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An Expansion Scheme for the Kirchhoff Theory of Acoustic Damping in Long Tubes Containing Rarefied Gases

K. Rathnam
Department of Automobile Technology, Anna University, Tiruchirappalli-620024, India.
rathnamku@yahoo.co.in

To Professor M. M. Oberai on the occasion of his seventy-seventh birthday

Summary
An expansion scheme is employed to solve the problem of the propagation of acoustic waves in an infinite cylindrical tube. In the governing equations all the terms of the linearised Navier-Stokes equations are retained and in the boundary conditions are incorporated the corrections due to the wall temperature fluctuation and the slip and temperature jump. The only assumption made is that the Stokes layer thickness is much smaller than the wave length Λ and this provides the expansion parameter. The emphasis is laid on the calculation of attenuation coefficient and explicit results are obtained both for large and small values of the tube radius. The scheme is shown to be valid for $\lambda \ll R$ where λ is the mean free path and R is the tube radius.

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1. Introduction

The study of the propagation of acoustic wave in tubes containing a gaseous medium in which the viscous and thermal effects/dissipations are not negligible is essentially a problem in gas dynamics. Special interest attaches to this problem if the velocity slip and temperature jump corrections are not ignorable. This case is the subject matter of the present investigation.

The damping of acoustic wave in long tubes due to friction at the side walls was made by Helmholtz [1], who published his results in 1863. The theory of acoustic damping in long cylindrical tubes that includes the previously neglected effects of heat conduction was given by Kirchhoff [2] in 1868. Lord Rayleigh [3] has included an account of these papers in his book on the 'Theory of Sound' Volume II. The classical theory of Kirchhoff is based on the exact solution of the linearised Navier-Stokes equations that take into consideration all diffusive effects in a simple gas, i.e. the first and second viscosity as well as heat conduction. In the special case of no tube walls, the so called Stokes-Kirchhoff theory of damping in an unbounded medium is obtained; the problem is reducible to the solution of a quartic. It is known now that for waves progressing in an unbounded atmosphere for long distances, certain nonlinear effects become ultimately important. These effects are taken into consideration in the modern theory of damping by Lighthill [4]. However, even within the frame work of the Stokes-Kirchhoff theory it is considered, in general, as convenient as well as physically significant to approximate the results under the assumption that the wave length is much larger than the mean free path. If the opposite limit (which lies anyway beyond the realm of a continuum theory) is excluded, simple results are obtained without the necessity of solving a quartic.

There exist in the literature a large number of papers on this subject, and a wide variety of different approximation has been used. Here, only one method will be discussed, which can be said to give a better approximation; it will lead to results valid for arbitrary ratios of Stokes layer thickness to tube radius, within the continuum limit. The only simplifying assumption will be that the Stokes layer thickness is small compared to the wave length. As in most cases the wave length is also some measure of the tube length, the approximation is said to be valid for "long tubes". It would be more precise to speak of long waves. It will be found that in the leading (or the zero order) terms of this approximation, the radial pressure and all diffusive gradients do not appear, just as in the Prandtl's boundary layer theory.

The leading term of the present approximation has been used as a working model by several authors; notably by Berg and Tijdeman [5] who quote Iberall [6] as the earliest reference in this regard. Rott [7] used the same model for the generalization of the Kirchhoff problem to the case of a gas with axial temperature stratification. In the present paper, first the exact problem is formulated taking into account the wall temperature fluctuations (which become

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significant at low temperature) and the velocity slip and the temperature jump. The phenomena of velocity slip and temperature jump are part of rarefied gas dynamics corresponding to a slight departure from the continuum limit. These effects which appear as small corrections to the continuum no velocity slip, no temperature jump boundary conditions are significant when the pressure and consequently density of the gas is low. Then, an expansion scheme is established and the zero-, the first- and the second order terms are calculated. The first order terms arise due to velocity slip and temperature jump, while the radial pressure gradient and axial diffusive gradients affect only the second order terms. Next, explicit calculations are made for the limiting cases of the wide tube and the narrow tube. In the final discussion, present results are compared with the results of other authors as well as the consideration of the limitations of the expansion scheme.

The comparison between the present approximation and the exact Kirchhoff theory will be part of the discussion. However, it might be helpful to point out in advance that it is no simple matter to obtain the present approximation from the completed solution of the exact theory. This is the consequence of the fact that for the Kirchhoff problem, the order of the two operations "solution" and "approximation" is not easily reversed. As will be seen presently, the approximate method consists essentially of integration alternately across and along the tube, with the boundary conditions fulfilled at each step by approximate solutions.

The exact Kirchhoff method calls for the establishment of the exact expressions for all the acoustic quantities first; then the fulfillment of the boundary conditions leads to a very difficult combination of algebraic and transcendental equations. There have been many attempts to extend the Kirchhoff theory to rarefied gas acoustic oscillations incorporating the velocity slip and the temperature jump corrections.

The contributions of Henry [8] and Weston [9] are notable. They have made an approximation for the propagation constant appearing in the viscous terms to simplify the determinant expansion. Such an approximation affects the analytical results. In reference [10], a method of reformulating the coefficient matrix has been detailed incorporating all the corrections in the Kirchhoff equations. Results are obtained for both the wide and narrow tubes without making any approximation of the type assumed by Henry and Weston.

As the determinant expansion of the coefficient matrix leads to a difficult combination of unmanageable terms, analytical results are furnished for making a comparison with the leading terms of the zero-, the first- and the second order of the present expansion scheme. Zwikker and Kosten [11] introduced a simpler, approximate method for the Kirchhoff theory of sound propagation in circular tube by treating the effects of viscosity and thermal conductivity separately and summarized results for 'low' and 'high' frequencies. An expression for the intermediate frequency was also given and we shall refer to it later.

Stinson [12] showed that the exact Kirchhoff solution can lead to the Zwikker and Koston solution when simplifying approximations, appropriate for certain choices of tube radius and sound frequency are made. Tijdeman [13] solved the full Kirchhoff equation numerically and presented results for the propagation constant in terms of four non- dimensional parameters. Beltman [14] extended the work of Tijdeman and shed some light on the applicability of various models by showing graphically the region of validity of the model in terms of the key parameters. The remark of the author on the use of the full linearised Navier-Stokes model under extreme conditions of low pressure lends credence to the present work.

Numerical solutions of the problem were published by Shields *et al.* [15] and Shields and Faughn [16]. The numerical solution has been obtained by an iterative method of the full Kirchhoff equations incorporating the major corrections mentioned in the present work and thus can be considered as the most complete solutions so far made. They reported accurate experimental results under low pressure conditions and compared them with the numerical solutions. These experimental results reported in their work are compared with the analytical results obtained in the present work. It may be mentioned that the proposed approximation representing a series solution of the exact Kirchhoff problem is obtained from the basic governing equations.

