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# Experimental and Theoretical Temperature Gradient Coefficient in Thermoacoustics

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**Summary**

The temperature gradient coefficient of a cotton wool stack is investigated at a nonzero temperature gradient. The direct measurement method of the temperature gradient coefficient is proposed and its experimental results are compared with theoretical calculations. Parallel, capillary-tube-based thermoacoustic theory for random porous media is applied for theoretical calculations. Comparison results show reasonable agreement in the temperature gradient coefficient, which is useful to determine the performance of thermoacoustic devices. These results lead to convincing the validation scheme of both the measurement methods and theoretical calculations.

## 1. Introduction

Symko *et al.* [1] have built small high frequency thermoacoustic refrigerators and prime movers whose stacks use fibrous materials such as steel, glass, or cotton wools. Parallel, capillary-tube-based extension of thermoacoustic theory for random porous media was theoretically developed and their potential applications were discussed [2]. Characterizations of fibrous materials have thus been in great demand in the context of the effective development of such thermoacoustic devices using fibrous materials.

The temperature gradient coefficient is considered for the complete characterization of stacks or regenerators for thermoacoustic devices with the temperature gradient, in addition to the zero temperature acoustic properties [3, 4, 2]. Its characteristics are closely related to the performance of thermoacoustic devices [5, 2]. It is a quantity determined as a function of frequency or wave number in the presence of a temperature gradient. The direct measurement of the temperature gradient coefficient has not been explored yet even for a regular geometry stack since the thermoacoustics theory [6, 7] developed. In this paper, the temperature gradient coefficient of cotton wool is experimentally and theoretically studied at a nonzero temperature gradient. At a finite temperature gradient, a measurement method for the temperature gradient coefficient is introduced. Parallel, capillary-tube-based thermoacoustic theory for random porous media [2] is applied for theoretical calculations in comparison with experimental data.

## 2. Parallel capillary-tube-based thermoacoustic theory for random porous media

Parallel, capillary-tube-based thermoacoustic theory for random porous media [2] is briefly described.

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In thermoacoustic theory [6, 7], the viscous wave number $\lambda_\mu$ is defined by

$$\lambda_\mu = R\sqrt{\frac{\rho_0\omega}{\mu}} = \frac{\sqrt{2}R}{\delta_\mu}, \tag{1}$$

and the thermal wave number $\lambda_T$ by

$$\lambda_T = R\sqrt{\frac{\rho_0\omega c_p}{\kappa}} = \frac{\sqrt{2}R}{\delta_T}. \tag{2}$$

$R$ is the tube radius, $\rho_0$ the gas density, $\omega$ the angular frequency of the sound, $\mu$ the gas viscosity, $c_p$ the heat capacity of the gas and $\kappa$ the thermal conductivity of the gas. $\delta_\mu$ is the viscous penetration depth of the gas and $\delta_T$ is the thermal penetration depth.

In the bulk porous medium extension from a single pore theory, the scaling parameters such as the viscous dynamic shape factor $n_\mu$, thermal dynamics shape factor $n_T$, and tortuosity factor $q$ are introduced. The dynamic shape factors are used to account for pore shapes, the effect of changes in pore cross sections along their lengths, and uncertainty in how the characteristic radius should be determined for random porous media. The tortuosity accounts for two effects. The pores themselves are longer so that viscous effects are larger in proportion to the length and the pressure gradient is also reduced by a factor proportional to the actual length.

The viscous and thermal dynamic shape factors, $n_\mu$ and $n_T$, are included in the dimensionless scaling parameters

$$\lambda_{\mu n} = \lambda_\mu / n_\mu, \tag{3}$$

and

$$\lambda_{Tn} = \lambda_T / n_T. \tag{4}$$

The tortuosity factor $q$ is used so that the particle velocity within a pore $u^*$ becomes $qu^*$ in the bulk porous medium.

$$u^* \to qu^*, \tag{5}$$

where $*$ reflects variable fluctuation.

