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On the Theory of Sound Transfer in Rigid-Frame Porous Materials

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Summary
The modeling of the macroscopic propagation of sound inside porous media is usually based on first order acoustical quantities. Single or multiple layers are treated as an equivalent impedance network which can be also solved with a transfer matrix formulation. In this work the one dimensional, time averaged, macroscopic sound transfer process in rigid-frame porous media is reinterpreted with the concepts developed in the field of sound intensity. An algebraic derivation shows that the active intensity is proportional to the spatial gradient of the sound energy density, so that the two quantities obey a diffusion equation. The name *sound conductivity* is suggested for the related coefficient, and the expressions of the sinks of sound energy and of the sources of reactivity in the material are formalized. Moreover a differential formulation for the energy densities inside the material is developed and solved analytically.

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1. Introduction

The acoustic properties of rigid-frame fluid-saturated porous materials are often modelled by means of an homogeneous equivalent fluid with complex, frequency dependent, characteristic impedance Z_C and complex propagation constant γ. Once these two quantities are obtained by either theoretical or empirical models, the porous layer is fully described and its behaviour under different boundary conditions both in simple or multilayered configurations can be predicted. Thus several relevant quantities can be calculated, such as the surface impedance, the absorption coefficient and the transmission coefficient. This procedure is based on the manipulation of first order acoustical quantities, namely pressure and particle velocity, which are written for the wave propagating inside the material [1]. A quite useful algebraic approach to the solution of this physical problem makes use of the so called "transfer matrix" method [2]. Using first order quantities does not give direct information on the effective flow of sound energy through the material and especially on the eventual reaction of the material to the energy flow itself. This reaction can be actually expected since the complex nature of the characteristic impedance Z_C will introduce a further phase shift between pressure and particle velocity, which has to be combined with the phase displacement due to the sound field without the material. This extra phase shift will affect significantly the energy flux, which is crucially dependent of the phase relationship between pressure and particle velocity. In the present work a different macroscopic approach is proposed: the time averaged propagation of sound inside the porous material will be treated as a pure flow of sound energy across it by means of the second order acoustical quantities, that is the active and reactive intensities and the sound energy densities. In particular the active intensity is responsible for the net transfer of energy, whereas the reactive component has had for many years a more debated and less clear nature [3]. Firstly, one could say that there is a general agreement on the physical significance of the reactive intensity, whose instantaneous value at a point describes the energy flowing back and forth thus procuring no net energy transfer [4]. Also the frequency dependent definition of the reactive intensity for harmonic sound fields was proposed as the imaginary part of the so-called complex intensity [5]. But the extension of these concepts to the more general time-dependent case gave rise to somewhat contrasting proposals in the past [6, 7]. In fact, although the different definitions of the reactive intensity output the same amplitude for the quantity in the case of a monochromatic sound field, the extension to a general sound field requires great care. As experimentally detailed in [8], when the time-dependent narrowband expression of the reactive intensity is employed and frequency bands are used, one can get results incongruous with the above physical picture of the localized oscillatory flux. In particular, if there are tonal components of the averaged vector reactive intensity adding with opposite sign in the same band (i.e. coming from different sound sources), an almost vanishing absolute value for the quantity is obtained in the same band. This is in contrast

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with the physical evidence of energy being localized in the space between the sources. Based on this criticism the time-domain definition and also the vectorial nature of the averaged reactive intensity were questioned, and a different general time-dependent definition was introduced that made use of a tensor for the representation of the localized flux phenomenon. To avoid confusion, the name of the quantity was revised from reactive intensity to "oscillating intensity". In this somehow complicated framework the conventional frequency dependent definition of the reactive intensity for harmonic sound fields can still be useful under certain assumptions. In fact to comply with the picture elaborated for general fields a physical significance is to be attributed to the amplitude of the quantity only, and its sign, though not arbitrary, has to be regarded primarily as the result of a convention. This approach will be followed in the present work, which will deal with harmonic sound fields only. In particular here the vectorial nature of the time-averaged reactive intensity will be formally useful in some part of the algebraic derivations, but no fundamental hypothesis will be based on it. After this necessary digression the first step of the work will be to work out the expressions of all the second order quantities in the case of the sound field inside a porous material according to the equivalent fluid approach. Then the second order quantities describing the sound transfer process will be analytically demonstrated to obey to a diffusion law. A consistent picture of the propagation process will be elaborated accordingly, and the relationships between the quantities involved will be derived. Furthermore a differential equation for the sound energy densities will be developed and solved.

2. Expression of the energetic quantities

Let us firstly consider a layer of rigid frame porous material of thickness l (see Figure 1) under the equivalent fluid assumptions. Its acoustical behaviour is governed by the equivalent fluid density $\tilde{\rho}$ and by the compressibility $\tilde{\chi}$. Referring to the definitions of the quantities dynamic tortuosity $\tilde{\alpha}$ [9] and dynamic compressibility $\tilde{\beta}$ [10], the equivalent fluid density can be expressed as $\tilde{\rho} = \tilde{\alpha}\,\rho_0/\phi$ with ρ_0 ambient density and ϕ porosity, and the compressibility is written as $\tilde{\chi} = \chi_0\phi\tilde{\beta}$ where χ_0 is the adiabatic compressibility. Then the characteristic impedance Z_C and the propagation constant γ are derived with the following equations:

$$\begin{aligned} Z_C &= \left|Z_C\right| \mathrm{e}^{\mathrm{i}\varphi} = \sqrt{\tilde{\rho}/\tilde{\chi}}, \\ \gamma &= \mathrm{i}\omega\sqrt{\tilde{\rho}\tilde{\chi}} = \alpha + \mathrm{i}k, \end{aligned} \tag{1}$$

where ω is the angular frequency. Restricting the analysis to a one dimensional propagation along the positive and negative x-axes, the first order quantities pressure p and particle velocity v may be written, in the case of an harmonic sound field as:

$$\begin{aligned} P(x) &= N\mathrm{e}^{(\alpha+\mathrm{i}k)X} + M\mathrm{e}^{-(\alpha+\mathrm{i}k)X}, \\ V(x) &= \frac{N}{|Z_C|}\mathrm{e}^{(\alpha+\mathrm{i}k)X}\mathrm{e}^{-\mathrm{i}\varphi} - \frac{M}{|Z_C|}\mathrm{e}^{-(\alpha+\mathrm{i}k)X}\mathrm{e}^{-\mathrm{i}\varphi}, \end{aligned} \tag{2}$$

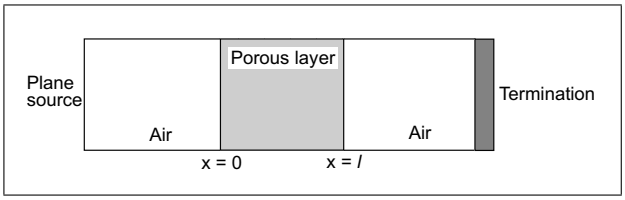

Figure 1. A simple sketch of the modelled tube with the inserted porous layer.

where $X = l - x$ is the distance inside the layer, N and M are the complex wave amplitudes and the time factor $\mathrm{e}^{\mathrm{i}\omega t}$ was dropped for convenience. As explained in [1] the two quantities in (2) (which one may call *equivalent pressure* and *equivalent velocity*), are continuous at the boundary between the porous layer and the air outside. The continuity of pressure and velocity implies that also sound intensity, which is their product, must be continuous at the same boundary. The derivation of suitable energy densities inside the porous layer is not as straightforward. This result, which is based on the results in [11] and on the algebra of phasors, is due to Lafarge [12] and its derivation is sketched in Appendix A1. Thus the working definitions of the time averaged macroscopic energy densities are

$$\begin{aligned} E_P &= \frac{1}{2}\chi_0\phi\Re e\{\tilde{\beta}\}\frac{|P(x)|^2}{2}, \\ E_k &= \frac{1}{2}\frac{\rho_0}{\phi}\Re e\{\tilde{\alpha}\}\frac{|V(x)|^2}{2}, \end{aligned} \tag{3}$$

where, in order to simplify the notation, one can re-express the coefficients as

$$\begin{aligned} \chi_r &= \Re e\{\tilde{\chi}\} = \Re e\{\chi_0\phi\tilde{\beta}\} = \chi_0\phi\Re e\{\tilde{\beta}\}, \\ \rho_r &= \Re e\{\tilde{\rho}\} = \Re e\{\frac{\rho_0}{\phi}\tilde{\alpha}\} = \frac{\rho_0}{\phi}\Re e\{\tilde{\alpha}\}. \end{aligned}$$

After some algebraic manipulation one obtains

$$\begin{aligned} E_P &= \frac{1}{4}\frac{\rho_r}{Z_r^2}\big(D_1 + D_2\big), \\ E_k &= \frac{1}{4}\frac{\rho_r}{|Z_c|^2}\big(D_1 - D_2\big), \end{aligned} \tag{4}$$

where $Z_r = \sqrt{\rho_r/\chi_r}$ and, for convenience, the following auxiliary quantities have been introduced:

$$\begin{aligned} D_1 &= |N|^2\mathrm{e}^{2\alpha X} + |M|^2\mathrm{e}^{-2\alpha X}, \\ D_2 &= 2\Re e\big\{N M^*\mathrm{e}^{2\mathrm{i}kX}\big\}. \end{aligned} \tag{5}$$

The superscript asterisk stands for the complex conjugation while $\Re e$ is the real part of the argument (the imaginary part will be indicated as $\Im m$).

It is to be noted that the definition in (3) imply a discontinuity in the energy density when passing from outside the material to the inside. In fact, if a point outside the layer is considered, the coefficients to be inserted in (3) become respectively those of air, that is $\chi_r \to \chi_0$ and $\rho_r \to \rho_0$. So, due to the continuity of pressure and velocity, it follows the discontinuity of the potential and kinetic energy densities at the boundaries.

After the energy densities, the definitions of the time averaged energy fluxes must be considered too. The working quantity is the so-called "complex intensity" C, from which the active I and reactive J intensities are derived respectively as the real and the imaginary part [5], that is

$$C = I + \mathrm{i}J, \quad I = {}^1\!/_2 \Re e\{PV^*\}, \quad J = {}^1\!/_2 \Im m\{PV^*\}. \quad (6)$$

By algebraic manipulation also these quantities can be readily expressed in terms of auxiliary functions as

$$\begin{aligned} I &= H_1 \cos\varphi + H_2 \sin\varphi, \\ J &= H_1 \sin\varphi - H_2 \cos\varphi, \end{aligned} \quad (7)$$

with H_1 and H_2 being equal to

$$\begin{aligned} H_1 &= \frac{|N|^2}{2|Z_C|} \mathrm{e}^{2\alpha X} - \frac{|M|^2}{2|Z_C|} \mathrm{e}^{-2\alpha X}, \\ H_2 &= \Im m\Big\{ \frac{NM^*}{|Z_C|} \mathrm{e}^{2\mathrm{i}kX} \Big\}. \end{aligned} \quad (8)$$

The definitions introduced and discussed above - equations (3) and (6) - can be employed to investigate the behaviour of the second order energetic quantities inside a porous layer.