2. Statement of the problem and the governing equations

A piston is exciting acoustic oscillations of frequency $\omega/2\pi$ in a perfect (thermally as well as calorically) gas contained in a semi-infinite cylindrical tube of radius R in the cylindrical coordinates (x, r) employed and is located at $x = 0$. Assuming that the time-dependence of all acoustic quantities is given by $\exp(\mathrm{i}\omega t)$, the linearised equation of continuity (taking into account the axial symmetry) and Navier-Stokes equations for momentum and energy are written as

$$\frac{\mathrm{i}\omega}{\rho_m}\rho + \frac{\partial U}{\partial x} + \frac{1}{r}\frac{\partial (rV)}{\partial r} = 0, \tag{1}$$

$$U - \frac{\mathrm{i}}{\omega\rho_m}\frac{\partial p}{\partial x} = \frac{\nu}{\mathrm{i}\omega}\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial U}{\partial r}\right\} + \frac{\partial^2 U}{\partial x^2} - \frac{\mathrm{i}\omega\alpha}{\rho_m}\frac{\partial \rho}{\partial x}\right], \tag{2}$$

$$V - \frac{\mathrm{i}}{\omega\rho_m}\frac{\partial p}{\partial r} = \frac{\nu}{\mathrm{i}\omega}\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial (rV)}{\partial r}\right\} + \frac{\partial^2 V}{\partial x^2} - \frac{\mathrm{i}\omega\alpha}{\rho_m}\frac{\partial \rho}{\partial r}\right], \tag{3}$$

$$\rho - \frac{1}{a^2}p = \frac{\nu}{\mathrm{i}\omega\sigma}\left[\frac{1}{r}\frac{\partial}{\partial r}\left\{r\frac{\partial}{\partial r}\left(\rho - \frac{\gamma}{a^2}p\right)\right\} + \frac{\partial^2}{\partial x^2}\left(\rho - \frac{\gamma}{a^2}p\right)\right]. \tag{4}$$

In writing equation (4) the acoustic temperature T is eliminated by using the following relation which is derived from the perfect gas equation

$$T = -\frac{T_m}{\rho_m}\left(\rho - \frac{\rho_m}{p_m}p\right) = -\frac{T_m}{\rho_m}\left(\rho - \frac{\gamma}{a^2}p\right). \tag{5}$$

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3. Boundary conditions

The radial velocity must vanish both at the wall and along the axis of the tube,

$$\left(V\right)_{r=R} = 0, \tag{6}$$

$$\left(V\right)_{r=0} = 0. \tag{7}$$

However, for writing the boundary conditions for U and ρ at the wall, it is proposed to include the wall temperature fluctuations as well as the slip and the temperature jump conditions.

3.1. The instantaneous wall temperature

To calculate the instantaneous wall temperature $T_w e^{i\omega t}$, the problem of heat conduction in the tube wall has to be solved. At the frequency used in practice the changes of temperature due to the vibrations of the gas will only penetrate a short distance into the tube wall. A conservative estimate of this depth of penetration in terms of the period of one oscillation τ_0 [s] and the thermal diffusivity α [m^2/s] can be made from the formula '$x = 1.6\sqrt{\pi\alpha\tau_0}$', given in the reference [17]. The tube wall thickness should be more than the penetration depth. For the frequency range of 8 kHz to 15 kHz, the maximum depth of penetration is found to be less than 1 mm for most of the tube materials. Hence, the axial heat conduction (in the wall) is neglected in comparison with the radial heat conduction and the temperature far from the inner wall remains unaffected by the fluctuating part of the wall temperature. If $T_s e^{i\omega t}$ is the instantaneous temperature at a point distance y from the inner surface, then

$$\frac{\partial^2 T_s}{\partial y^2} + \frac{1}{R+y}\frac{\partial T_s}{\partial y} = \frac{i\omega\rho_s c_s}{k_s} T_s = b^2 T_s, \tag{8}$$

where ρ_s, c_s and k_s are the density, specific heat and thermal conductivity of the wall material. Equation (8) is to be solved subject to the boundary conditions

$$T_s = T_w \qquad \text{at} \quad y = 0, \tag{9}$$

and the temperature far from the inner wall remains unaffected by the fluctuating part of the wall temperature (i.e.) $T_s = 0$. Hence, the solution is

$$T_s = T_w \frac{H_0^1\left[ib(R+y)\right]}{H_0^1\left[ibR\right]}, \tag{10}$$

where H_0^1 is the zero order Hankel function of the first kind. It may be mentioned in this context that Rott [18], in his analysis of thermally driven acoustic oscillations, obtained similar equation (cf. equation 40 of his paper). Next the heat flux at $r = R$ from the gas side is equated to that at $y = 0$ into the wall material; i.e.

$$-k\left(\frac{\partial T}{\partial r}\right)_{r=R} = -k_s\left(\frac{\partial T_s}{\partial r}\right)_{y=0}$$

or

$$T_w = \frac{ik}{bk_s}\frac{H_0^1\left[ibR\right]}{H_1^1\left[ibR\right]}\left(\frac{\partial T}{\partial r}\right)_{r=R}. \tag{11}$$

For ordinary values of ω, b is very large (for example cf. Henry [8] for glass $|b| \sim 10^3\,\mathrm{cm}^{-1}$) so that equation (11) may be rewritten as

$$T_w = -\frac{1}{b}\frac{k}{k_s}\left(\frac{\partial T}{\partial r}\right)_{r=R}\left[1 - \frac{1}{2bR} + O\left(\frac{1}{b^2R^2}\right)\right]. \tag{12}$$

In the following $|1/bR|$ is neglected in comparison with 1 and retain the condition

$$T_w = -\frac{1}{b}\frac{k}{k_s}\left(\frac{\partial T}{\partial r}\right)_{r=R} = -\frac{k}{\sqrt{i\omega k_s\rho_s c_s}}\left(\frac{\partial T}{\partial r}\right)_{r=R}, \tag{13}$$

which is the same as obtained by Henry by neglecting the second term on the L.H.S of equation (8) i.e. by making use of the approximation that the wall is plane. The fact that heat penetrates the wall to a depth of $|1/b|$ provides justification for neglecting axial conduction (in the wall) in comparison with the radial conduction.

3.2. Temperature jump

The condition for the temperature jump at the wall may be stated as

$$\left(T\right)_{r=R} - T_w = -F_1\lambda\left(\frac{\partial T}{\partial r}\right)_{r=R}, \tag{14}$$

where $F_1 = [(2-\beta)/\beta][2\gamma/\sigma(\gamma+1)]$ (which contains the thermal, or energy, accommodation coefficient β). The energy accommodation coefficient β is defined by $\beta = (E_i - E_r)/(E_i - E_w)$ where E_i is the energy brought up to a unit area of surface per second by the incident stream, E_r is the energy carried away by the reflected molecules, and E_w is the energy the latter stream would carry away if it carried the same mean energy per molecule as does a stream issuing from a gas in equilibrium at the wall temperature. On substituting for T_w from equation (13) and using equation (5), the temperature jump condition simplifies to

$$\left(\rho - \frac{\gamma}{a^2}p\right) = -\left[\frac{k}{\sqrt{i\omega k_s\rho_s c_s}} + F_1\lambda\right]\left(\frac{\partial\rho}{\partial r} - \frac{\gamma}{a^2}\frac{\partial p}{\partial r}\right)_{r=R}. \tag{15}$$

3.3. Velocity slip

Finally, the condition for the velocity slip at the wall is written as

$$\left(U\right)_{r=R} = -E_1\lambda\left(\frac{\partial U}{\partial r}\right)_{r=R}, \tag{16}$$

where $E_1 = (2-\alpha_1)/\alpha_1$ (which contains the tangential momentum accommodation coefficient α_1). The tangential momentum accommodation coefficient α_1 is defined as the fraction of molecules diffusely reflected from the wall.