The viscous function is expressed in cylindrical coordinates by

$$F(\lambda_{\mu n}) = 1 - \frac{2}{\lambda_{\mu n}\sqrt{\mathrm{i}}}\left[\frac{J_1(\sqrt{\mathrm{i}}\lambda_{\mu n})}{J_0(\sqrt{\mathrm{i}}\lambda_{\mu n})}\right], \tag{6}$$

and the thermal function by

$$F(\lambda_{Tn}) = 1 - \frac{2}{\lambda_{Tn}\sqrt{\mathrm{i}}}\left[\frac{J_1(\sqrt{\mathrm{i}}\lambda_{Tn})}{J_0(\sqrt{\mathrm{i}}\lambda_{Tn})}\right]. \tag{7}$$

The thermoacoustic wave equation for a random porous medium results in

$$\frac{\rho_0}{F(\lambda_{\mu n})}\frac{\mathrm{d}}{\mathrm{d}z}\left(\frac{F(\lambda_{\mu n})}{\rho_0}\frac{\mathrm{d}p(z)}{\mathrm{d}z}\right) + 2\alpha_b(\lambda_{\mu n}, \lambda_{Tn})\frac{\mathrm{d}p(z)}{\mathrm{d}z} + k_b^2(\lambda_{\mu n}, \lambda_{Tn})p(z) = 0, \tag{8}$$

where $z$ is the wave propagation direction. The propagation constant squared $k_b^2$ is given by

$$k_b^2(\lambda_{\mu n}, \lambda_{Tn}) = \frac{q^2\omega^2}{c^2}\frac{1}{F(\lambda_{\mu n})}\left[1 + \left(\gamma - 1\right)\left(1 - F(\lambda_{Tn})\right)\right], \tag{9}$$

where $c$ is the sound velocity. The temperature gradient coefficient $\alpha_b$ is expressed by

$$\alpha_b(\lambda_{\mu n}, \lambda_{Tn}) = \frac{\beta}{2}\frac{\mathrm{d}T_0}{\mathrm{d}z}\frac{F(\lambda_{Tn})/F(\lambda_{\mu n}) - 1}{(1 - N'_{Pr})}, \tag{10}$$



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where $\beta$ is the thermal expansion coefficient. The modified Prandtl number $N'_{Pr} = \lambda^2_{Tn}/\lambda^2_{\mu n} = N_{Pr} n^2_\mu / n^2_T$ is introduced in $\alpha_b$.

The characteristic impedance of the bulk medium leads, under the assumption of the short stack, to

$$Z_c = \frac{\omega\rho(\omega)}{k_b(\omega)} = \frac{q^2}{\Omega}\frac{\rho_0}{F(\lambda_{\mu n})k_b(\omega)}. \tag{11}$$

The normalized characteristic impedance is defined by $Z_{cn} = Z_c/\rho_0 c$.

## 3. Direct measurement method of temperature gradient coefficient

The temperature gradient coefficient expressed by (10) is an acoustic parameter used to determine the acoustic gain for a given temperature gradient in a stack. The thermoacoustic conversion of heat to acoustic work or acoustic work to heat transport in a standing wave device is principally determined by the magnitude of the imaginary part of the temperature gradient coefficient [5]. The temperature gradient coefficient is an acoustic property characterized at a nonzero temperature gradient while the viscous function (6), the thermal function (7), and the propagation constant (9) are acoustic properties characterized at a zero temperature gradient.

Two step measurements are conducted to determine the temperature gradient coefficient. Before carrying out the direct measurements of the temperature gradient coefficient, measurements of two microphone impedance tube at a zero temperature gradient [8] are conducted to determine acoustic properties characterized at a zero temperature gradient. Experimental instrumentations at a nonzero temperature gradient are similar to those at a zero temperature gradient except applying a temperature gradient between the hot and cold heat exchangers.

The direct measurement method of the temperature gradient coefficient is introduced under the assumption of the short stack approximation compared with the derived wave length.

### 3.1. Impedance translation through a stack with zero and nonzero temperature gradients

The specific acoustic impedance is defined by $Z(z) = p/u$. Two microphone impedance tube method is applied to obtain data of specific acoustic impedance on cotton wool. The impedance tube is consisted of the compliance chamber and the stack part. The experimental apparatus is shown in Figure 1. The right hand side of the stack section in Figure 1 is open to air so that the resonator is a quarter-wavelength resonator.

At a zero temperature gradient, the specific acoustic impedance at the left hand side of the stack in Figure 1 is expressed by

$$Z_l = \frac{p_l}{u_l} = \mathrm{i}\rho_0 c \frac{H_{21}\sin(kz_1) - \sin(kz_2)}{\cos(kz_2) - H_{21}\cos(kz_1)}, \tag{12}$$

where $H_{21}$ is the transfer function between two microphone positions, $z_1$ and $z_2$, with respect to the left hand side of stack $z = 0$, and $k$ is the propagation constant for a gas inside a rigid tube [8].

