities such as Young's modulus and the shear modulus, Poisson's ratio and material damping must be known when dealing with noise problems in analytical and numerical calculations. By classical methods, often only a single number is obtained which is real and independent of frequency. Sometimes, however, this is not sufficient. If a material behaves visco-elastically, we can use N-parameter-models to describe its behaviour. Then, the aforesaid properties can be described as complex numbers and in dependence of frequency, often with just a fistful of parameters which are valid for a wide frequency range. This paper gives a short introduction into the theory of visco-elastic materials and presents three different measurement examples of parameter identification.

PACS no. 43.40.+s, 43.58.+z, 46.35.+z, 46.80.+j

1. Introduction

The measurement of mechanical material parameters is a recurrent task in acoustics. When dealing with structure-borne sound, quantities such as Young's modulus, the shear modulus, Poisson's ratio and material damping must be known. Proper material characteristics are essential for the success of computational calculations on the basis of finite element methods (FEM), boundary element methods (BEM) or statistical energy analysis (SEA). Often, the quantities are required as complex numbers.

Therefore, one finds several normative references which describe different experimental methods to measure the mechanical characteristics of a material. We can distinguish between static measurement methods, to determine quantities under static conditions, and dynamic measurement methods, to determine quantities under oscillating conditions. In acoustics, our interest is focused mainly on the latter methods. For instance, in ASTM D 7121 and in the German standard DIN 53512 [1], a method is given for measuring the resilience of rubber by means of a Schob-type rebound pendulum. The German standard DIN 53513 [2] furnishes experimental setups for measuring the dynamic Young's modulus, the shear modulus and material damping for elastomers. Here, the measurement is carried out at 10 Hz, with forced vibrations. If frequency-dependent data are required, however, it is hardly possible to use these methods. ISO 9052-1 [3] allows the measurement of the dynamic stiffness for materials used under floating floors. Only the first resonance of a mass-spring system is used to determine the stiffness of a material test piece. To determine the dynamic mechanical properties of plastics, different methods can be found in ISO 6721 (see [4, 5, 6]). In this standard, the common basic measurement setups are compiled as resonance methods, forced vibration methods and free vibration methods. Deformation forms are torsional vibration, bending vibration and tensile vibration. The dynamic moduli as well as material damping are determined for particular frequencies. Complex moduli can be calculated from the measured quantities. For instance, in [7], free vibrations of crystal silicon test-pieces are used to determine the elastic modulus with the objective to develop an elastic modulus reference material.

Most of these 'classical' methods offer the possibility of dynamic measurement for only one frequency, rarely for several ones. The unreflected use of the word 'dynamic' misleads one to assume that the measured results are valid for all frequencies, not only for the measured ones. Some methods allow the measurement frequency to be changed within a certain range by changing the length or thickness of the test piece, but a lot of separate measurements are necessary for a continuous-like frequency dependence.

The term *visco-elastic* is used in some of the above-mentioned standards to account for the dissipation of the vibration energy in a real damped material. But none of these standards exploits the full power of the visco-elastic material model, although the visco-elastic material description is well-known in the field of mechanical engineering. For instance, Bormann [8] uses the visco-elastic description of elastomers to characterise bearings in rotor dynamics. Al-Khoury et al. identify the material complex moduli of visco-elastic layered media such as asphalt con-

Received 12 July 2007,
accepted 10 March 2008.

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crete [9]. Singh et al. use, among others, visco-elasticity to describe the non-linear behaviour of polyurethane foams [10]. Even more examples for the use of visco-elasticity of typical engineering materials can be found in literature. Short introductions were written by Tschoegl [11] and Williams [12]. More detailed works are, for instance, Christensen [13] and Ferry [14], the latter with a closer look to polymers which is the viscoelastic material *par excellence* [11]. Further introductions can be found in [15] or [16] and there are also some normative references, e.g. [17] and [18].

For a large class of materials, a visco-elastic characterisation can be made, leading to a small set of real parameters. Once these parameters are identified, all mechanical properties – including Young's modulus, the shear and the compression modulus, Poisson's ratio and material damping – can be calculated continuously for a wide frequency range by just a few simple formulas.

The frequency-dependence is interesting to predict the dynamical behaviour of structures, especially in computer-aided simulations. Moreover, the detailed information about a material provides an insight into the molecular structure (see e.g. [14]) which is interesting for the laboratory design of a new material or for quality control in the production process. One can conceive further applications where detailed data about the mechanical behaviour of a material are required. The origin of this work was the need for complex and frequency-dependent information about some materials that were used for the construction of downscaled models of a building acoustics wall test facility [19].

There are commercial devices for measurement of viscoelastic material properties, called *rheometers*. They are used in quality control and development for the characterisation of liquids or mainly flexible solids such as fluids, dispersions, gels, adhesives, melts of polymers, rubber and asphalts. Usually, the viscosity of a liquid-like material is determined, thus such specific devices are called *viscometers*. Some rheometers are able to measure dynamic properties but the frequency range is usually limited to 100 Hz or below. Moreover, the devices are normally built as rotational rheometers, so only shear properties are found. Due to these limitations, such a device is not suited for detailed investigations for frequencies in the kHz-range into complete mechanical behaviour of specific materials whose shape is not standardised.

Especially in research, the more time-consuming way of *measurement by hand* is interesting. As an advantage, common basic measurement setups can be used. Unfortunately, higher computational effort is necessary for parameter identification, but this disadvantage will decrease in the future as the computational power is continuously increasing.

This paper deals with the characterisation of different materials – silicone, aluminium and acrylic glass – as linear visco-elastic materials by means of a set of N parameters. Different measurement setups are used to find different complex moduli, including material damping.

We will start with a brief introduction into the linear theory of visco-elastic solids. Since the theory is usually dealt with in the time domain, the connection to the frequency domain will be explained, as this is the domain that is commonly used in acoustics. We will discuss in which way it is possible to calculate different quantities from the determined parameters and give a survey of how the data are to be measured and analysed in order to find an adequate set of parameters. Finally, as practical examples, three different characterisation measurements will be presented.

2. A survey of the linear visco-elastic theory

2.1. Memory functions in the time domain

A visco-elastic material will always remember strains it has suffered in the past. The momentaneous state of such a material is not only influenced by the deformations it undergoes at that special point in time, but also by deformations it has undergone in the past. In this paper, we will restrict ourselves to the linear behaviour. According to [20] and [21], the general material law of an inhomogeneous, anisotropic, linear visco-elastic material can – for local behaviour – be written as:

$$\sigma_{ij}(\underline{x}, t) = \int_{-\infty}^{t} G_{ijkl}(\underline{x}, t-\tau) \cdot \frac{\partial}{\partial \tau} \epsilon_{kl}(\underline{x}, \tau) \mathrm{d}\tau. \quad (1)$$

All quantities depend on the location $\underline{x}$ and the time t. σ_{ij} is the 3×3 tensor of stress, G_{ijkl} is the $3 \times 3 \times 3 \times 3$ tensor of the *memory functions*, and ϵ_{kl} is the 3×3 tensor of strain. In equation (1), we can see that the momentaneous stress at a time t is determined by the strains suffered at all times τ in the past up to the momentaneous point in time t.