The reason for the occurrence of the phenomena of velocity slip and temperature jump is that, under conditions

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of low pressure and hence low density, some average properties of the gas at the wall surface may not be the same as those corresponding to the properties of the surface; for example, (a) the velocity of gas molecules tangential to the surface may not be exactly zero at the surface and (b) the temperature of the gas at the surface may not be exactly equal to the temperature of the surface. As stated in the introduction these corrections are significant only when the pressure of the gas is low. Equations (6), (7), (15) and (16) give the four boundary conditions for the system represented by equations (1), (2), (3) and (4). In addition the pressure condition is also known at $x = 0$.

4. The parameter ε and non-dimensional coordinates

As stated in the introduction, present study will be confined to the situation when

$$\sqrt{\nu/\omega} \ll \Lambda = a/\omega, \tag{17}$$

so that, if the parameter ε is defined as

$$\varepsilon = \sqrt{\omega\nu/\mathrm{i}a^2}, \tag{18}$$

with the specification, here and in the subsequent, that $\sqrt{i} = \mathrm{e}^{\mathrm{i}\pi/4}$, then $|\varepsilon| \ll 1$. It may be noted that $|\varepsilon^2|$ is proportional to the ratio of ω to the collision frequency of the gas molecules.

Next the coordinates (x, r) are changed to dimensionless form (ξ, η) via the transformations

$$\xi = x\frac{\omega}{a}, \qquad \eta = \frac{1}{\varepsilon}\frac{\omega}{a}r = \sqrt{\frac{\mathrm{i}\omega}{\nu}}r. \tag{19}$$

On incorporating the transformations (19) into the system represented by equations (1), (2), (3) and (4) and the boundary conditions (6), (7), (15) and (16), the problem is governed by the system

$$\varepsilon\Big(\frac{\mathrm{i}a}{\rho_m}\rho + \frac{\partial U}{\partial\xi}\Big) + \frac{1}{\eta}\frac{\partial}{\partial\eta}\big(\eta V\big) = 0, \tag{20}$$

$$-\frac{1}{\eta}\frac{\partial}{\partial\eta}\Big(\eta\frac{\partial U}{\partial\eta}\Big) + U = \frac{\mathrm{i}}{a\rho_m}\frac{\partial p}{\partial\xi} + \varepsilon^2\frac{\partial^2 U}{\partial\xi^2} - \alpha\frac{\mathrm{i}a}{\rho_m}\varepsilon^2\frac{\partial\rho}{\partial\xi}, \tag{21}$$

$$\frac{\partial p}{\partial\eta} = \mathrm{i}a\rho_m\varepsilon\Big[-V + \frac{\partial}{\partial\eta}\Big(\frac{1}{\eta}\frac{\partial(\eta V)}{\partial\eta}\Big) + \varepsilon^2\frac{\partial^2 V}{\partial\xi^2} - \alpha\frac{\mathrm{i}a}{\rho_m}\varepsilon\frac{\partial\rho}{\partial\eta}\Big], \tag{22}$$

$$-\frac{1}{\sigma}\frac{1}{\eta}\frac{\partial}{\partial\eta}\Big(\eta\frac{\partial\rho}{\partial\eta}\Big) + \rho = \frac{1}{a^2}p - \frac{\gamma}{a^2\sigma}\frac{1}{\eta}\frac{\partial}{\partial\eta}\Big(\eta\frac{\partial p}{\partial\eta}\Big) + \frac{1}{\sigma}\varepsilon^2\frac{\partial^2}{\partial\xi^2}\Big(\rho - \frac{\gamma}{a^2}p\Big), \tag{23}$$

subject to the boundary conditions when

$$\eta = 0, \quad V = 0: \qquad U, \rho \text{ finite} \tag{24}$$

$$\eta = \eta_w \quad V = 0: \tag{25}$$

$$U = -\mathrm{i}\varepsilon E\frac{\partial U}{\partial\eta}, \tag{26}$$

$$\rho - \frac{\gamma}{a^2}p = -\Big(\mathrm{i}\varepsilon F + \frac{K}{\sqrt{\sigma}}\Big)\Big(\frac{\partial\rho}{\partial\eta} - \frac{\gamma}{a^2}\frac{\partial p}{\partial\eta}\Big), \tag{27}$$

where

$$E = \frac{2-\alpha_1}{\alpha_1}\sqrt{\frac{\pi\gamma}{2}}, \tag{28}$$

$$F = \frac{2\gamma}{\sigma(\gamma+1)}\frac{2-\beta}{\beta}\sqrt{\frac{\pi\gamma}{2}}, \tag{29}$$

$$K = \sqrt{\frac{k\rho c_p}{k_s\rho_s c_s}}. \tag{30}$$

It may be noted from equation (30) that while at ordinary temperatures K is a negligible quantity, it becomes large at low temperatures due to T^3 dependence of c_s. In writing the boundary conditions the following expression is used for λ, the mean free path of the gas molecules:

$$\lambda = \frac{\mu}{p_m}\sqrt{\frac{\pi R_G T_m}{2}}. \tag{31}$$

5. Expansion in powers of ε

To solve the system represented by equations (20), (21), (22) and (23) the variables U, V, ρ and p are expanded in powers of ε. On observing from equation (20) that $V \sim o(\varepsilon)$ and from equation (22) $\partial p/\partial\eta \sim o(\varepsilon^2)$, the appropriate expansions are

$$U(\xi,\eta;\varepsilon) = U_0(\xi,\eta) + \varepsilon U_1(\xi,\eta) + \varepsilon^2 U_2(\xi,\eta), \tag{32}$$

$$V(\xi,\eta;\varepsilon) = \varepsilon\big(V_0(\xi,\eta) + \varepsilon V_1(\xi,\eta) + \varepsilon^2 V_2(\xi,\eta) + \ldots\big), \tag{33}$$

$$\rho(\xi,\eta;\varepsilon) = \rho_0(\xi,\eta) + \varepsilon\rho_1(\xi,\eta) + \varepsilon^2\rho_2(\xi,\eta), \tag{34}$$

$$p(\xi,\eta;\varepsilon) = P_0(\xi,\eta) + \varepsilon P_1(\xi,\eta) + \varepsilon^2\big(P_2(\xi) + p_2(\xi,\eta)\big). \tag{35}$$

Substituting the above expansions in the system represented by equations (20), (21), (22) and (23) and in the boundary conditions represented by equations (24), (25), (26) and (27) and separating the coefficients of various powers of ε, the following zero-, first-, and second order problems are obtained.