At a nonzero temperature gradient, the specific acoustic impedance $Z_l$ on the left hand side of the stack is expressed in terms of the known impedance of the right hand side $Z_r$, according to Raspet *et al.* [9] It leads to

$$Z_l = \frac{p_l}{u_l} = \frac{C}{k\Omega}\frac{Z_r\left(1 - \frac{Ds}{2} - \frac{k^2s^2}{4}\right) - \mathrm{i}\frac{C}{k\Omega}ks}{-\mathrm{i}Z_r ks + \frac{C}{k\Omega}\left(1 - \frac{k^2s^2}{4}\right) + \frac{C}{k\Omega}\frac{D}{2}s}, \tag{13}$$

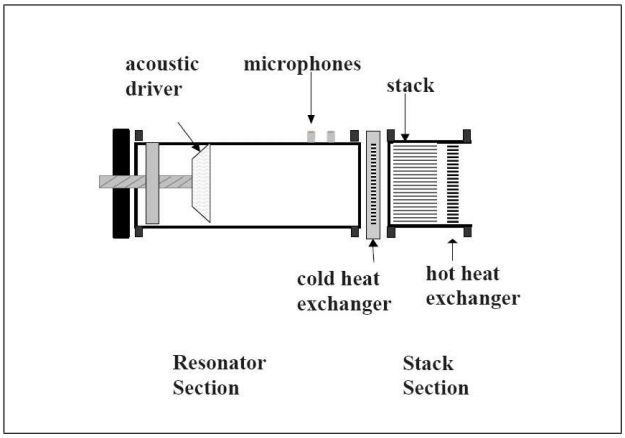



Figure 1. Schematic diagram of two microphone impedance tube method.

where $\Omega$ is the porosity of the stack and $s$ the stack length. The parameters $C$ and $D$ are given by

$$C = \frac{\omega\rho_0}{F(\lambda_{\mu n})}, \tag{14}$$

and

$$D = \frac{1}{T_0}\frac{\mathrm{d}T_0}{\mathrm{d}z}\frac{F(\lambda_{Tn}/F(\lambda_{\mu n}) - 1}{1 - N'_{Pr}}, \tag{15}$$

where $T_0$ is the mean temperature of working gas.

The development of (13) assumes that the stack is short with respect to wavelength. It also assumes that the values of fluid properties $c_p, \rho, \mu$, and $K$ at the center of the stack represent the values across the stack and their corrections for variation are higher order neglected. This assumption is typically satisfied for most thermoacoustic applications. The Euler finite difference approximation can be used to overcome the breakdown of the short stack assumption. The stack is divided into smaller segments and each segment is then treated by the translation theorem, under the assumption that all temperature dependent values are evaluated at the mean temperature of the segment.

### 3.2. Measurement method of the temperature gradient coefficient

In the case of a vanishingly short stack [9], if Rott's boundary condition [6] is applied, (13) leads to

$$\frac{p_l}{u_l} = \frac{p_r}{u_r}\mathrm{e}^{-Bs}, \tag{16}$$

where $Z_r = p_r/u_r$ is the specific acoustic impedance at the right hand face of the stack. The coefficient $B$ is given by

$$B = \frac{1}{s}\alpha\frac{\triangle T}{T_m}, \tag{17}$$

where $\alpha$ is the normalized temperature gradient coefficient and $T_m$ is the mean temperature of working gas at the mid point of the stack. To obtain the temperature gradient coefficient $\alpha_b \simeq 2\triangle T/s\alpha = 2T_m B$, two measurements of the specific acoustic impedance $Z_l = p_l/u_l$, $Z^1_l$ and $Z^2_l$, are performed at two different temperature differences, $\triangle T_1$ and $\triangle T_2$, when the mean temperatures of working gas at the mid point of the stack are the same. From (16) and (17), the normalized temperature gradient coefficient is derived by

$$\alpha = -\left[\ln\frac{Z^1_l}{Z^2_l}\right]\frac{T_m}{\triangle T_1 - \triangle T_2}, \tag{18}$$