Knowing a material's properties means to know its memory functions $G_{ijkl}(\underline{x}, t)$. Due to the symmetries $\sigma_{ij} = \sigma_{ji}$ and $\epsilon_{kl} = \epsilon_{lk}$, the tensor G_{ijkl} has, in general, 21 independent components. According to Morro [22], certain cases of thermodynamic laws (Clausius-Duhem inequality) implicate monotonicity of the memory function. Materials with non-monotonic memory functions may be possible, e.g. by terms of inertia on a molecular level. Here, we keep to monotonically decreasing memory functions which implies fading memory. As deformations which will take place in the future are not able to affect the current behaviour of the material, $G_{ijkl}(\underline{x}, t)$ is defined zero for negative times.

For simplification, we restrict ourselves to homogeneous materials. Thus, we can omit the dependence on the location $\underline{x}$ in all quantities.

To be able to characterise a material completely, we have to determine the 21 continuous memory functions $G_{ijkl}(t)$ by experiments. A measurement, however, delivers only a discrete set of measurement points in time and we therefore have to discretise the continuous functions. Usually, this is done by Prony series as follows:

$$G_{ijkl}(t) = \sum_{p=0}^{n} a_{ijkl}^{p} \cdot \mathrm{e}^{-b_{ijkl}^{p} \cdot t}, \quad (2)$$

with p being an upper index and no exponent. By means of this formula, for each combination of i, j, k, l, the $n+1$ real material constants a^p_{ijkl} and the n real time constants b^p_{ijkl} have to be determined for each of the 21 components in a measurement. b^0_{ijkl} is defined zero. Then, for the term with $p = 0$, only a^0_{ijkl} remains, which is called the *static modulus* and can be measured in static experiments. For a purely elastic material, no memory exists, which means that there are no terms with $p > 0$. In that case, the 21 material constants a^0_{ijkl} are the only constants to be determined.

By inserting the discretisation (2) into equation (1), we find an alternative formulation of the material law:

$$\begin{aligned} \sigma_{ij}(t) &= \sum_{p=0}^{n} a^p_{ijkl}\left(\epsilon_{kl}(t) - v^p_{ijkl}(t)\right) \qquad (3)\\ \dot{v}^p_{ijkl}(t) &= b^p_{ijkl} \cdot \left(\epsilon_{kl}(t) - v^p_{ijkl}(t)\right), \quad p = 0, \ldots, n. \end{aligned}$$

The new variables v^p_{ijkl} include the memory integral of equation (1):

$$v^p_{ijkl}(t) := \int_{\tau=-\infty}^{t} b^p_{ijkl} \cdot \mathrm{e}^{-b^p_{ijkl}\cdot(t-\tau)} \cdot \epsilon_{kl}(\tau)\,\mathrm{d}\tau, \qquad p = 0, \ldots, n. \qquad (4)$$

The variables v^p_{ijkl} relate to the strains suffered in the past. In equation (3), the stress σ_{ij} depends on the outer strain ϵ_{kl} and on the new variables v^p_{ijkl}, having the dimension of a strain. Just as ϵ_{kl} is the strain which acts from outside of the materials, we can see v^p_{ijkl} as the pth inner strain acting with the pth material constant onto the material. Therefore, we call the $n+1$ new variables v^p_{ijkl} the *inner strain variables*.

Equation (1) turned into a set of common differential equations of first order (equations (3)) which can be solved by finding the real coefficients a^p_{ijkl} and b^p_{ijkl}. Since the $n+1$ equations in the second line of equation (3) describe the evolutionary behaviour of the material, these equations are called the *evolution equations*.

2.2. Complex modulus in the frequency domain

Up to now, we have used the time domain to formulate the material law. In acoustics, however, usually the frequency domain is preferred. Furthermore, motion is, in general, harmonic in acoustics. Hence, we can write the stress and strain functions as:

$$\sigma_{ij}(t) = \hat{\sigma}_{ij} \cdot \mathrm{e}^{\mathrm{i}\omega t}; \epsilon_{kl}(t) = \hat{\epsilon}_{kl} \cdot \mathrm{e}^{\mathrm{i}\omega t}, \qquad (5)$$

where $\hat{\sigma}_{ij}$ and $\hat{\epsilon}_{ij}$ are the stress and the strain amplitudes, respectively, i is the complex unit and ω the angular frequency. We still restrict ourselves to homogeneous materials, without dependence on location. Using harmonic stress- and strain-functions in equation (1), we find the stress- and strain-tensors connected by a $3\times3\times3\times3$ tensor with complex and frequency-dependent components:

$$\hat{\sigma}_{ij} = G^*_{ijkl}(\omega) \cdot \hat{\epsilon}_{kl} \qquad (6)$$

The quantity found is the *complex modulus* G^*_{ijkl} which is well known in acoustics. Its relation to the tensor of memory functions is given by:

$$G^*_{ijkl}(\omega) = \mathrm{i}\omega \int_{-\infty}^{+\infty} G_{ijkl}(t)\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t \qquad (7)$$

Equation (7) reads as a Fourier transformation, multiplied by a factor iω. Hence, we find the memory functions directly coupled to the well-known complex modulus. Both contain the same information on the material. The discretisation (2) of the memory functions leads to a discretisation of the continuous complex modulus (see equation 7):

$$G^*_{ijkl}(\omega) = \sum_{p=0}^{n} a^p_{ijkl} \frac{\mathrm{i}\omega}{\mathrm{i}\omega + b^p_{ijkl}}. \qquad (8)$$

The $n+1$ real material constants a^p_{ijkl} and the n real time constants b^p_{ijkl} are the same as in equation (2). Remember that b^0_{ijkl} is defined zero and that a^0_{ijkl} is the static modulus.

For a homogeneous anisotropic material, we have to identify $n+1$ material and n time constants for each of the 21 components of the $3\times3\times3\times3$ tensor $G_{ijkl}(t)$ in the time domain, or equivalently, the same constants for the $3\times3\times3\times3$ tensor $G^*_{ijkl}(\omega)$ in the frequency domain. This is the measurement task.

2.3. Isotropic material

When we restrict ourselves to isotropic materials, only two independent components of the $3\times3\times3\times3$ tensors remain. Different formulations of the independent components are possible and can be transferred with each other. Common formulations are the Lamé's constants λ and μ or the moduli of compression K and shear G. This proceeding can be found in literature in connection with the theory of elasticity (see, for instance, [23] or [24]).

A homogeneous isotropic material is totally characterised if we know the material and time constants of two of the afore-said moduli. For instance, using the moduli of compression and shear, the equations (8) turn into

$$K(\omega) = \sum_{p=0}^{n} K_p \cdot \frac{\mathrm{i}\omega}{\mathrm{i}\omega + b_p}, \qquad (9)$$

$$G(\omega) = \sum_{q=0}^{m} G_q \cdot \frac{\mathrm{i}\omega}{\mathrm{i}\omega + b_q}. \qquad (10)$$

In this case, a total characterisation of a material is provided by the $(n+1)$ real constants K_p, the n real time constants b_p, the $(m+1)$ real constants G_q and the m real time constants b_q. Choosing n and m large enough allows a good discretisation of the complex moduli K and G.