3. The sound transfer

In order to derive the general solution of the problem it is firstly necessary to compute the gradients of the potential and kinetic energy densities, which have to be regarded as the main indicators of the spatial variation of the quantities. By differentiating (4) one can write

$$\begin{aligned} \nabla E_P &= \frac{1}{4} \frac{\rho_r}{Z_r^2} \big(\nabla D_1 + \nabla D_2 \big), \\ \nabla E_k &= \frac{1}{4} \frac{\rho_r}{|Z_c|^2} \big(\nabla D_1 - \nabla D_2 \big), \\ \nabla E &= \nabla E_P + \nabla E_k. \end{aligned} \quad (9)$$

The problem becomes now the calculation of the gradient of the auxiliary quantities D_1 and D_2, which is easily accomplished. Furthermore, by comparison with (8), one can write the following chain of equalities:

$$\begin{aligned} \nabla D_1 &= 2\alpha \big(|M|^2 \mathrm{e}^{-2\alpha X} - |N|^2 \mathrm{e}^{2\alpha X} \big) = -4\alpha |Z_c| H_1, \\ \nabla D_2 &= 2k \cdot 2\Im m\big\{ NM^* \mathrm{e}^{2\mathrm{i}kX} \big\} = 4k |Z_c| H_2, \end{aligned}$$

and the gradient of the sound energy is expressed accordingly as

$$\begin{aligned} \nabla E = &-\alpha |Z_c| \Big(\frac{\rho_r}{Z_r} + \frac{\rho_r}{|Z_c|^2} \Big) H_1 \\ &+ k |Z_c| \Big(\frac{\rho_r}{Z_r} - \frac{\rho_r}{|Z_c|^2} \Big) H_2. \end{aligned} \quad (10)$$

At this stage the system (7) has to be inverted in order to express the auxiliary quantities H_1 and H_2 as functions of I and J as

$$H_1 = I \cos\varphi + J \sin\varphi, \quad H_2 = I \sin\varphi - J \cos\varphi.$$

Then H_1 and H_2 are substituted in (10) and after some algebraic processing an equation relating the gradient of the energy density and the two energy fluxes is obtained which reads

$$\nabla E = -\chi_I I - \chi_J J = -\Re e\big\{ \chi^* C \big\}. \quad (11)$$

The complex quantity $\chi = \chi_i + \mathrm{i}\chi_J$ is here introduced and named complex *acoustical resistivity*. The real and imaginary parts of χ, which have physical dimensions [s/m^2], depend only on the material characteristics. They can be easily computed by comparing (10) and (11) and are equal to

$$\begin{aligned} \chi_I &= \frac{\rho_r}{Z_r^2} \Re e\big\{ \gamma Z_C \big\} + \frac{\rho_r}{|Z_c|^2} \Re e\big\{ \gamma^* Z_C \big\} \\ &= -\omega \Im m\big\{ \tilde{\rho} \tilde{\chi} \big\} = \frac{2\alpha k}{\omega}, \\ \chi_J &= \frac{\rho_r}{Z_r^2} \Im m\big\{ \gamma Z_C \big\} + \frac{\rho_r}{|Z_c|^2} \Im m\big\{ \gamma^* Z_C \big\} = 0. \end{aligned} \quad (12)$$

The result can be better understood and made operative if an alternative form of (11) written as

$$I = -D_T \nabla E, \quad (13)$$

where the inverse of the non vanishing real part of the acoustical resistivity coefficient is termed *sound transfer coefficient* or *sound conductivity*

$$D_T = \frac{1}{\chi_I} \quad [\mathrm{m}^2/\mathrm{s}].$$

After a laborious derivation, the result (13) deserves some discussion. Firstly it is demonstrated that the time averaged energetic quantities describing the propagation of sound inside the porous layer obey, on the macroscopic scale, a diffusion equation. The spatial gradient of the sound energy drives the whole process and the minus in front of it in (13) stresses that a flux is generated whenever in the material one passes from a region of higher to one of lower energy density. Moreover it is possible to associate the gradient of E with the net active part only. The local oscillations of energy occurring in time, whose amount is described by the averaged amplitude of J, do not enter in the balance defining the net transfer of energy. Physically speaking D_T accounts for the attitude of the material to host or not the sound transfer or, in other words, to be permeated by sound. The ideal substance in this respect is of course air itself since, if energy losses are neglected, air shows a vanishing resistivity and an infinite conductivity (see also Appendix A2).