Zero order problem

$$\frac{\partial(\eta V_0)}{\partial\eta} = -\eta\Big(\frac{\mathrm{i}a}{\rho_m}\rho_0 + \frac{\partial U_0}{\partial\xi}\Big), \tag{36}$$

$$-\frac{1}{\eta}\frac{\partial}{\partial\eta}\Big(\eta\frac{\partial U_0}{\partial\eta}\Big) + U_0 = \frac{\mathrm{i}}{a\rho_m}\frac{\mathrm{d}P_0}{\mathrm{d}\xi}, \tag{37}$$

$$-\frac{1}{\sigma}\frac{1}{\eta}\frac{\partial}{\partial\eta}\Big(\eta\frac{\partial\rho_0}{\partial\eta}\Big) + \rho_0 = \frac{1}{a^2}P_0, \tag{38}$$

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when

$$\eta = 0, \qquad V_0 = 0; \quad U_0, \rho_0 \text{ finite}, \tag{39}$$
$$\eta = \eta_w \qquad V_0 = 0, \tag{40}$$
$$U_0 = 0, \tag{41}$$
$$\rho_0 = \frac{\gamma}{a^2} P_0 - \frac{K}{\sqrt{\sigma}} \frac{\partial \rho_0}{\partial \eta}. \tag{42}$$

First order problem

$$\frac{\partial(\eta V_1)}{\partial \eta} = -\eta\Big(\frac{\mathrm{i}a}{\rho_m}\rho_1 + \frac{\partial U_1}{\partial \xi} \Big), \tag{43}$$
$$-\frac{1}{\eta}\frac{\partial}{\partial \eta}\Big(\eta \frac{\partial U_1}{\partial \eta}\Big) + U_1 = \frac{\mathrm{i}}{a\rho_m}\frac{\mathrm{d}P_1}{\mathrm{d}\xi}, \tag{44}$$
$$-\frac{1}{\sigma}\frac{1}{\eta}\frac{\partial}{\partial \eta}\Big(\eta \frac{\partial \rho_1}{\partial \eta}\Big) + \rho_1 = \frac{1}{a^2} P_1, \tag{45}$$

when

$$\eta = 0, \qquad V_1 = 0; \quad U_1, \rho_1 \text{ finite}, \tag{46}$$
$$\eta = \eta_w \qquad V_1 = 0, \tag{47}$$
$$U_1 = -\mathrm{i}E \frac{\partial U_0}{\partial \eta}, \tag{48}$$
$$\rho_1 = \frac{\gamma}{a^2} P_1 - \frac{K}{\sqrt{\sigma}} \frac{\partial \rho_1}{\partial \eta} - \mathrm{i}F \frac{\partial \rho_0}{\partial \eta}. \tag{49}$$

Second order problem

$$\frac{\partial(\eta V_2)}{\partial \eta} = -\eta\Big(\frac{\mathrm{i}a}{\rho_m}\rho_2 + \frac{\partial U_2}{\partial \xi} \Big), \tag{50}$$
$$-\frac{1}{\eta}\frac{\partial}{\partial \eta}\Big(\eta \frac{\partial U_2}{\partial \eta}\Big) + U_2 = \tag{51}$$
$$\frac{\mathrm{i}}{a\rho_m}\Big(\frac{\mathrm{d}P_2}{\mathrm{d}\xi} + \frac{\partial p_2}{\partial \xi} \Big) + \frac{\partial^2 U_0}{\partial \xi^2} - \alpha \frac{\mathrm{i}a}{\rho_m}\frac{\partial \rho_0}{\partial \xi},$$
$$-\frac{1}{\sigma}\frac{1}{\eta}\frac{\partial}{\partial \eta}\Big(\eta \frac{\partial \rho_2}{\partial \eta}\Big) + \rho_2 = \frac{1}{a^2}\big(P_2 + p_2\big) \tag{52}$$
$$-\frac{\gamma}{a^2\sigma}\frac{1}{\eta}\frac{\partial}{\partial \eta}\Big(\eta \frac{\partial p_2}{\partial \eta}\Big) + \frac{1}{\sigma}\frac{\partial^2}{\partial \xi^2}\Big(\rho_0 - \frac{\gamma}{a^2} p_0\Big),$$
$$\frac{\partial^2 p_2}{\partial \eta^2} = -\mathrm{i}a\rho_m V_0 + \mathrm{i}a\rho_m \frac{\partial}{\partial \eta}\Big[\frac{1}{\eta}\frac{\partial}{\partial \eta}(nV_0)\Big] \tag{53}$$
$$+ \alpha a^2 \frac{\partial \rho_0}{\partial \eta},$$

when

$$\eta = 0, \qquad V_2 = 0; \quad U_2, \rho_2 \text{ finite}, \tag{54}$$
$$\eta = \eta_w \qquad V_2 = 0, \tag{55}$$
$$U_2 = -\mathrm{i}E \frac{\partial U_1}{\partial \eta}, \tag{56}$$
$$\rho_2 = \frac{\gamma}{a^2}\big(P_2 + p_2\big) - \mathrm{i}F \frac{\partial \rho_1}{\partial \eta} \tag{57}$$
$$-\frac{K}{\sqrt{\sigma}}\Big(\frac{\partial \rho_2}{\partial \eta} - \frac{\gamma}{a^2}\frac{\partial p_2}{\partial \eta} \Big).$$

6. Solution

The second and third equations of the zero- and first order problem [respectively of the systems (37) and (38) and (44) and (45)] are integrated using the boundary conditions at the wall to obtain the solutions of the zero- and first order problems for U and ρ. Substituting the expressions so obtained for these two variables respectively in equations (36) and (43) the solution for V_0 and V_1 are obtained using the condition at $\eta = 0$. Finally, satisfying the boundary conditions at $\eta = \eta_w$ will give second order differential equations for P_0 and P_1. Thus the solutions for the zero- and first order problems are obtained and are given below:

Zero-order solution

$$U_0 = \frac{\mathrm{i}}{a\rho_m}\frac{\mathrm{d}P_0}{\mathrm{d}\xi}\left[1 - \frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right], \tag{58}$$
$$\rho_0 = \frac{P_0}{a^2}\left[1 + \frac{\gamma - 1}{\vartheta}\frac{J_0(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right], \tag{59}$$
$$V_0 = -\frac{\mathrm{i}}{a\rho_m}\left\{ P_0\left[\frac{\eta}{2} + \frac{\gamma - 1}{\vartheta}\frac{J_1(\mathrm{i}\sqrt{\sigma}\eta)}{\mathrm{i}\sqrt{\sigma}J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right] \right.$$
$$\left. + \frac{\mathrm{d}^2 P_0}{\mathrm{d}\xi^2}\left[\frac{\eta}{2} - \frac{J_1(\mathrm{i}\eta)}{\mathrm{i}J_0(\mathrm{i}\eta_w)}\right]\right\}. \tag{60}$$

P_0, itself is determined from the homogeneous differential equation

$$\big(1 - f\big)\frac{\mathrm{d}^2 P_0}{\mathrm{d}\xi^2} + \Big[1 + \frac{\gamma - 1}{\vartheta} f^*\Big] P_0 = 0. \tag{61}$$

In the above

$$f = \frac{2J_1(\mathrm{i}\eta_w)}{\mathrm{i}\eta_w J_0(\mathrm{i}\eta_w)}, \quad f^* = \frac{2J_1(\mathrm{i}\sqrt{\sigma}\eta_w)}{\mathrm{i}\sqrt{\sigma}\eta_w J_0(\mathrm{i}\sqrt{\sigma}\eta_w)},$$
$$\text{and} \quad \vartheta = 1 + K\frac{\sqrt{\sigma}\eta_w}{2} f^*. \tag{62}$$

The solution of equation (61) which corresponds to positive damping is given by

$$P_0 = A_0 \mathrm{e}^{-\mathrm{i}M_0 \xi}, \tag{63}$$

where

$$M_0 = \sqrt{\big(1 + (\gamma - 1)f^*/\vartheta\big)/(1 - f)}, \tag{64}$$

and the root is to be so chosen that $\Im m\{M_0\} < 0$. The constant A_0 is determined from the pressure condition at $\xi = 0$, the location of the piston.