The different moduli relate to each other. In the further context, we will need Young's modulus E which can be calculated by the moduli of compression and shear by

$$E(\omega) = \frac{9K(\omega)G(\omega)}{3K(\omega) + G(\omega)}. \qquad (11)$$

Thereby, all quantities are complex. Furthermore, the Poisson ratio μ is given by the moduli by

$$\begin{aligned}\mu(\omega) &= \frac{1}{2}\cdot\frac{E(\omega)}{G(\omega)} - 1 \\ &= \frac{1}{2}\cdot\frac{3K(\omega) - 2G(\omega)}{3K(\omega) + G(\omega)}.\end{aligned} \tag{12}$$

As it is defined by two complex quantities, the Poisson ratio, too, is a complex quantity.

For thermodynamic reasons, Young's modulus and the Poisson ratio are restricted to the following values:

$$\begin{aligned}-1 < |\mu| < 0.5 \quad &\Leftrightarrow \quad |K| > 0, \\ |G| > 0 \quad &\Leftrightarrow \quad 0 < |E| < 3|G|.\end{aligned} \tag{13}$$

A separate damping can be calculated for each modulus. In general, damping is defined by the ratio of the imaginary part $\Im$ and the real part $\Re$ of a modulus. For the three introduced moduli K, G and E, we find the damping coefficients:

$$\eta_K = \frac{\Im[K]}{\Re[K]}; \quad \eta_G = \frac{\Im[G]}{\Re[G]}; \quad \eta_E = \frac{\Im[E]}{\Re[E]}. \tag{14}$$

Different moduli represent different forms of motions. In general, these motions could base on different microscopic mechanisms. This is the reason why the damping coefficients could differ for different moduli. In practice, it is often assumed that the damping coefficients are equal for all moduli. But the experimental results presented in chapter 5 show the opposite.

3. The canonical illustration

In the following, we present a more intuitive way to understand a visco-elastic material, i.e. by the theory of mechanical networks.

Let's take any of the independent components of the tensor $G^*_{ijkl}(\omega)$ and designate it as a general modulus D. D can, for example, be the shear modulus, the compression modulus, Young's modulus or one of Lamé's constants.

The behaviour of a purely elastic solid material can be illustrated by a spring. The current stress σ and the current strain ϵ are linked by the real spring constant k as $\sigma = k\epsilon$. This relation is known as Hooke's law, and therefore a solid that can be represented as a spring is called a *Hooke's solid*. No deformation energy is dissipated.

In an equivalent way, a purely viscous liquid can be illustrated by a single damper. The current stress σ is linked to the current strain velocity $\dot{\epsilon}$ by the real damper constant d as $\sigma = d\dot{\epsilon}$. Such a liquid is called a *Newton's liquid*. All deformation energy is dissipated.

A visco-elastic material is both solid and liquid. The deformation energy is partially saved and partially dissipated. As Hooke's solid and Newton's liquid are idealised models, a visco-elastic model is a good approximation for a lot of materials. A large number of them can be discretised as in chapter 2. Since the material behaviour can then be predicted by a set of N parameters, such a material is called an *N-parameter material*. The N parameters can be interpreted as the constants of N ideal spring- or damper-elements which form a network by serial and parallel connection. The large number of possible spring-damper constellations can be abstracted by two general configurations which are called the *canonical illustrations*: the *generalised Kelvin-Voigt model* and the *generalised Maxwell model*. In the further context, we will use the general Maxwell model.

A spring serially connected to a damper forms a so-called *Maxwell element*. The general Maxwell model consists of a number of Maxwell elements that are connected in parallel. For materials which are more solid than liquid, one Maxwell element reduces to a single spring.

To connect the stress and the strain acting on such a visco-elastic material, we want to construct an equivalent modulus for the spring- and damper-elements. The equivalent modulus for an ideal spring is just the real spring constant k:

$$D_k = k \quad \text{(ideal spring)} \tag{15}$$

The equivalent modulus for an ideal damper can be calculated when we assume harmonic deformation. Then, the equivalent modulus for the damper element with the damper coefficient d is purely imaginary:

$$D_d = \mathrm{i}\omega d \quad \text{(ideal damper)} \tag{16}$$

For a Maxwell element, we find the equivalent modulus by summing up the reciprocals of the individual moduli that are connected in series. It is a complex quantity,

$$\begin{aligned}D_{\text{Maxwell}} &= \left(D_k^{-1} + D_d^{-1}\right)^{-1} \\ &= k\cdot\frac{\mathrm{i}\omega}{\mathrm{i}\omega + k/d} \quad \text{(Maxwell element)}\end{aligned} \tag{17}$$

Due to the parallel connection, we can sum up the single spring, which we denote by index 0, and the n Maxwell elements, which we denote by index $p = 1, \ldots, n$, to the total equivalent modulus of the N-parameter model:

$$D = k_0 + \sum_{p=1}^{n} k_p \frac{\mathrm{i}\omega}{\mathrm{i}\omega + k_p/d_p}. \tag{18}$$

Comparing equation (18) to equation (8) gives us an idea of the n+1 real material constants a^p_{ijkl} as the real constants of the n+1 springs in the generalised Maxwell model. Similarly, the n non-zero time constants b^p_{ijkl} correspond to the ratio of the spring and damper constants of the n Maxwell elements. The N-parameter-model is characterised by the $N = (n+1) + n = 2n + 1$ spring and damper constants of the generalised Maxwell model.

A lot of solid visco-elastic materials behave liquid-like when it comes to fast relaxations. Then one of the Maxwell elements reduces to a single damper. This damper covers also relaxations which are too fast for the measurement devices or setups. For that reason, its real damper constant is

assigned as d_{fast}. Usually, this fast damper is placed right from the beginning in the generalised Maxwell model and the identification process must show whether it is necessary or not. Then, equation (18) is extended by a purely imaginary term $i\omega d_{fast}$,

$$D = k_0 + \sum_{p=1}^{n} k_p \frac{i\omega}{i\omega + k_p/d_p} + i\omega d_{\text{fast}}. \qquad (19)$$

Figure 1 shows a sketch of the general model of a visco-elastic solid including a fast damper, according to equation (19).

By means of the Maxwell model, we also get an idea of the inner variables. The inner strain variables in equations (3) and (4) can be understood as the strains that drop at the individual Maxwell elements. In chapter 5, we will use inner deformations. We can think of these also as the parts of the outer deformation that split at the particular Maxwell elements. There is an analogy to electrical networks. We can set mechanical stress and deformation as electrical voltage and charge, respectively, and exchange connections in parallel and in series. Then, the spring constants correspond to reciprocal capacities and the dampers correspond to electrical resistors (see, for instance, [25] or [17]).