In order to exemplify the behaviour of the coefficient D_T the theoretical case of a layer consisting in a stack of identical cylinders perpendicular to the surface can be easily simulated according to the Rayleigh model [1, eq. 4.105, 4.106, 4.115]. The physical parameters of this model for the stack of parallel cylindrical pores are the flow resistivity σ and the open porosity ϕ since the system has unit tortuosity. Actually, in a physical porous sample, the two parameters are inter-dependant and they depend

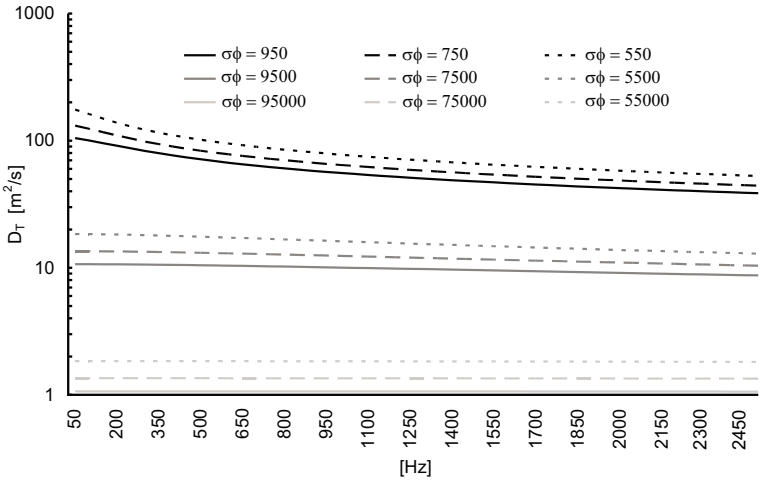

Figure 2. Calculated values of the sound conductivity D_T as function of the flow resistivity and open porosity in a stack of cylinders [1, eq. 4.105, 4.106, 4.115]. The parametric values of the curves are the combination ofσ = 10^3, 10^4, 10^5 Pa s/m^2 and ϕ = 0.55, 0.75 , 0.95.

on the microstructural arrangement. On the other hand, in the model of parallel cylinders once ϕ is fixed, several σ can be obtained depending on the diameter of the cylinders employed (the greater the diameter the lower the resistivity). With these concepts in mind the model can be used to accomplish a simple parametric exercise by selecting ϕ = 0.55, 0.75, 0.95 and three increasing flow resistivity values $\sigma = 10^3, 10^4, 10^5$. Nine layers are thus simulated by means of $\tilde{\rho}$ and $\tilde{\chi}$ obtained for the related equivalent fluid equations. The investigated frequency range is in this case from 50 Hz to 2500 Hz. Figure 2 shows the obtained theoretical curves of D_T, whose axis is represented in a logarithmic scale. It is seen that, as one might expect, a lower σ implies an higher sound conductivity. The parameter shows a lower limit equal to unity which is reached for highest values of the product of flow resistivity and porosity. Moreover the obtained curves for lower σ are decreasing with increasing frequency meaning a stronger permeation of sound in the material at lower frequencies. In this system of small tubes if a fixed value of σ is examined, then it is seen that the impact of a higher ϕ is that of decreasing the sound conductivity D_T, but this effect is more limited in the case of the higher σ values. As outlined above, higher σ values imply difficulties in permeating the material with sound energy and thus D_T values approximating unity may indicate scarce sound absorption performance as lamented for materials with excessive flow resistivity.

4. Physical interpretation and source terms

The physical interpretation of the above findings can be explored if we go back to the layer of material inserted in the tube and make an integration over its boundaries. Mathematically speaking this operation can be indicated by the following surface integrals:

$$\iint_{\Sigma} I \,\mathrm{d}s = -D_T \iint_{\Sigma} \nabla E \,\mathrm{d}s, \tag{14}$$

where the coefficients can be taken out of the integral since in the above hypothesis they do not depend on the position. Equation (14) can be easily interpreted as a conservation equation for the sound power. The term on the left-hand side is the sound power dissipated inside the material as it is the net balance of incoming and outgoing energy flux. The term on the right-hand side is the power globally transferred in the medium between the input and output sections. Equation (14) can be further processed by applying the Gauss theorem and turning the surface integral into a volume one as follows:

$$\iiint_{V} \nabla I \,\mathrm{d}V = -D_T \iiint_{V} \nabla^2 E \,\mathrm{d}V. \tag{15}$$

The integrand functions can be thus equated on the two sides and the following equation is derived:

$$\nabla^2 E = \chi_I Q, \tag{16}$$

where

$$Q = -\nabla I.$$

Even if not effective in the context of the flux regulation, it can be remarked that a similar divergence can be defined for the reactive intensity too, which reads

$$R = -\nabla J.$$

The above quantities Q and R can be expressed analytically by careful handling of the divergence of the intensities. The results are:

$$\begin{aligned} Q &= \frac{2\Re e\{\gamma^* Z_C\}}{\rho_r}\frac{Z_r^2}{|Z_C|^2}E_P + \frac{2\Re e\{\gamma Z_C\}}{\rho_r}E_K \\ &= -2\omega\Big(\Im m\{\tilde{\chi}\}\,|P(x)|^2 + \Im m\{\tilde{\rho}\}\,|V(x)|^2\Big), \\ R &= \frac{2\Im m\{\gamma^* Z_C\}}{\rho_r}\frac{Z_r^2}{|Z_C|^2}E_P + \frac{2\Im m\{\gamma Z_C\}}{\rho_r}E_K \\ &= -2\omega\big(E_P - E_K\big). \end{aligned} \tag{17}$$

The two derived quantities can be regarded respectively as the "sink" term of the sound energy in the material (Q) and

the "source" term for reactivity inside the material (R). They both depend on the space variable through the energy densities, so that Q and R correctly account for the two phenomena which are actually distributed within the material, that is energy dissipation (Q) and energy storage (R). While the physical meaning of Q is well established, the interpretation of R requires specifications. A source of reactivity or of flux localization can be considered as a physical mechanism causing the phase displacement between pressure and particle velocity, which is at the basis of the instantaneous reactive flux itself. This is somehow dual to the concept of source activity hidden in the definition of Q, which in turn, by means of the phase alignment of pressure and velocity, causes the absorption of sound in the material.