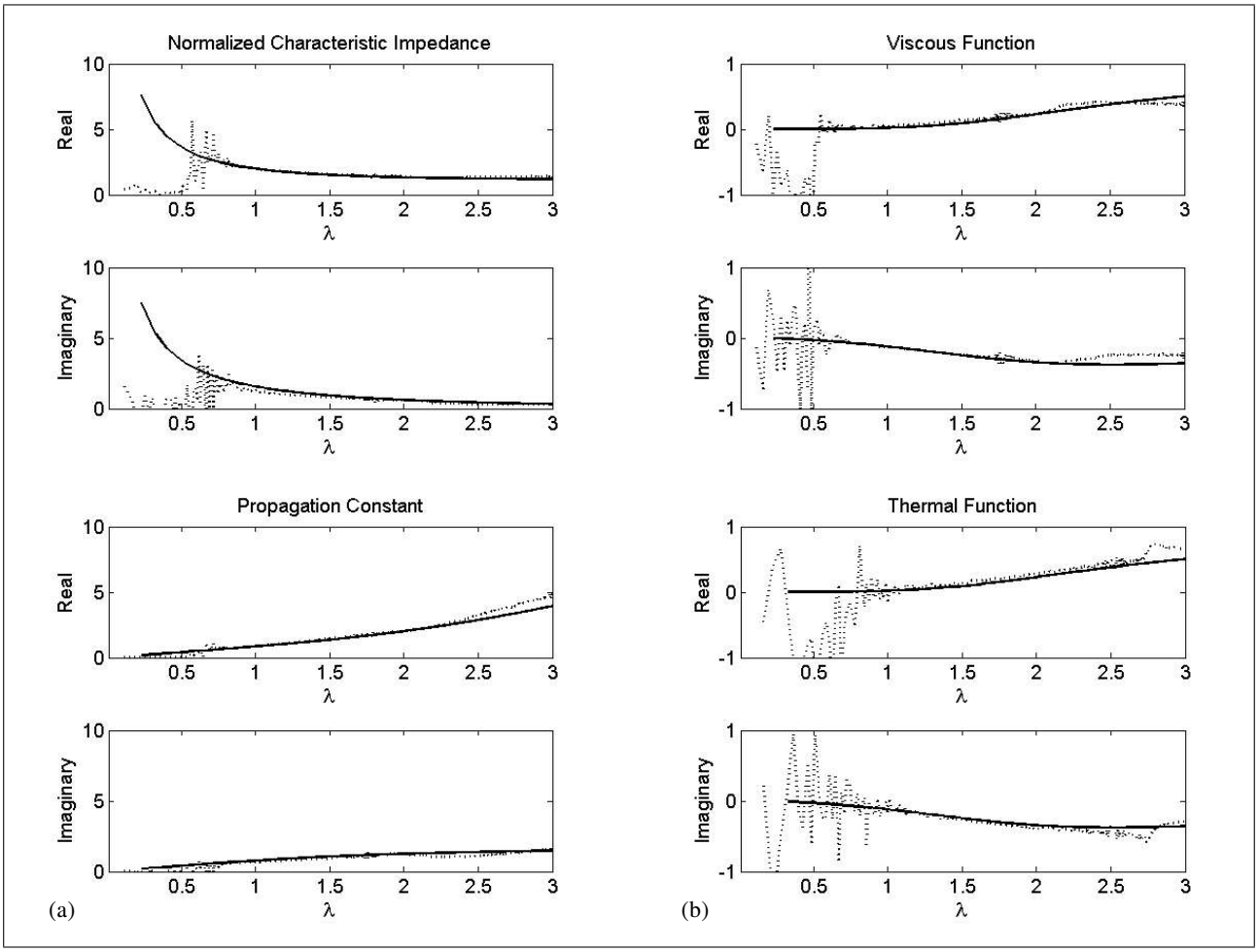



Figure 2. Comparisons of the calculated (solid line) and experimentally (dotted line) derived acoustic properties for cotton wool stack: a) Normalized characteristic impedance and propagation constant. b) Viscous function and thermal function. The thermal function is shown as a function of $\lambda_T$ and the other properties are shown as a function of $\lambda_\mu$.

and the temperature gradient coefficient is given by

$$\alpha_b = 2T_m B = -2\left[\ln \frac{Z_l^1}{Z_l^2}\right]\frac{1}{s}. \tag{19}$$

In these measurements, it is assumed that the values of $Z_r = p_r/v_r$ for the different temperatures are the same. The values of $Z_l^1$ and $Z_l^2$ are obtained by using (13) and $Z_r$ at two different temperatures.