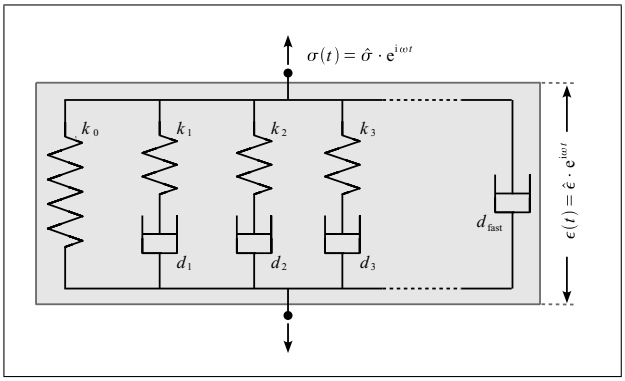

Figure 1. General Maxwell model of a visco-elastic solid with an additional fast damper. Harmonic deformation is assumed. The complex modulus $G^*_{ijkl}(\omega)$ is an equivalent modulus consisting of springs and dampers.

4. General approach to identify the visco-elastic material parameters

The visco-elastic parameters are identified by fitting a calculated system response to its measured equivalent. This can be done both in the time domain and in the frequency domain. The examples given in this paper are processed in the frequency domain.

In general, for measurements, a common basic setup as the ones specified in ISO 6721 [4, 5, 6], for instance, are used. When processing in the time domain, usually free resonance systems are used. One vibration sensor is placed at a convenient point on the test piece. The test piece is moved out of its position of rest and let loose to swing free while the system response is recorded. In the frequency domain, two sensors are used at two different points of the test piece and the transfer function is measured in between. The test piece is excited in such a way that it follows a forced movement.

Usually, the modulus for a special deformation is sought-after and the material is assumed to be isotropic. Deformation types which have proved suitable for measurements are tension and bending bars. Thus, Young's modulus can be obtained by means of both a tension probe and a thin bar; the shear modulus can be determined from torsional deformation or on thick bending bars.

For analysis, a computational model of the experimental setup is prepared. An adequate physical model must be found which accounts for all effects that influence the motion of the test piece. The aim is to calculate a system response that is equivalent to the measured one. This can be done by analytical solution of the equations of motion as well as by use of numerical methods as FEM. A mixture of analytical and numerical method is possible, too. Doing this, the visco-elastic material law must be used for the stress-strain relation. Usually, the material law is not used with outer and inner strains as is done in equation (3), but with outer and inner displacement variables.

The computed system response then only depends on geometrical quantities and on material characteristics, including the visco-elastic material parameters which have to be determined. Up to now, the number n of the parameter pairs a^p_{ijkl} and b^p_{ijkl} – i.e. the number of Maxwell elements – is not specified. For the first computation, we need a starter set of parameters that can be taken from literature, from estimations or by other – maybe classical, e.g. statical – experiments. Within a fitting loop, the computed system response is repeatedly compared with the measured one and the set of material parameters is optimised until a satisfying state of agreement of the computed and measured system response is reached. The fitting process can be done using a standard optimisation routine. Prevalently, the Levenberg-Marquardt or the Simplex algorithm is used. We have to take care not to exceed the physical limits given in chapter 2.3 while identifying the parameters. This can be achieved by using penalty terms or transformations. The number n of the Maxwell elements is determined in the optimisation process, too, whereby preferably kept low.

A properly measured transfer function is the prerequisite for finding a proper set of parameters. Therefore, the test piece has to be dimensioned in such a way that the system response is distinct from the noise floor and from other bad influences. The boundary conditions turned out to be essential. An ideal clamped end is hard to realize, so free ends are preferred. Symmetrical setups usually have good boundary conditions. The type of excitation signal is arbitrary as the transfer function is determined between the sensor positions. The only condition is that it is a harmonic signal and that the amplitudes are not too huge, so that the test piece behaves in a linear way. Furthermore, the sensors should not exert an influence on the movement of the test piece. Non-contact sensors can be used like Doppler

Laser vibrometers. The measurements described in chapter 5 were performed by means of lightweight accelerometers. The advantage when measuring the acceleration instead of the velocity or the displacement is that the system response decreases less in frequency – by a factor ω or ω^2, respectively – than in the case of velocity or displacement. The additional mass which the sensors inflict upon the system can be balanced by means of dummy masses. Moreover, the new generation of lightweight accelerometers generates only a low device noise.

5. Measurement examples

In the following, we will describe three different experiments which have been carried out to find Young's modulus of silicone and of aluminium and Young's modulus and the shear modulus of acrylic glass. Different physical models and different experimental setups were used. All measurements were carried out at room temperature. The application of inner variables for the analytical model of the setup is demonstrated in the first example.

5.1. Young's modulus for silicone

The measurement task is to find the frequency-dependent Young's modulus of silicone. For this purpose, common silicone is cast into a cylindrical test piece of 22 mm in diameter. After hardening, two cylindrical metal blocks are glued to both ends of the test piece, one at each end, using silicone of the same origin. Since the silicone test piece acts like a spring and the metal blocks like stiff masses, the constellation works as a mass–spring–mass system. The system is suspended vertically on a shaker. The upper mass is rigidly attached to the shaker pike, so that it follows the pike's forced motion. Accelerometers are attached to both masses, and the transfer function is measured from the upper mass to the lower mass. Due to the fact that the silicone test piece acts as a dilated bar in only one dimension, Young's modulus can be identified directly. Figure 2 shows the experimental setup.

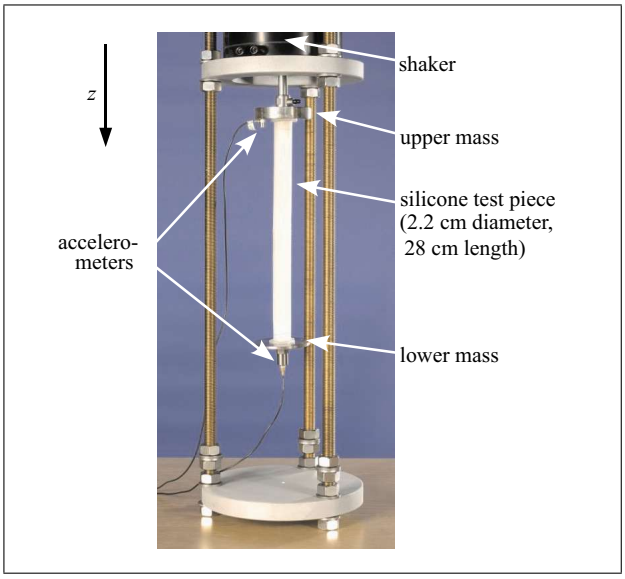

Figure 2. The silicone test piece is mounted between two metal blocks and is suspended on a shaker. The transfer function is measured from the upper to the lower metal block with two accelerometers.