As a comparison with air one can see that, if losses are neglected, equations (17) becomes $Q = 0$ and $R = -2\omega(E_{P0} - E_{K0})$ (with E_{P0} and E_{K0} being the respective energy densities in air), and R coincides with the one calculated in [4]. This confirms the relevance of the sources of reactivity and prompts also for further research on the nature and on the significance of the time dependent nature of the energy flux in the material.

5. The differential field equation for the energy densities

Equation (16) for the energy density can serve as the basis for the prediction of the whole energetic field. This equation is hardly solved in the presented form since the dependence of E is expressed simultaneously in terms of both E_P and E_K which together enter in the definition of Q. What has to be done is thus to decouple (16) into two separate equations for E_P and E_K. To accomplish this task one must recall (9) and operate separately the Gauss theorem on each energy density. By doing so the following system is obtained:

$$\begin{aligned}\nabla^2 E_P &= 2(\alpha^2 - k^2)E_P + 2(\alpha^2 + k^2)\frac{|Z_C|^2}{Z_r^2}E_K,\\ \nabla^2 E_K &= 2(\alpha^2 + k^2)\frac{|Z_C|^2}{Z_r^2}E_P + 2(\alpha^2 - k^2)E_K.\end{aligned} \tag{18}$$

The problem has turned into the solution of a system of two ordinary second order differential equations. It turns out that, by expressing E_P (E_K) as function of E_K and $\nabla^2 E_K$ (E_P and $\nabla^2 E_P$), one can easily derive a single fourth order differential equation,

$$\nabla^4 E_d - 4(\alpha^2 - k^2)\nabla^2 E_d - 16\alpha^2 k^2 E_d = 0. \tag{19}$$

This equation can be regarded as the field equation for the energy densities since, in this very form, it is valid for E, E_P and E_K. For this reason a subscript "d" has been inserted. The general solution of (19) is achieved [13] as a sum of trigonometric and hyperbolic functions. If the required four coefficients are named $C_{1..4}$, the solution can be handed out as follows:

$$\begin{aligned}E_d(x) &= C_1\cosh(2\alpha x) + C_2\sinh(2\alpha x)\\ &\quad + C_3\cos(2kx) + C_4\sin(2kx).\end{aligned} \tag{20}$$

To obtain the specific solutions inside the porous layer one must introduce the values of $E_d(x)$ and $\partial E_d/\partial x = E_d'(x)$ at the interior side of the material boundaries at $x = 0$ and $x = l$. The same procedure, which is carried out in Appendix A2, procures specific coefficients for each of the three sound energy densities. The three solutions derived by (20) are able to fully predict the behaviour of the energy densities inside the porous layer.

Furthermore, once the energy densities are known, also the energy fluxes can be derived. Again this can be shown after some algebra involving the auxiliary functions H_1 and H_2. The active intensity is already available from (13) whereas the reactive intensity is

$$\begin{aligned}J &= -\frac{\Re e\{\gamma^* Z_C\}}{2\rho_r\alpha k}\frac{Z_r^2}{|Z_C|^2}\nabla E_P + \frac{\Re e\{\gamma Z_C\}}{2\rho_r\alpha k}\nabla E_K\\ &= \frac{\omega}{2\alpha k}\left(\frac{\Im m\{\tilde{\chi}\}}{\chi_r}\nabla E_P - \frac{\Im m\{\tilde{\rho}\}}{\rho_r}\nabla E_K\right).\end{aligned} \tag{21}$$

It is worthwhile to investigate what the solutions will look like if air is considered. This is done in the Appendix A3.

6. Discussion

The analytical derivation above showed that the sound energy density and the active intensity, which describes the flow of energy inside a porous layer, obey a diffusion equation. The permeation of the sound inside the material is regulated by a material specific sound transfer coefficient D_T. Unfortunately the diffusion analogy is not working as such in lossless media (i.e. air if heat and viscous losses are neglected) because in this case D_T would equal infinity thus indicating an ideal ability of this medium to host the sound transfer. Inside the material there are instantaneous oscillations of energy procuring no net energy flux.

This developed approach can be regarded as an alternative to the use of impedance networks or transfer matrixes for both single and multilayered structures, and provides information consistent with former methods. In this method the porous layer is characterized by its real values D_T and the numerical approach does not involve the complex algebra of equivalent networks but the solution of simple partial differential equations.

From the experimental point of view the direct measurement of D_T will require several precautions due to the intrinsic non linear nature of second order quantities inside the materials, especially in the higher frequency range. Nevertheless, as stated in equation (12), the quantity χ_I can be achieved as function of the real and imaginary parts of the propagation constant γ, whose measurement methods are nowadays sufficiently reliable [14].