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First order solution

$$U_1 = \frac{\mathrm{i}}{a\rho_m}\left\{\frac{\mathrm{d}P_1}{\mathrm{d}\xi}\left[1-\frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right] + \frac{\mathrm{i}E\eta_w}{2}\frac{\mathrm{d}P_0}{\mathrm{d}\xi}f\frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right\}, \quad (65)$$

$$\rho_1 = \frac{1}{a^2}\left\{P_1\left[1+\frac{\gamma-1}{\vartheta}\frac{J_0(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right] - \frac{\mathrm{i}\sigma F\eta_w}{2}\frac{\gamma-1}{\vartheta^2}P_0 f^*\frac{J_0(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right\}, \quad (66)$$

$$V_1 = -\frac{\mathrm{i}}{a\rho_m}\left\{\frac{\mathrm{d}^2P_1}{\mathrm{d}\xi^2}\left[\frac{\eta}{2}-\frac{J_1(\mathrm{i}\eta)}{\mathrm{i}J_0(\mathrm{i}\eta_w)}\right] + P_1\left[\frac{\eta}{2}+\frac{\gamma-1}{\vartheta}\frac{J_1(\mathrm{i}\sqrt{\sigma}\eta)}{\mathrm{i}\sqrt{\sigma}J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right] + \frac{\mathrm{i}E\eta_w}{2}f\frac{\mathrm{d}^2P_0}{\mathrm{d}\xi^2}\frac{J_1(\mathrm{i}\eta)}{\mathrm{i}J_0(\mathrm{i}\eta_w)} - \frac{\mathrm{i}\sigma F\eta_w}{2}\frac{\gamma-1}{\vartheta^2}f^*P_0\frac{J_1(\mathrm{i}\sqrt{\sigma}\eta)}{\mathrm{i}\sqrt{\sigma}J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right\}. \quad (67)$$

P_1 satisfies the non-homogeneous differential equation

$$(1-f)\frac{\mathrm{d}^2P_1}{\mathrm{d}\xi^2}+\left[1+\frac{\gamma-1}{\vartheta}f^*\right]P_1 = -\frac{\mathrm{i}E\eta_w}{2}f^2\frac{\mathrm{d}^2P_0}{\mathrm{d}\xi^2}+\frac{\mathrm{i}\sigma F\eta_w}{2}\frac{\gamma-1}{\vartheta^2}(f^*)^2P_0, \quad (68)$$

so that

$$P_1 = A_1\mathrm{e}^{-\mathrm{i}M_0\xi} - A_0\left(\frac{E\eta_w}{4}f^2M_0 + \frac{\sigma F\eta_w}{4}\frac{\gamma-1}{\vartheta^2}(f^*)^2\frac{1}{M_0}\right)\frac{\xi\mathrm{e}^{-\mathrm{i}M_0\xi}}{1-f}.$$

The constant of integration A_1 must vanish as the pressure condition at $\xi = 0$ is independent of ε.

Hence,

$$P_1 = B_1A_0\xi\mathrm{e}^{-\mathrm{i}M_0\xi}, \quad (69)$$

where

$$B_1 = \frac{-\left(\frac{E\eta_w}{4}f^2M_0+\frac{\sigma F\eta_w}{4}\frac{\gamma-1}{\vartheta^2}(f^*)^2\frac{1}{M_0}\right)}{1-f}. \quad (70)$$

It is obvious that in the absence of velocity slip and temperature jump (i.e. $E = 0 = F$) P_1 and consequently, U_1, ρ_1 and V_1 vanish.

Second order solution

In this case, the equation $\partial p_2/\partial\eta$ is solved first to obtain p_2 as

$$p_2 = P_0\left[-\frac{\eta^2}{4}+\frac{\gamma-1}{\vartheta}\left(1+\alpha-\frac{1}{\sigma}\right)\frac{J_0(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right] - \frac{\eta^2}{4}\frac{\mathrm{d}^2P_0}{\mathrm{d}\xi^2}. \quad (71)$$

Then, the same procedure is followed as for the zero- and first order problems to obtain solutions for U_2, ρ_2 and V_2. These solutions involve lengthy expressions and are not included here(for details of the solution, see Appendix).

P_2 satisfies the non-homogeneous differential equation

$$(1-f)\frac{\mathrm{d}^2P_2}{\mathrm{d}\xi^2}+\left[1+\frac{\gamma-1}{\vartheta}f^*\right]P_2 = L_1\frac{\mathrm{d}^2P_1}{\mathrm{d}\xi^2}+L_2P_1+L_3\frac{\mathrm{d}^4P_0}{\mathrm{d}\xi^4}+L_4\frac{\mathrm{d}^2P_0}{\mathrm{d}\xi^2}+L_5P_0, \quad (72)$$

where

$$L_1 = -\frac{\mathrm{i}E\eta_w}{2}f^2, \quad (73)$$

$$L_2 = \frac{\mathrm{i}\sigma\eta_w}{2\vartheta^2}F(\gamma-1)(f^*)^2, \quad (74)$$

$$L_3 = -(1-f)+\frac{\eta_w^2}{8}-\frac{\eta_w^2}{4}f+\frac{\eta_w^2}{4}f^2, \quad (75)$$

$$L_4 = -E^2\frac{\eta_w^2}{4}f^3+(1+\alpha)(1-f)+\frac{\eta_w^2}{4}\left(\frac{1}{2}-f\right) + \frac{\gamma-1}{\vartheta\sigma}(f^*-f)+\frac{\eta_w^2}{8}+\frac{\gamma-1}{\vartheta\sigma}(1-f^*) + \frac{\gamma-1}{\vartheta}\frac{\eta_w^2}{4}f^*-\frac{\gamma-1}{4\vartheta^2}\eta_w^2(f^*)^2 + \frac{K(\gamma-1)}{\vartheta\sqrt{\sigma}}\frac{\eta_w}{2}\left(1-\frac{1}{\vartheta}\right)f^*, \quad (76)$$

$$L_5 = \frac{\eta_w^2}{8}-\frac{\gamma-1}{\sigma}-\frac{(\gamma-1)^2}{\vartheta}\left(1+\alpha-\frac{1}{\sigma}\right)(1-f^*) + \frac{\gamma-1}{\vartheta}\left[\frac{1}{\sigma}+\frac{\eta_w^2}{4}-\frac{\gamma}{\vartheta}\left(1+\alpha-\frac{1}{\sigma}\right) + \frac{\sigma(\gamma-1)}{4\vartheta}\left(1+\alpha-\frac{1}{\sigma}\right)\eta_w^2f^* + \frac{F^2\sigma^2\eta_w^2}{4\vartheta^2}(f^*)^2\right]f^* + \frac{K(\gamma-1)}{\vartheta\sqrt{\sigma}}\left[\frac{\eta_w}{2}+\frac{\gamma-1}{2\vartheta}\sigma\left(1+\alpha-\frac{1}{\sigma}\right)\eta_w - \frac{\gamma\sigma}{2\vartheta}\left(1+\alpha-\frac{1}{\sigma}\right)\eta_wf^*\right]f^*. \quad (77)$$