For the analytical model, we use a bar with dilatation in direction z (see Figure 2). The equation of motion results from the balance of forces of the infinitesimal mass element

$$\rho \ddot{u}_z = \frac{\partial}{\partial z}\sigma_{zz}, \tag{20}$$

whereby ρ is the material density and u_z the dilatation in direction z. We consider the material silicone to be isotropic and the stress state to be a uniaxial normal stress state in direction z. The equations (3) then reduce to:

$$\begin{aligned} \sigma_{zz} &= \sum_{p=0}^{n} a_E^p \left(\epsilon_{zz} - v_{zz}^p\right) \\ \dot{v}_{zz}^p &= b_E^p \left(\epsilon_{zz} - v_{zz}^p\right), \quad p = 0, \ldots, n, \end{aligned} \tag{21}$$

whereby the second line shows the $n+1$ evolution equations for the inner strain variables v_{zz}^p. For small deformations, the strain is defined by (see, for instance, [23]).

$$\epsilon_{ik} := \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right), \tag{22}$$

which leads to

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{23}$$

for the momentaneous stress state. Equivalent to the definition (22) for the strain ϵ_{ik} from the deformations u_i, we can define the inner strain variables from the inner dilatation variables:

$$v_{ik}^p := \frac{1}{2}\left(\frac{\partial y_i^p}{\partial x_k} + \frac{\partial y_k^p}{\partial x_i}\right), \tag{24}$$

which leads to

$$v_{zz}^p = \frac{\partial y_z^p}{\partial z} \tag{25}$$

for the stress state in the silicone test piece. Using equations (21), (23) and (25) in (20), we find the equation of motion with deformations as

$$\begin{aligned} \rho \ddot{u}_z &= \sum_{p=0}^{n} a_E^p \partial_z^2 \left(u_Z - y_z^p\right) \\ \dot{y}_z^p &= b_E^p \left(u_Z - y_z^p\right), \quad p = 0, \ldots, n, \end{aligned} \tag{26}$$

with $n+1$ evolution equations for the inner dilatations. ∂_z is equal to $\partial/\partial z$ and means the derivative with respect to z.

We assume that the shaker excites the test piece with a low amplitude so that linear oscillation can be assumed. Then, we can do a harmonic ansatz for the time dependence of the outer and inner dilatation variables u_z and y_z^p, respectively,

$$u_z(x,t) = \tilde{u}_z(x) \cdot e^{i\Omega t};$$
$$y_z^p(x,t) = \tilde{y}_z^p(x) \cdot e^{i\Omega t}. \quad (27)$$

We use Ω to denote the exciter angular frequency, and we use the evolution equations to express the inner dilatations by the outer dilatations

$$\tilde{y}_z^p(x) = \frac{b_p}{i\Omega + b_p}\tilde{u}_z(x), \quad p = 0, \ldots, n, \quad (28)$$

and we obtain the equation of motion as

$$-\Omega^2 \rho \tilde{u}_z(x) = \sum_{p=0}^{n} a_E^p \frac{i\Omega}{i\Omega + b_p} \cdot \partial_z^2 \tilde{u}_z(x). \quad (29)$$

Now, equation (29) can be solved as usual with an exponential ansatz for the locus $\tilde{u}_z(x)$ and with accurate boundary condition.

Then, the transfer function can be calculated from the top to the bottom of the silicone test piece if the material parameters a_E^p and b_E^p are known. We use a starter set of material parameters, calculate the transfer function, compare it to the measured transfer function and adjust the set of parameters. Loop for loop, we go through a non-linear optimisation process to find the optimal set of parameters with a minimum number n of Maxwell elements. The single spring that is to be seen in the generalised Maxwell model is always used because it makes the material solid. Also, the single damper from equation (19) is set a priori.

The transfer function was measurable from about 10 Hz to about 550 Hz. Due to the high damping in the test piece, the maximum length sequence (MLS) technique was used. Several resonances were found. Figure 3 shows the measured and the calculated transfer functions. We find good agreement up to about 350 Hz for this case. The final material parameters listed in Table I are found by combining several measurements. The transfer function was measured for three different test piece lengths from about 5 cm to about 30 cm and for different foot masses. When identifying the parameters, for each of the measured situations an adequate transfer function is computed and an overall comparison is made for the fit. Besides the separate spring, two Maxwell elements and an additional damper were identified. In this way, we found a 6-parameter model for silicone which is accurate in a range from 10 Hz to 550 Hz.

Figure 4 and Figure 5 show the real part of Young's modulus and tensional damping, both computed from the 6 parameters found. Young's modulus increases by more than 10% in the valid frequency range, from about 9 MPa to about 10.2 MPa. This is approximately the range that can be found in technical papers of several silicone manufacturers. But direct comparisons are difficult because usually there is no frequency dependence stated in literature.

Table I. Identified visco-elastic material parameters of silicone, valid for frequencies between 10 Hz and 550 Hz.

index p	E_p [N/m^2]	b_p [1/s]	E_{fast} [Ns/m^2]
0	8.92e6	0	23.97
1	7.22e5	125.4	
2	6.04e5	504.1	

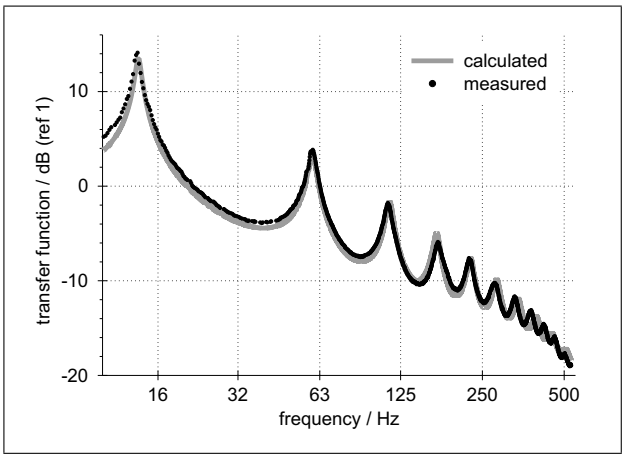

Figure 3. Transfer function with a silicone test piece of 28 cm length and a lower mass of 133 g; measured and calculated with final parameters.

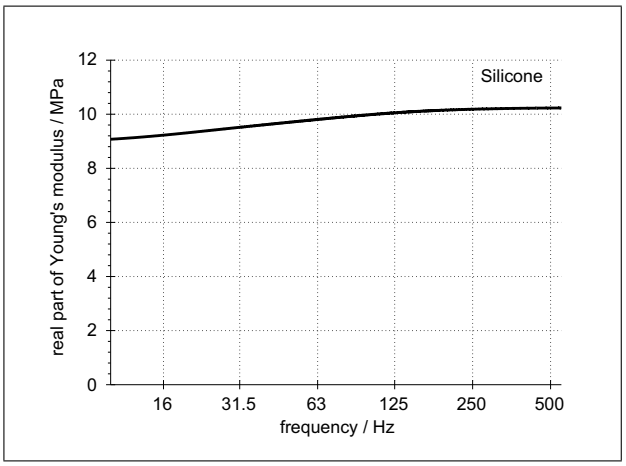

Figure 4. Real part of Young's modulus for silicone according to the identified parameters of Table I.

The damping has a maximum of about 5.5% at 30 Hz and decreases – up to 550 Hz – to about 2%.

5.2. Young's modulus for aluminium

Young's modulus of aluminium is investigated.