Moreover the experimental validation of the theory would face the doubtless trouble of obtaining robust and

effective measurements of the second order quantities inside a porous layer. In the past the procedure of measuring sound pressure inside sound absorbing samples was developed as the most straightforward way to measure Z_C and γ (this procedure was known as Scotts's method and is reviewed in ref. [15]). In the present case the measurements would be more difficult, as they should accomplish both the sound pressure and the particle velocity at the same position inside the porous sample. On the other hand the experimental validation of the present work, which will be accomplished in the near future, will benefit from sound particle velocity measurements obtained by means of impulse responses already established for air in [16]. This method can be also applied with sequential measures done with one microphone only, resolving the phase errors and correcting the finite difference errors to achieve reliable values for pressure, velocity and the second order quantities.

Finally it is to be recalled that the above derivations are valid in the presented form under the assumption of plane wave propagation and time averaged behaviour. Removing these constraints should probably involve changes in the expression of D_T and the picture of the sound transfer might include a role for the reactive intensity to take into account at each position the twofold nature of the instantaneous energy flux.

7. Conclusions

In the present work second order quantities were used to reinterpret the known results of the macroscopic propagation of sound in rigid frame porous media. The theory is based on the continuity of pressure and velocity at the layer boundaries and, to express the second order energetic quantities, the characteristic impedance and propagation constant are necessary. Though within the limits of a one dimensional, time averaged propagation (and having neglected the heat and viscous losses in air), several interesting findings have been achieved and discussed. In particular they can be outlined as follows:

- a gradient of the sound energy density inside a porous layer generates a net energy flux which is fully accounted for by the active intensity and the related constant is real valued. There is a formal analogy with the heat conduction described by the Fourier law where the role of D_T is taken over by the heat conductivity;
- there is no dependence of the time-averaged reactive flux from the gradient of the energy density;
- the internal sinks of sound energy and the sources of reactivity have been formalized in terms of energetic quantities;
- all of the second order quantities can be predicted thanks to a differential formulation for the energy densities inside the material.

As discussed above, the experimental validation of the theory, to be accomplished in the near future, will imply sound intensity and sound energy densities measurements inside a porous layer.

Appendix

A1. Derivation of the energy densities

Naming a a quantity at the microscopic level in the porous medium it has firstly to be pointed out that its macroscopic counterpart can be obtained by a spatial average over a ball whose radius is much smaller than the wavelength, but long enough to smooth out the spatial inhomogeneities of the medium. This averaging process can be worked out in two ways depending on the volume normalization. In the former case the total volume V is used and the average is indicated by $\langle a \rangle$, whereas in the latter case the average is normalized to the volume V_f of the fluid phase only, and the average is referred to as $\overline{a}$. Since the two normalization volumes are related by the medium porosity, that is $V_f = \phi V$, then one has $\langle a \rangle = \phi \overline{a}$. The same rules apply to the quantities sound pressure and particle velocity, whose macroscopic values are related to the spatial averages according to the equalities below,

$$\boldsymbol{V} = \langle \boldsymbol{v} \rangle; \qquad P = \overline{p} = \frac{\langle p \rangle}{\phi}. \tag{A1}$$

In order to achieve the suitable macroscopic expression of the energy density in the porous layer it is possible to start from the related microscopic one reported in [11],

$$e = \frac{1}{2}\rho_0 v^2 + \frac{1}{2}\chi_0 p^2 + \frac{1}{2}\frac{\rho_0 T_0}{C_{P0}} s^2, \tag{A2}$$

where s is the entropy, T_0 and C_{P0} are respectively the equilibrium temperature and constant pressure specific heat. Firstly it is possible to apply the spatial average to (A.2) and the result reads

$$\langle e \rangle = \frac{1}{2}\rho_0 \langle v^2 \rangle + \frac{1}{2}\chi_0 \langle p^2 \rangle + \frac{1}{2}\frac{\rho_0 T_0}{C_{P0}} \langle s^2 \rangle.$$

Then the next step is to consider the time average of the above energy density $\langle e \rangle$, given that the quantities can be represented in the usual formalism of complex amplitudes. The space and time-averaged microscopic sound energy density in the porous layer reads

$$\overline{\langle e \rangle}^{\text{t.a.}} = \frac{1}{2}\rho_0 \frac{\langle v v^* \rangle}{2} + \frac{1}{2}\chi_0 \frac{\langle p p^* \rangle}{2} + \frac{1}{2}\frac{\rho_0 T_0}{C_{P0}} \frac{\langle s s^* \rangle}{2}.$$

This expression can be reworked by using equation (A1) and the general thermodynamic relation expressing entropy as a function of pressure and temperature ($\tau\, C_p/T_0 - p\,\beta_0/\rho_0 = s$). What is achieved is

$$\begin{aligned} \overline{\langle e \rangle}^{\text{t.a.}} = {} & \frac{1}{2}\rho_0 \frac{\langle v v^* \rangle}{\langle v \rangle \langle v^* \rangle} \frac{|V|^2}{2} + \frac{1}{2}\chi_0 \phi^2 \frac{\langle p p^* \rangle}{\langle p \rangle \langle p^* \rangle} \frac{|P|^2}{2} \\ & + \frac{1}{2}\frac{\rho_0 T_0}{C_{P0}} \frac{1}{2} \left\langle \left(\frac{C_p}{T_0}\tau - \frac{\beta_0}{\rho_0} p \right) \left(\frac{C_p}{T_0}\tau^* - \frac{\beta_0}{\rho_0} p^* \right) \right\rangle. \end{aligned} \tag{A3}$$