### 3.3. Instrumentation of two microphone impedance tube method

Measurements of two microphone impedance tube [8] are performed by the technique of using swept-sine mode. The cold and hot heat exchangers are inserted at the left and right hand sides of stack, respectively.

Two microphones of Endevo Model 8510-1 piezo-resistive transducers in an impedance tube are used for measuring acoustic disturbances. The acoustic impedance tube is comprised of two sections of stainless steel pipe.

The resonator and stack sections have lengths of 1.2 m, and 0.15 m, respectively. The nominal inside diameter of the pipe is 10.2 cm and the nominal outside diameter is 11.4 cm. The microphone separation of 229 mm is chosen. The left end of the resonator section in Figure 1 is closed and the right hand side of the stack section is open. Air is used as working gas. It has a pressure of 14.696 psi, the ratio of specific heat is 1.4, and the speed of sound is 340 m/s at 15 °C.

A Stanford Research Systems Model SR785 Dynamic Signal Analyzer is used as both the source and receiver of acoustic signals. The source is amplified by a Techron 5530 Power Supply Amplifier. The sine-sweep mode is fed into the speaker assembly which consists of an Audax model AP100G0 102 mm woofer mounted inside an aluminum collar.

The hot heat exchanger is made of a nichrome ribbon arrangement. The cold heat exchanger is a copper fin-tube arrangement which is assembled by two tubes circulated by water with a Cole Parmer constant temperature circulator. A Xantrex XFR 60-20 DC power supply is used to power the hot heat exchanger. Temperature differences of 0 to 195 °$C$ are applied between the cold and hot heat exchangers.

## 4. Temperature gradient coefficient of cotton wool

Parallel, capillary-tube-based extension of thermoacoustic theory for a fibrous material [2] is subject to test its validity at the nonzero temperature gradient as well as thermoacoustic theory for a regular pore stack is. The direct verification at a finite temperature is required even though the normalized temperature gradient coefficient indirectly determined at the zero temperature gradient has showed reasonable validity.

Cotton wool is chosen as a fibrous material stack for this purpose. To determine the input parameters for theoretical calculations, measurable quantities are measured at both zero and nonzero temperature gradients.

Estimation methods for the characteristic radius, tortuosity, and dynamic shape factors of fibrous materials with open cells have not been established yet. To resolve the problem, probable values of the tortuosity and dynamic shape factors are hinted by referring to the values of random porous materials such as reticulated vitreous carbon and aluminum foam in the reference [2]. Probable characteristic radius is suggested by considering the density, porosity, compression level, etc. of fibrous materials. These estimated values are tested by theoretical and experimental comparisons of the characteristic impedance and propagation constant or the viscous and thermal functions.

In measurements of a zero temperature gradient, two microphone impedance tube method [8] shown in Figure 1 is used to measure the specific acoustic impedance (12). The specific acoustic impedance is converted to the characteristic impedance and propagation constant or the viscous and thermal functions [8, 3, 4]. The detailed experimental procedures will be reported elsewhere.

At a zero temperature gradient, experimentally and theoretically derived acoustic properties for the cotton wool stack are compared as a function of wave number in Figure 2. The normalized characteristic impedance and propagation constant are shown in the figure and their corresponding viscous function and thermal function are also shown. Specifications of the cotton wool stack are represented by the porosity 0.95 and characteristic radius 0.09 mm. The sample length is 120 mm. Theoretical calculations use thermoacoustic parameters of the viscous dynamic shape factor 0.25, thermal dynamic shape factor 0.15, and tortuosity factor 1.02. For air gas, the modified Prandtl number is 1.12.

The input parameters determined by measurements at the zero temperature gradient as described above are used for theoretical calculations of the temperature gradient coefficient (10) with the help of the information of the temperature gradient. The direct measurement results of the temperature gradient coefficient (19) are derived by using the data of specific acoustic impedance (13) obtained in terms of two microphone impedance tube method of Figure 1 in the presence of the temperature gradient. The experimental results of the temperature gradient coefficient are then compared with its theoretical calculations.

The directly measured temperature gradient coefficients for the cotton wool stack are compared as a function of frequency and wave number with theoretical calculations in Figure 3. The viscous wave number (1), instead of (3), is used in the horizonal axis. To obtain the temperature gradient coefficient from (18), two data of specific acoustic impedance at different temperature differences with the same mean temperature of working gas are used.