The relevant general solution of P_2 is

$$P_2 = A_2\mathrm{e}^{-\mathrm{i}M_0\xi} + \frac{A_0\mathrm{e}^{-\mathrm{i}M_0\xi}}{(1-f)2\mathrm{i}M_0}\left[\left(L_1M_0^2-L_2\right)B_1\left(\frac{\xi^2}{2}+\frac{\xi}{2\mathrm{i}M_0}\right) + \left(2\mathrm{i}M_0B_1L_1-M_0^4L_3+M_0^2L_4-L_5\right)\xi\right]. \quad (78)$$

It may be observed that due to η dependence of p_2, the quantity (P_2+p_2) cannot satisfy the piston-pressure condition at $\xi = 0$ unless the piston wall departs from the plane by a quantity which is ε^2 times a suitable function of η or else one may solve, separately, the problem of the boundary layer on the piston wall and then affect the matching at the edge of the boundary layer. Instead, the pressure condition at $\xi = 0$ is satisfied and work subsequently with

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$(P_2 + \overline{p}_2)$ where

$$\overline{p}_2 = \frac{1}{\pi\eta_w^2}\int_0^{\eta_w} p_2(2\pi\eta)\,\mathrm{d}\eta \qquad (79)$$
$$= A_0 \mathrm{e}^{-\mathrm{i}M_0\xi}\left[\frac{\eta_w^2}{8}(M_0-1) + \frac{\gamma-1}{\vartheta}\Big(1+\alpha-\frac{1}{\sigma}\Big)f^*\right].$$

The constant A_2 of equation (78) is obtained by satisfying the pressure condition at $\xi = 0$.

Hence,

$$A_2 = -A_0\left[\frac{\eta_w^2}{8}(M_0-1) + \frac{\gamma-1}{\vartheta}\Big(1+\alpha-\frac{1}{\sigma}\Big)f^*\right], \qquad (80)$$

and consequently

$$P_2 + \overline{p}_2 = A_0\mathrm{e}^{-\mathrm{i}M_0\xi}\big(B_{21}\xi + B_{22}\xi^2\big), \qquad (81)$$

where

$$B_{21} = \Big[\frac{(L_1M_0^2 - L_2)}{2\mathrm{i}M_0}B_1 + 2\mathrm{i}M_0B_1L_1 - M_0^4L_3 + M_0^2L_4 - L_5\Big] \cdot \Big[(1-f)2\mathrm{i}M_0\Big]^{-1} \qquad (82)$$

and

$$B_{22} = \frac{(L_1M_0^2 - L_2)}{(1-f)4\mathrm{i}M_0}. \qquad (83)$$

7. Recasting as exponential

As stated earlier, the sign of root in equation (64) has to be so chosen that $\Im m\{M_0\} < 0$ i.e. P_0 corresponds to positive damping. Once this choice has been made the next task is to study the sign and the relative magnitude of εP_1 and $\varepsilon^2(P_2 + \overline{p}_2)$. But the forms of equations (69) and (81) do not permit such direct comparison to be possible due to the occurrence of factorsξ and ξ^2. For a proper comparison to be possible one must verify whether

$$P_0 + \varepsilon P_1 + \varepsilon^2(P_2 + \overline{p}_2) + \cdots = A_0\mathrm{e}^{-\mathrm{i}M_0\xi}\big(1 + \varepsilon B_1\xi + \varepsilon^2\big(B_{21} + B_{22}\big)\xi^2 + \dots\big) \qquad (84)$$

can be expressed as

$$A_0\mathrm{e}^{-\mathrm{i}M\xi} = A_0\mathrm{e}^{-\mathrm{i}(M_0+\varepsilon M_1+\varepsilon^2M_2+\dots)\xi}. \qquad (85)$$

The condition that the right hand side of equation (84) can be recast in the form of the right hand side of equation (85) is that

$$B_{22} = B_1^2/2, \qquad (86)$$

and from the expressions for B_1, L_1, L_2 and B_{22} it may be verified that the condition (86) is indeed satisfied.

Hence

$$P_0 + \varepsilon P_1 + \varepsilon^2(P_2 + \overline{p}_2) + \cdots = A_0\mathrm{e}^{-\mathrm{i}(M_0+\varepsilon M_1+\varepsilon^2M_2+\dots)\xi} \qquad (87)$$

where

$$M_1 = \mathrm{i}B_1 \qquad (88)$$

and

$$M_2 = \mathrm{i}B_{21}. \qquad (89)$$

Thus, the problem is reduced to studying the sign and the relative (to $\Im m\{M_0\}$) magnitude of $\Im m\{\varepsilon M_1\}$ and $\Im m\{\varepsilon^2 M_2\}$.

8. Explicit results for wide and narrow tubes

The expressions for B_1 and B_{21} are too involved to separate, for the general case (i.e. for any η_w), the real and imaginary parts. In this section the present study considers the two limiting cases: when $|\eta_w| \gg 1$ and $|\eta_w| \ll 1$. In each case results are obtained explicitly to emphasize the contribution to $\Im m\{M_0\}$, $\Im m\{\varepsilon M_1\}$ and $\Im m\{\varepsilon^2 M_2\}$ of the following factors:

- Factor I: Wall temperature fluctuations,
- Factor II: Velocity slip and temperature jump,
- Factor III: Radial pressure gradient and axial (viscous and heat) dissipation.

The contribution of the factor I begins with M_0, of the factor II with M_1, and of the factor III with M_2. In the next section present study focuses the bounds between which η_w (or equivalently R) must lie in order to satisfy a necessary condition for the validity of the expansion scheme.