To complete the derivation it is necessary to make use of a series of exact identities that can be proved (see [10],

[17, 18]) reported below,

$$\frac{\overline{vv^*}}{\overline{v}\,\overline{v^*}} = \Re e\{\tilde{\alpha}\}, \quad \frac{\overline{\tau\tau^*}}{\overline{\tau}\,\overline{\tau^*}} = \Re e\{\tilde{\alpha}'\}, \quad \frac{\overline{\tau}}{\overline{p}} = \frac{\beta_0 T_0}{\rho_0 C_p}\frac{1}{\tilde{\alpha}'},$$
$$\tilde{\beta} = \gamma - \frac{\gamma-1}{\tilde{\alpha}'}, \qquad \frac{\overline{pp^*}}{\overline{p}\,\overline{p^*}} = 1, \qquad \frac{\overline{\tau^* p}}{\overline{\tau^*}\,\overline{p}} = 1,$$

(the two last identities express that $p - P \ll P$). The third term on the right of equation (A2) reduces to

$$\frac{1}{2}\frac{\rho_0 T_0}{C_{P0}}\frac{1}{2}\left\langle\left(\frac{C_p}{T_0}\tau - \frac{\beta_0}{\rho_0}p\right)\left(\frac{C_p}{T_0}\tau^* - \frac{\beta_0}{\rho_0}p^*\right)\right\rangle = \frac{1}{2}(\gamma-1)\chi_0\phi\left(1 - \frac{\Re e\{\alpha'\}}{|\alpha|^2}\right)\frac{|P|^2}{2},$$

where the general thermodynamic identity $\gamma - 1 = (T_0/C_p)(\beta_0^2/\rho_0\chi_0)$ has been used. Now, by gathering with the other two terms one can accomplish the final result of equation (3):

$$\begin{aligned} E &= \frac{1}{2}\chi_0\phi\left[\gamma - \left((\gamma-1)\frac{\Re e\{\alpha'\}}{|\alpha|^2}\right)\right]\frac{|P|^2}{2} \\ &\quad + \frac{1}{2}\frac{\rho_0}{\phi}\Re e\{\tilde{\alpha}\}\frac{|V|^2}{2} \\ &= \frac{1}{2}\chi_0\phi\Re e\{\tilde{\beta}\}\frac{|P|^2}{2} + \frac{1}{2}\frac{\rho_0}{\phi}\Re e\{\tilde{\alpha}\}\frac{|V|^2}{2}. \end{aligned}$$

A2. Derivation of the $C_{1\ldots4}$ coefficients

The solution of equation (19), which is valid for potential, kinetic and total energy densities inside the layer can be pursued with the help of [16] and has the general form of (20),

$$\begin{aligned} E_d(x) &= C_1\cosh(2\alpha x) + C_2\sinh(2\alpha x) \\ &\quad + C_3\cos(2kx) + C_4\sin(2kx). \end{aligned}$$

The working solution has to be elaborated by setting the proper conditions at the two boundaries of the porous layer, found at $x = 0$ and at $x = l$. The values required are those on the interior boundaries (suffix i). For the potential and kinetic energy densities they are obtained from those on the air side of the layer (suffix o) by the simple equations

$$E_d^i(0) = \delta E_d^o(0), \qquad E_d^i(l) = \delta E_d^o(l),$$

with $\delta = \chi_r/\chi_0$ when $E_d \equiv E_P$ and $\delta = \rho_r/\rho_0$ when $E_d \equiv E_K$. This simplification does not apply to the first order derivatives of the energy densities which are also required to run equation (20). They can be evaluated by inverting a system made by equations (13) and (21) and by estimating the sound intensities just inside the layer. Once all of these boundary values are available the suffix i can be dropped and they can be termed as

$$E_d(0) = A, \quad E_d(l) = B, \quad E_d'(0) = C, \quad E_d'(l) = D.$$

Also some of the mathematical functions involved are replaced with letters in order to make the exposition more compact,

$$\cosh(2\alpha l) = TH, \qquad \sinh(2\alpha l) = SH,$$
$$\cos(2kl) = T, \qquad \sin(2kl) = S.$$

With the above specifications the introduction of the boundary values provides a system of four equations with four unknowns,

$$\begin{aligned} A &= C_1 + C_3, \\ B &= C_1 TH + C_2 SH + C_3 T + C_4 S, \\ C &= 2\alpha C_2 + 2kC_4, \\ D &= 2\alpha SHC_1 + 2\alpha THC_2 + 2kSC_3 + 2kTC_4. \end{aligned} \tag{A4}$$

Solution of (A4) can be handled manually with some effort and the solution gives the required $C_{1\ldots4}$ coefficients that are reported here for documentation:

$$\begin{aligned} C_1 &= \frac{(2\alpha AT - 2\alpha B + SHC) + 2(\alpha S - kSH)C_4}{2\alpha(T - TH)}, \\ C_2 &= \frac{C - 2kC_4}{2\alpha}, \\ C_3 &= \frac{(2\alpha B - 2\alpha ATH - SHC) - 2(\alpha S - kSH)C_4}{2\alpha(T - TH)}, \\ C_4 &= \Big[2\alpha(T - TH)D - A\big(4\alpha(\alpha SHT + kSTH)\big) \\ &\quad + B\big(4\alpha(\alpha SH + kS)\big) - C\big(2\alpha THT + 2kSHS - 2\alpha\big)\Big] \\ &\quad \cdot \Big[4(\alpha S - kSH)(\alpha SH + kS) + 4k\alpha(T - TH)^2\Big]^{-1} \end{aligned}$$