A thin bar of aluminium (2.5 mm in thickness x 25 mm in width x 1 m in length) is placed horizontally in such a way on the pike of a shaker that it finds itself in a balanced position (see Figure 6). The centre of the bar is mounted rigidly to the shaker and follows the enforced motion. The accelerometers are mounted at the excited centre of the bar as well as at one end of the bar. To maintain the symmetry, a compensation mass or a third accelerometer is mounted

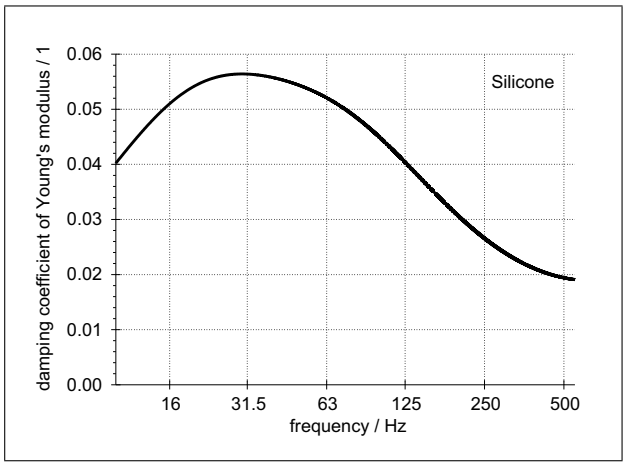

Figure 5. Damping of Young's modulus for silicone according to the identified parameters of Table I.

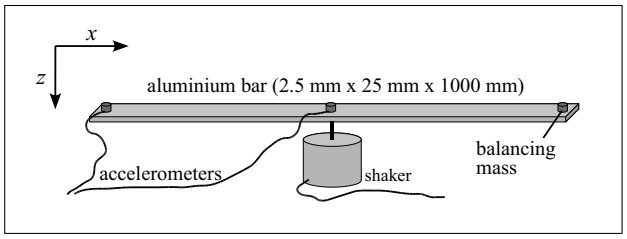

Figure 6. Experimental setup for the aluminium test piece. The thin bar acts like a Bernoulli beam.

at the other end of the bar. The transfer function from the centre to one end of the bar is measured.

Thanks to the small thickness of the aluminium bar, we can use a *Bernoulli* beam as physical model. This means, we neglect any shear deformation. The bending of the bar then results only from uniaxial strain in direction x which varies for locations in x (for orientation, see Figure 6). The bending stiffness is only influenced by Young's modulus. This is why Young's modulus can be identified directly.

The equation of motion follows from the balance of forces:

$$A\rho\ddot{u}_z = \partial_x^2 \int_A \sigma_{xx} z \, \mathrm{d}A. \tag{30}$$

Here, A is the cross-sectional area of the bar, ρ is the density of the aluminium and ∂_x is the derivative with respect to x. We assume that aluminium is isotropic. Applied to the uniaxial normal stress state in direction x, the visco-elastic material law (3) reduces to

$$\begin{aligned} \sigma_{xx} &= \sum_{p=0}^{n} a_E^p \left(\epsilon_{xx} - v_{xx}^p\right) \\ \dot{v}_{xx}^p &= b_E^p \left(\epsilon_{xx} - v_{xx}^p\right), \quad p = 0, \ldots, n. \end{aligned} \tag{31}$$

For the Bernoulli beam, the outer strain results from the deformation in direction z,

$$\epsilon_{xx} = -z\partial_x^2 u_z \tag{32}$$

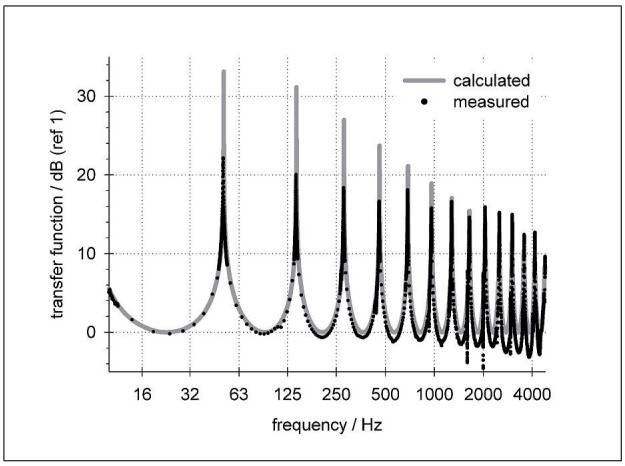

Figure 7. Transfer function of the thin aluminium bar; measured and calculated with the final parameters of Table II.

Table II. Identified parameters for aluminium.

index p	E_p [N/m^2]	b_p [1/s]	E_{fast} [Ns/m^2]
0	68.13e9	0	2.82e4

Equivalently, we define the inner strains v_{xx}^p from the inner deformations y_z^p in direction z :

$$v_{xx}^p = -z\partial_x^2 y_z^p. \tag{33}$$

Using the geometrical moment of inertia $I_y = \int_A z^2 \, \mathrm{d}A$; the equation of motion can now be written with outer and inner deformations only,

$$\begin{aligned} A\rho\ddot{u}_z &= -I_y \sum_{p=0}^{n} a_E^p \partial_x^4 \left(u_z - y_z^p\right) \\ \dot{y}_z^p &= b_E^p \left(u_z - y_z^p\right), \quad p = 0, \ldots, n, \end{aligned} \tag{34}$$

with $n+1$ evolution equations for the inner deformations.

The further procedure is equivalent to chapter 5.1 and results in a calculated transfer function from the centre of the bar to its ends that only depends on the dimensions of the bar and on ρ and the visco-elastic parameters a_E^p and b_E^p.

A proper transfer function was measured from 10 Hz to about 5 kHz (see Figure 7). The identification process led to a Kelvin-Voigt model consisting of a spring and a damper connected in parallel. In fact, this 2-parameter-model is typical for metals. Young's modulus of about 68.1 GPa is invariant for all frequencies and agrees with values from literature (see, for instance, [26] or [27]). The damping is purely viscous, which means that it increases linearly with velocity and therefore with frequency. The loss factor is nearly zero for low frequencies and about 0.013 for 5 kHz. Table II indicates the identified parameters.

5.3. Young's modulus and the shear modulus for acrylic glass

Young's modulus and shear modulus for acrylic glass are examined jointly. Assuming that the material is isotropic,

all mechanical properties are defined by the two complex moduli.

The same setup as the one in chapter 5.2 for the aluminium bar was used. A bar of acrylic glass with a quadratic cross-sectional area of about 25 mm x 25 mm and a length of about 2 m is mounted symmetrically on the pike of a shaker (Figure 8). The bar is exited to perform bending vibrations. The transfer function is measured from the centre of the bar to one end.

Because the bar is thick compared to the bending wavelength in the centre and in the upper frequency range measured, we cannot use the simple Bernoulli model. The bending stiffness is influenced by Young's modulus as well as by the shear modulus. We account for this by using the *Timoshenko bar*. The equations of motion derive from the balance of forces for rotational and for translational motion:

$$\begin{aligned}\rho I_y \ddot{\beta} &= A \sum_{p=0}^{n} a_G^p \left((u_z - y_z^p)' - (\beta - \tilde{\beta}^p) \right) \\ &\quad + I_y \sum_{p=0}^{n} a_E^p \left(\beta - \tilde{\beta}^p \right)'' , \\ \rho \ddot{u}_z &= \sum_{p=0}^{n} a_G^p \left((u_z - y_z^p)'' - (\beta - \tilde{\beta}^p)' \right) , \qquad (35) \\ \dot{y}_z^p &= b^p \left(u_z - y_z^p \right) , \qquad p = 0, \ldots, n, \\ \dot{\tilde{\beta}}_z^p &= b^p \left(\beta_z - \tilde{\beta}_z^p \right) , \qquad p = 0, \ldots, n. \qquad (36)\end{aligned}$$

Here, ρ is the density of the acrylic glass, I_y is the geometrical moment of inertia, A is the cross-sectional area, β is the angle of bending, $\tilde{\beta}^p$ is the equivalent inner variable for the bending angle and the dash $'$ is the derivative with respect to x. The equations (36) define the evolution equations for the inner displacement and the inner bending angle. As we identify two moduli at the same time, we need twice the number of inner variables and evolution equations.