8.1. Wide tube

When $|\eta_w| \gg 1$, the relevant expansions are

$$f = \frac{2J_1(\mathrm{i}\eta_w)}{\mathrm{i}\eta_w J_0(\mathrm{i}\eta_w)} = \frac{2}{\eta_w} - \frac{1}{\eta_w^2} - \frac{1}{4\eta_w^3} + \dots,$$
$$f^* = \frac{2J_1(\mathrm{i}\sqrt{\sigma}\eta_w)}{\mathrm{i}\sqrt{\sigma}\eta_w J_0(\mathrm{i}\sqrt{\sigma}\eta_w)} = \frac{2}{\sqrt{\sigma}\eta_w} - \frac{1}{\sigma\eta_w^2} - \frac{1}{4\sigma\sqrt{\sigma}\eta_w^3} + \dots,$$
$$\frac{1}{\vartheta(\eta_w)} = \frac{1}{1+K}\left[1 + \frac{K}{2\sqrt{\sigma}(1+K)\eta_w} + \dots\right],$$

so that

$$\frac{1 + \frac{\gamma-1}{\vartheta}f^*}{1-f} = 1 + \frac{4C_0}{\eta_w} + \frac{3 + \frac{4(\gamma-1)}{\sqrt{\sigma}(1+K)} - \frac{\gamma-1}{\sigma(1+K)^2}}{\eta_w^2} + \dots,$$

where

$$C_0 = \frac{1}{2}\left[1 + \frac{\gamma-1}{\sqrt{\sigma}(1+K)}\right]. \qquad (90)$$

Hence,

$$\Im m\{M_0\} = -\frac{\sqrt{2}C_0}{|\eta_w|} - \frac{1}{|\eta_w^2|}\left[2C_0 - \frac{\gamma(\gamma-1)}{2\sigma(1+K)^2}\right] + \dots \qquad (91)$$

$$\Im m\{\varepsilon M_1\} = \Big|\frac{\varepsilon}{\eta_w^2}\Big|\frac{1}{\sqrt{2}}\left[E(1+2C_0) - \frac{(\gamma-1)F}{(1+K)^2} \cdot \left(\frac{\gamma}{\sqrt{\sigma}(1+K)} - 1\right)\right] + O\Big(\Big|\frac{\varepsilon}{\eta_w^4}\Big|\Big) \qquad (92)$$

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and

$$\Im m\{\varepsilon^2 M_2\} = -\frac{|\varepsilon^2|}{2}\Big[\frac{3}{2}+\alpha+\frac{\gamma-1}{\sigma}+\frac{(\gamma-1)^2}{2\sigma(1+K)^2} + \frac{\gamma-1}{\sqrt{\sigma}(1+K)}\Big] + O\Big(\Big|\frac{\varepsilon^2}{\eta_w}\Big|\Big). \quad (93)$$

$O(|\varepsilon^2/\eta_w|)$ contains (among others) the terms

$$+\Big|\frac{\varepsilon^2}{\eta_w}\Big|\frac{1}{\sqrt{2}}\Big[E^2+\frac{(\gamma-1)\sqrt{\sigma}F^2}{(1+K)^3}\Big]. \quad (94)$$

8.2. Narrow tube

When $\eta_w \ll 1$, the relevant expansions are

$$f = \frac{2J_1(\mathrm{i}\eta_w)}{\mathrm{i}\eta_w J_0(\mathrm{i}\eta_w)} = 1-\frac{\eta_w^2}{8}+\frac{\eta_w^4}{48}+\ldots,$$
$$f^* = \frac{2J_1(\mathrm{i}\sqrt{\sigma}\eta_w)}{\mathrm{i}\sqrt{\sigma}\eta_w J_0(\mathrm{i}\sqrt{\sigma}\eta_w)} = 1-\frac{\sigma\eta_w^2}{8}+\frac{\sigma^2\eta_w^4}{48}+\ldots,$$
$$\frac{1}{\vartheta(\eta_w)} = 1-\frac{K\sqrt{\sigma}\eta_w}{2}+\frac{K^2\sigma\eta_w^2}{4}+\ldots,$$

so that

$$\frac{1+\frac{\gamma-1}{\vartheta}f^*}{1-f} = \frac{8\gamma}{\eta_w^2}\Big[1-\frac{(\gamma-1)K}{2\gamma}\sqrt{\sigma}\eta_w + \Big(\frac{1}{6}-\frac{(\gamma-1)\sigma(1-2K^2)}{8\gamma}\Big)\eta_w^2+\ldots\Big].$$

Hence,

$$\Im m\{M_0\} = -\frac{2\sqrt{\gamma}}{|\eta_w|}+O(|\eta_w|), \quad (95)$$
$$\Im m\{\varepsilon M_1\} = \Big|\frac{\varepsilon}{\eta_w^2}\Big|4E\sqrt{\gamma}+O(|\varepsilon|). \quad (96)$$

$O(|\varepsilon|)$ contains (among others) the term

$$-|\varepsilon|\frac{\gamma-1}{2\sqrt{\gamma}}F\sigma \quad (97)$$

and

$$\Im m\{\varepsilon^2 M_2\} = -\Big|\frac{\varepsilon^2}{\eta_w^3}\Big|12E^2\sqrt{\gamma}-\Big|\frac{\varepsilon^2}{\eta_w}\Big|\gamma\sqrt{\gamma}\Big(\frac{5}{3}+\alpha\Big) + O(|\varepsilon^2\eta_w|). \quad (98)$$

In $O(|\varepsilon^2\eta_w|)$, whose contributions are less, the temperature jump contributes the term

$$+|\varepsilon^2\eta_w|F^2\sigma^2\frac{(\gamma-1)(3\gamma+1)}{16\gamma\sqrt{\gamma}}. \quad (99)$$

Table I. Values of ε for different gases at normal density for a frequency of 1 MHz.

Fluid	ε	Fluid	ε
Air	0.0277	Hydrogen	0.00616
Argon	0.0281	Neon	0.033
Carbon dioxide	0.0256	Nitrogen	0.0272
Helium	0.00374	Oxygen	0.0293

9. Limitations of the present expansion scheme

No effort will be made to answer the question whether the series represented by equations (32), (33), (34) and (35) converge and if so, whether they converge to the Kirchhoff's exact solution. Instead, one may find the limits on the order of magnitude of R such that beyond these limits the present expansion scheme becomes invalid; even this consideration will be limited to the results obtained in the last section. To this end it may be observed that a necessary condition for the validity of the scheme is that

$$\big|\Im m\{\varepsilon^2 M_2\}\big| \ll \big|\Im m\{\varepsilon M_1\}\big| \ll \big|\Im m\{M_0\}\big|. \quad (100)$$

For a wide tube, it is to be noted from the expressions (91), (92) and (93) that while $|\Im m\{\varepsilon M_1\}|$ is always less than $|\Im m\{M_0\}|$, the condition $|\Im m\{\varepsilon^2 M_2\}| \ll |\Im m\{M_0\}|$ stipulates that $|\varepsilon| \ll 1$. Similarly, for a narrow tube one may observe from the expressions (95), (96) and (98) that the two orders of magnitude inequalities of (100) are satisfied if and only if $|\varepsilon| \ll |\eta_w|$ or $\lambda \ll R$. Of course, the condition $\lambda \ll R$ is the natural limit of the continuum theory and $|\varepsilon| \ll 1$ is the assumption proposed while introducing the expansion scheme. The parameter $|\varepsilon|$ may be interpreted either as the ratio of the Stokes layer thickness $\sqrt{\nu/\omega}$ to the wave length (a/ω) or to the square root of the ratio of the frequency of oscillations to that of collisions of the gas molecules.

Table I furnishes numerical values of the parameter ε for gases at normal density for a frequency of 1 MHz.