A3. Solution for the medium air

It is indeed interesting to complement the proposed analysis with the study of the form that the above equation assume in air. This is valuable to check the consistency of the above theory with available results. Formally the equivalence is done by substituting as follows:

$$\gamma_0 = \mathrm{i}k_0, \qquad Z_C = Z_0 = \rho_0 c_0.$$

With this substitution one easily sees that also χ_I vanishes so that we immediately can verify for air that the total energy density (equations 11 and 16) reduces to

$$\nabla E = 0, \qquad \nabla^2 E = 0. \tag{A5}$$

This result fully complies with the total energy density being a constant in the air portion of the system. A deeper insight is anyway gained when (18) is considered and solved also for E_P and E_K. With the above simplifications the system now reads

$$\begin{aligned} \nabla^2 E_P &= -2k^2 E_P + 2k^2\frac{|Z_C|^2}{Z_r^2}E_K, \\ \nabla^2 E_K &= 2k^2\frac{|Z_C|^2}{Z_r^2}E_P - 2k^2 E_K, \end{aligned} \tag{A6}$$

and the respective fourth order ordinary differential equation is obtained in the following form,

$$\nabla^4 E_d + 4k^2\nabla^2 E_d = 0.$$

The solution is immediate since this equation can be rewritten and a simple oscillation equation is left. Formally one can write

$$\nabla^2 E_d = \Phi, \qquad \nabla^2\Phi + 4k^2\Phi = 0.$$

And, provided that suitable constant boundary condition are defined, the two subsequent solutions for ϕ and E_d give

$$\begin{aligned}\Phi(x) &= G_1 e^{2ikx} + G_2 e^{-2ikx},\\ E_d(x) &= -\frac{1}{4k^2}\left[L_1\cos(2kx) + L_2\sin(2kx)\right].\end{aligned}$$

This implies that the energy densities follow the well known combination of $\sin(kx)^2$ and $\cos(kx)^2$ course as expected for the concurrent presence of plane progressive waves in opposite directions building a standing wave pattern.

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References

[1] J. F. Allard: Propagation of sound in porous media,. Elsevier Applied Science, London, 1993. Chapter 2 and 4.

[2] B. H. Song, J. S. Bolton: A transfer matrix approach for estimating the characteristic impedance and wave numbers of limp and rigid porous materials. J. Acoust. Soc. Am. **107** (2000) 1131–1152.

[3] F. Jacobsen: Sound intensity and its measurement and applications. Note No. 2216, 10–11, Oersted-DTU, Acoustics Technology, Lyngby (Denmark), 2002.

[4] F. J. Fahy: Sound intensity. II. edition. E&FN Spon, London, 1995. Chapter 4.

[5] G. W. Elko: Simultaneous measurement of the complex acoustics intensity and the acoustic energy density. Proc. of 2nd Int. Congress on Acoustic Intensity, CETIM, Senlis, France, September 1986.

[6] F. Jacobsen: A note on instantaneous and time-averaged active and reactive sound intensity. J. Sound Vib. **147** (1991) 489–496.

[7] D. Stanzial, N. Prodi, G. Schiffrer: Reactive acoustic intensity for general fields and energy polarization. J. Acoust. Soc. Am. **99** (1996) 1868–1876.

[8] D. Stanzial, N. Prodi: Measurement of newly defined intensimetric quantities and their physical interpretation. J. Acoust. Soc. Am. **102** (1997) 2033–2039.

[9] D. L. Johnson, J. K. Koplik, R. Dashen: Theory of dynamic permeability and tortuosity in fluid-saturated porous media. J. Fluid Mech. **176** (1987) 379–402.

[10] D. Lafarge, P. Lemarinier, J. F. Allard, V. Tarnow: Dynamic compressibility of air in porous structures at audible frequencies. J. Acoust. Soc. Am. **102** (1997) 1995–2006.

[11] A. D. Pierce: Acoustics: An introduction to its physical principles and applications. American Institute of Physics, New York. P. 516, eq. 10-2.5a, 1989.

[12] D. Lafarge. private communication with the author.

[13] A. D. Polyanin, V. F. Zaitsev: Handbook of exact solutions for ordinary differential equations. CRC Press, 1995. Chapter 4.

[14] P. Bonfiglio: Advanced experimental techniques for the description of rigid frame porous media. PhD Thesis, Facoltà di Ingegneria, Università di Ferrara, Italy, 2006. Chapter 3.

[15] C. D. Smith, T. L. Parrott: Comparison of three methods for measuring acoustics properties of bulk materials. J. Acoust. Soc. Am. **74** (1983) 1577–1582.

[16] P. Bonfiglio, N. Prodi, A. Farina, F. Pompoli: On the use of a p-u sound intensity probe for the qualification of complex surface properties. Proc. of 13 Inter. Conf. on Sound and Vib. ICSV, Wien, July 2006.

[17] D. Lafarge: Propagation du son dans les matériaux poreux à structure rigide saturés par un fluide visco-thermique. PhD Thesis, Université du Maine, 1993.

[18] A. Cortis, D. M. J. Smeulders, J. L. Guermond, D. Lafarge: Influence of pore roughness on high-frequency permeability. Phys. of Fluids **15** (2003) 1788–1775.