The measured results of the temperature gradient coefficient represented in Figure 3 show reasonable agreement with the theoretical calculations. The comparison results show better agreement at the bigger wave number or frequency than at the smaller one. These illustrate the validity of parallel, capillary tube-based thermoacoustic theory is verified through the measurement method of the temperature gradient coefficient in the presence of the temperature gradient.

To obtain more reliable data of the temperature gradient coefficient around a lower frequency region for a fixed sample, three factors are important. They are the low frequency response of a speaker, two microphone spacing, and the temperature gradient.

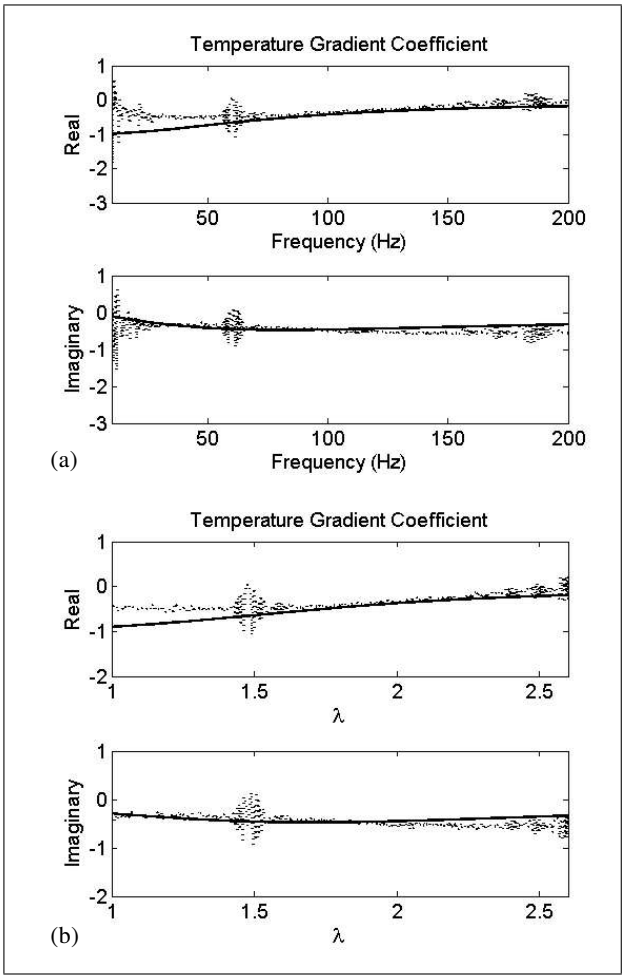



Figure 3. Experimental (dotted line) and theoretical (solid line) temperature gradient coefficient for cotton wool stack in the presence of finite temperature gradient. a) As a function of frequency. b) As a function of wave number $\lambda_\mu$.

Better low frequency response of a speaker and high temperature gradient give more rigorous data. The frequency range also changes by adjusting the microphone spacing. However, inappropriate microphone spacing makes more fluctuations in the data around the average value. For a fixed experimental apparatus, a sample with a smaller characteristic radius can provide better results at the lower frequency region. In addition to these factors, systematic estimation methods to determine the characteristic radius, tortuosity, and dynamic shape factors of fibrous materials with open cells are required to be established.

Through these comparisons, it is verified that parallel, capillary-tube-based thermoacoustic theory is useful in predicting acoustic properties of fibrous materials including the temperature gradient coefficient. The advantages of a fibrous material stack in developing thermoacoustic devices may be demonstrated through the applications of these theoretical and experimental investigations.

## 5. Conclusions

The temperature gradient coefficient, which is a unique property in thermoacoustics, of a fibrous material stack such as cotton wool was theoretically and experimentally studied because of their increasing demand in thermoacoustic devices. The direct measurement method of the temperature gradient coefficient

is first introduced. Its measurement results are obtained in the presence of the temperature gradient using a two microphone impedance tube method under the assumption of the short stack approximation compared with the derived wave length. Parallel, capillary-tube-based extension of thermoacoustic theory for random porous media is applied for theoretical calculations of the acoustic properties. Comparison results show reasonable agreement in the temperature gradient coefficient at a nonzero temperature gradient. These results thus lead to convincing the validation scheme of both the measurement methods and theoretical calculations. The advantages of fibrous material stack in developing thermoacoustic devices may be further studied based on these theoretical and experimental investigations since the temperature gradient coefficient plays an important role in the performance of thermoacoustic devices.

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