Instead of Young's modulus, the compression modulus is used for identification. This is done because when using Young's modulus and the shear modulus, the parameters may exceed the physical limits of chapter 2.3. By combining the compression modulus and the shear modulus, the volume-changing and the shape-changing deformations are separated. The material parameters then automatically fulfil physical boundary conditions. According to equation (11), Young's modulus is calculated from the identified compression modulus and the identified shear modulus. We assume that the time constants and the number of parameters are the same for compression modulus and shear modulus. Furthermore, we do not vary the time constants, but use fix values. According to the general rule that the range of influence of a Maxwell element spans over about one decade on the time- or frequency-axis, two points per decade are sufficient. We choose not to optimise the number of Maxwell elements. As a result, the identification process is more stable.

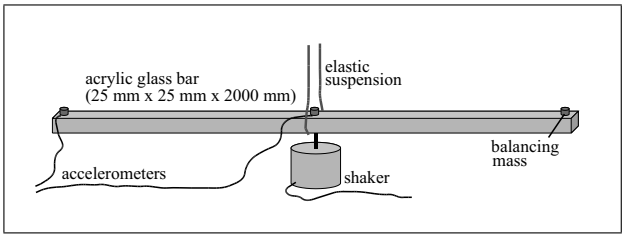

Figure 8. Experimental setup for the acrylic glass bar. The thick bar is treated as a Timoshenko bar.

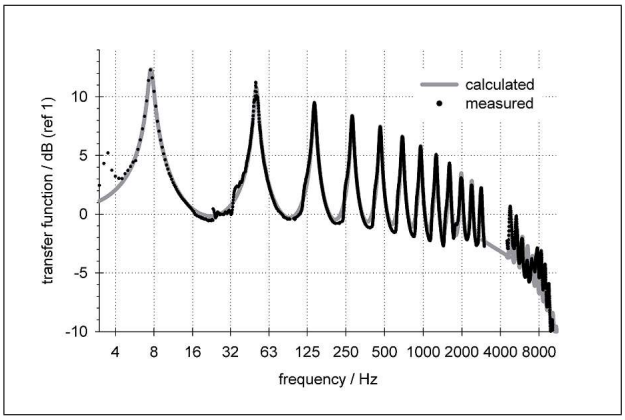

Figure 9. Transfer function for acrylic glass probe; measured and calculated with the final parameters of Table III.

The transfer function was measured from about 3 Hz to about 10 kHz. Above 1 kHz, the bending wavelength is so small that the shear modulus has a significant influence on the bending motion. Below 1 kHz, the acrylic bar acts as a Bernoulli bar. In the frequency range from about 3 kHz to about 4 kHz, the measured transfer function is polluted and is cut out for the fitting process (Figure 9). The gap is covered by the Maxwell elements to the left and to the right of the blank region because the Maxwell elements influence a frequency range of about a decade each. The data gap is small, compared to a decade, and the identification process is not disturbed.

The parameter identification led to 11 Maxwell elements each for the compression modulus and for the shear modulus, plus additional spring and additional fast parameters. Thus, we found acrylic glass as a 26-parameter material in the specified frequency range – the fixed time constants not included. The identified parameters are listed in Table III.

The real part of Young's modulus is found to increase slowly from about 4.5 GPa to about 5.5 GPa in the measurement range; the real part of the shear modulus increases, too, from about 1.5 GPa to about 2 GPa (Figure 10). This can be compared to values in literature. For instance in [24], the visco-elastic parameter identification for acrylic glass leads to a Young's modulus which is only in the range from about 2.5 GPa to 3.5 GPa. But the values are measured at a test piece temperature of about 105 °C. Due to the high temperature sensibility of acrylic glass, the two results do not conflict. The single-number values in the literature vary from 3 GPa to 6 GPa, which is exactly

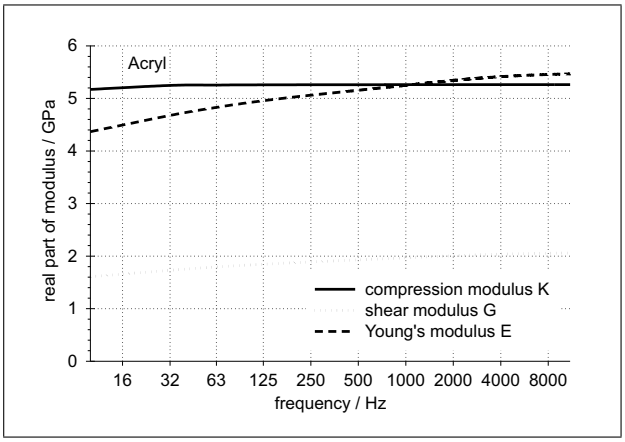

Figure 10. Real part of compression, shear and Young's modulus as calculated from the identified parameters of Table III.

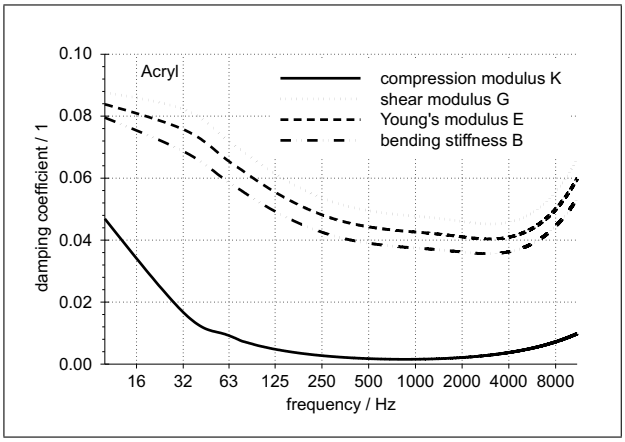

Figure 11. Damping of compression, shear and Young's modulus as calculated with the final parameters of Table III.

Table III. Identified visco-elastic parameters of acrylic glass for frequencies from 3 Hz to 11 kHz. K_{fast} = 741.01 Ns/m^2, G_{fast} = 1600.6 Ns/m^2.

index p	K_p [N/m^2]	G_p [N/m^2]	b_p [1/s]
0	4.06e9	1.43e9	0
1	9.92e-1	5.96e5	1
2	3.01e8	1.58e7	3.16
3	5.37e8	3.12e7	1e1
4	3.58e8	1.12e8	3.16e1
5	3.55e4	1.41e8	1e2
6	9.27e5	1.07e8	3.16e2
7	3.63e6	6.51e7	1e3
8	3.25e5	5.80e7	3.16e3
9	1.43e4	8.25e7	1e4
10	4.66e4	2.64e7	3.16e4
11	1.25e5	1.23e-6	1e5

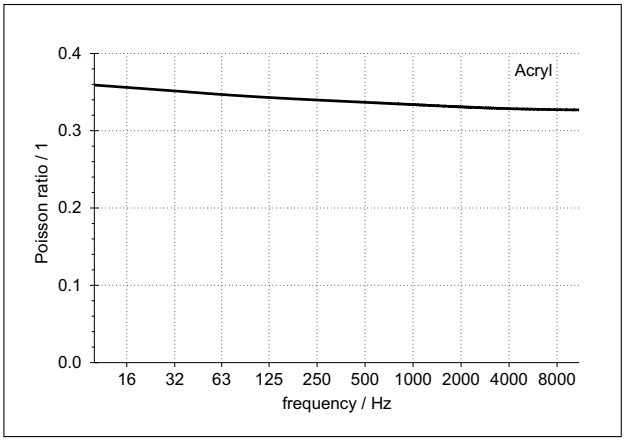

Figure 12. Poisson's ratio of acrylic glass as calculated with the final parameters of Table III.

the range we found in this paper. The tensional damping decreases from about 9% to about 2.5% in the measurement range; the shear damping varies between 4.5% and 7.5% (Figure 11). Since Young's modulus and the shear modulus are known for acrylic glass, complex Poisson's ratio can be calculated according to equation (12). Figure 12 shows the absolute value of the ratio varying between 0.35 and 0.45. The imaginary part of Poisson's ratio is about −0.007, varying only slightly in the measured frequency range.

5.4. Uncertainties

Since the identification of the material parameters is an extensive process, uncertainties have not been finally determined yet. As a first approximation, an error propagation is calculated with the uncertainties for the geometry and the mass of the experimental test piece. For Young's modulus of the silicone and aluminium identification, we achieve an uncertainty value of about 5%. For acrylic glass, the uncertainty value for the shear modulus is about 7% and the value for the compression modulus is about 70%. This large value is due to the fact that mainly shape deformation is performed by the acrylic bar. Since Young's modulus is hardly influenced by the compression modulus, its resulting uncertainty value is about 9%.

These uncertainty values are almost constant in the examined frequency range, with the exception of the range between the resonances of the test pieces where the deformation is small. At these small frequency ranges, the uncertainty can increase significantly, especially for low frequencies. These high values, however, do not affect the said uncertainties of the moduli because the Maxwell elements cover these small frequency regions. In general, the typical uncertainties of the N-parameter-method are comparable to those of the classical methods. This is, not least, due to the fact that the same experimental setups are used.

6. Conclusions

In this paper we described in which way frequency-dependent, dynamic mechanical material properties can be determined by means of the N-parameter-method.

We gave a brief introduction into the theory of visco-elastic material modelling, and joined the definition in the time domain to the – for the acoustician – more familiar

frequency domain. General relations for an isotropic material have been presented and the canonical illustration as an intuitive mechanical network model has been shown. The general approach for determining the parameters of a material was described. Finally, three examples of an experimental identification were given.

We showed that, compared to the classical methods we introduced at the beginning of this paper, the N-parameter-method offers significant advantages.

Whereas classical methods usually permit measurements at a single frequency, the identification of the visco-elastic material parameters permits the calculation of continuous frequency-dependent mechanical properties. The complex moduli can be computed from a fistful of real parameters. Damping and Poisson's ratio follow from the moduli thus calculated.

For the different mechanical quantities, no large data sets are needed. This is ideal for numerical calculations because any mechanical quantity can be computed for any frequency in the area of validity with just a few simple formulas; only the N parameters have to be known, maybe in the form of a material database.

The parameters are real material properties and as such independent of the macroscopic shape. The behaviour of any structure can be calculated when the mechanical properties are known.

In the last example, chapter 5.3, we have shown that gaps in the measurement data have virtually no negative influence on the identification process as long as the gap is not too large. This is due to the fact that when applying the N-parameter-method, the measured data are used for a fitting process by which parameters are found that are valid in a wide frequency range – usually the range of validity is said to be a decade. Thus, a gap in the input data only has a negative effect if it can not be covered by parameters that are placed to the left and to the right of the gap. Using classical methods, a gap in the measurement range means that the sought-for quantity cannot be determined there.

At first sight, the effort required to obtain the visco-elastic parameters is greater, compared to classical methods. If only one single value is needed for approximations, classical methods are, indeed, faster. Moreover, the frequency dependence found for the moduli of silicone is only about 10%, for acrylic glass it is only about 20%, each in the corresponding frequency range. From the acoustician's point of view, this variation is negligible and a single number value can be used. The damping coefficients vary to a greater extent: almost by a factor of 3 for silicone and by about a factor of 2 for acrylic glass. This corresponds to some dB and can no longer be neglected in calculations which are more demanding. The frequency dependence of other materials may exceed this; literature very rarely contains any information on this, especially not in the field of building acoustics. If complex quantities are required and the frequency dependence is not estimated as vanishing, then it is advisable to use the N-parameter-method. For a highly challenging material characterisation, the investigation with classical methods is just as time-consuming as carrying out experiments with the N-parameter-method. Indeed, the carrying out of experiments with the N-parameter-method takes less time because only one proper measurement is necessary. Therefore, it is true that the numerical identification process takes some time, but the computing power is continuously increasing so that in the future, far shorter computation times can be expected.

For the experimental part of the identification, the common basic setups can be used, for instance the tensional bar or the bending bar with free or forced motion. Thereby, with the N-parameter-method, the same materials can be investigated as with classical methods. The experimental setup should be as simple as possible because the most important point in the measurement are proper boundary conditions in order to obtain a high-quality system response. Constellations of test pieces with free ends symmetrically mounted onto the exciter have proven their suitability. Undesired motion, for instance from other vibration types, must be avoided.

For the numerical part of the parameter identification, an adequate physical model of the setup is necessary. All physical effects that influence the motion of the test piece have to be accounted for. For this purpose, the motion should be as simple as possible – feasible, it should be reduced to one uniaxial deformation. Since not all setups which are deemed suitable from a theoretical point of view are suitable for the experiments, a compromise has to be found.

The investigations can be carried out in both the time-domain and the frequency-domain. In acoustics, the moduli are usually wanted in the frequency-domain. This is why one would prefer to do the identification in the frequency-domain, too. But in general, the parameters for the time-domain and the frequency-domain description are the same and can be used for both the memory functions and the complex moduli.

All in all, we can say that the linear visco-elastic N-parameter-modelling is an excellent method for material characterisation. Quite a number of materials can be regarded as behaving visco-elastically, and most of them can be characterised with only a few parameters. The advantages of the method introduced in this paper compared to classical methods balance the numerical effort. The N-parameter-method is a real alternative, especially for highly challenging material characterisation.

Acknowledgement

The authors would like to sincerely thank Werner Scholl, Volker Wittstock and Michael Vorländer for their support, inspiring discussions and the review of this paper.

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