The present expansion scheme works for frequencies less than 1 MHz. However, the density of the gas is also to be considered before making such a statement. Hence, the frequency alone cannot be a deciding factor for the applicability of the expansion scheme.

10. Discussion

10.1. Wide tube

In the zero order, the only modification is that the Kirchhoff's constant $(1/2)(1+(\gamma-1)/\sqrt{\sigma})$ is changed to C_0 defined by equation (90). The effect of wall temperature fluctuation parameter K is to decrease the thermal contribution to damping. The expression C_0 agrees with those obtained by Henry ([8], cf. his equation 46) and by Weston ([9] cf. his equation 77) except that he has treated K

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as small and equated $1/(1+K)$ to $1-K$. The damping is positive but small, the attenuation being $\exp(-O(|1/\eta_w|))$.

In the first order (cf. expression 92) the slip (as represented by the parameter E) has, as expected, a negative contribution to damping. The sign of contribution of the temperature jump (as represented by the parameter F) depends on the magnitude of K. At ordinary temperature K is small and the temperature jump has positive contribution. When $K > \gamma/\sqrt{\sigma} - 1$, the contribution becomes negative. The contribution of both these factors to the powers of e is $O(|\varepsilon/\eta_w^2|)$. Henry has studied the effect of slip and temperature jump in wide tubes and concluded (cf. his equation 33 and corrected equation 46; as also Weston's statement following his equation 74) that to the order ε, slip and temperature jump make no contribution to damping. They did not carry out the Bessel function expansion to the term $1/\eta_w^2$.

In the second order (cf. expressions 93 and 94), the most dominant term is due to free space classical attenuation; the result is positive damping and the contribution to the powers of e is $O(|\varepsilon^2|)$. The slip and temperature jump, both reduce the damping but their effect is much smaller.

10.2. Narrow tube

In the zero order the damping is very large, the attenuation factor being $\exp(-O(|1/\eta_w|))$ (cf. equation 95). Such highly damped waves are observed in capillary tubes.

In the first order (cf. equations 96 and 97), the slip has a large and negative contribution to damping, the contribution to the argument of exponential being $O(|\varepsilon/\eta_w^2|)$. This factor agrees with that obtained by Weston (cf. his equation 75). The contribution of the temperature jump is positive but of much smaller order.

In the second order (cf. equations 98 and 99), the most dominant term is due to slip, the next being due to radial pressure gradient and finally the temperature jump; the respective signs of contribution to damping are positive, positive and negative.

10.3. Comparison with experimental data

The expressions for the attenuation factor/absorption coefficient m_1 are rewritten for the wide tube case from the results of equations (91), (92) and (93) as

$$\begin{aligned} m_1 = -\frac{\omega}{a}\Bigg[&\frac{\sqrt{\nu}}{R\sqrt{\omega}}\left(\frac{1}{\sqrt{2}}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}\right)\right) \\ &+\frac{\nu}{R^2\omega}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}-\frac{\gamma(\gamma-1)}{2\sigma}\right) \\ &-\frac{\nu\sqrt{\nu}}{R^2\sqrt{\omega}a}\sqrt{\frac{\pi\gamma}{4}}\left(\left(2+\frac{\gamma-1}{\sqrt{\sigma}}\right)\left(\frac{2-\alpha_1}{\alpha_1}\right)\right. \\ &\left.-(\gamma-1)\frac{2\gamma}{\sigma(\gamma+1)}\left(\frac{\gamma}{\sqrt{\sigma}}-1\right)\left(\frac{2-\beta}{\beta}\right)\right) \\ &+\frac{1}{2}\frac{\omega\nu}{a^2}\left(\frac{3}{2}+\alpha+\frac{\gamma-1}{\sigma}+\frac{(\gamma-1)^2}{2\sigma}+\frac{\gamma-1}{\sqrt{\sigma}}\right)\Bigg] \end{aligned} \tag{101}$$

It is to be noted that the expressions given in equation (94) are not included as there are some more terms of the order $O(|\varepsilon^2/\eta_w|)$ in the second order solutions. Also, the parameter K is neglected as its numerical value is small at room temperature (typical vaues are 0.00026 for Ar contained in brass tube and 0.0004 for N_2 contained in stainless tube).

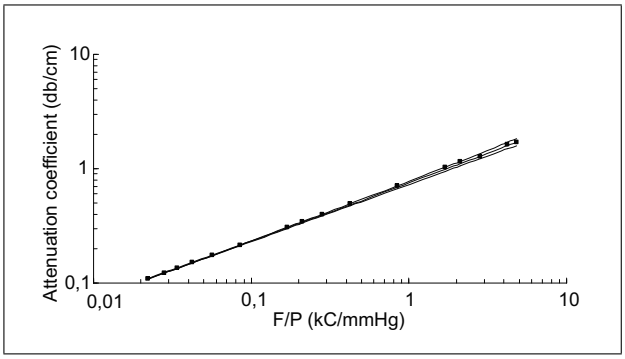


Figure 1. Sound absorption in argon (25.6 C). The measurements shown by filled squares were taken from reference[15]. The lower curve represents equation (102). The middle curve represents equation (104). The upper curve is obtained from equation (105).

In seeking to compare the results of present work with the experimental data, it is usual to include the contributions of Kirchhoff tube absorption coefficient α_t and classical absorption coefficient α_c. They are

$$\alpha_t = -\frac{\omega}{a}\left[\frac{\sqrt{\nu}}{R\sqrt{\omega}}\left(\frac{1}{\sqrt{2}}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}\right)\right)\right], \tag{102}$$

$$\alpha_c = -\frac{\omega}{a}\left[\frac{\omega\nu}{a^2}\frac{1}{2}\left(1+\alpha+\frac{\gamma-1}{\sigma}\right)\right]. \tag{103}$$

The total absorption in a tube is the sum of tube absorption and the classical absorption,

$$\begin{aligned} \alpha_{\text{total}} = \alpha_t + \alpha_c = -\frac{\omega}{a}\Bigg[&\frac{\sqrt{\nu}}{R\sqrt{\omega}}\left(\frac{1}{\sqrt{2}}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}\right)\right) \\ &+\frac{\omega\nu}{a^2}\frac{1}{2}\left(1+\alpha+\frac{\gamma-1}{\sigma}\right)\Bigg]. \end{aligned} \tag{104}$$

The contributions of the velocity slip and the temperature jump corrections to the attenuation rate are found to be less than 1 per cent at the pressure 3.5 mm Hg. Hence, the appropriate expressions for computing the theoretical values are as

$$\begin{aligned} m_1 = -\frac{\omega}{a}\Bigg[&\frac{\sqrt{\nu}}{R\sqrt{\omega}}\left(\frac{1}{\sqrt{2}}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}\right)\right) \\ &+\frac{\nu}{R^2\omega}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}-\frac{\gamma(\gamma-1)}{2\sigma}\right) \\ &+\frac{1}{2}\frac{\omega\nu}{a^2}\left(\frac{3}{2}+\alpha+\frac{\gamma-1}{\sigma}+\frac{(\gamma-1)^2}{2\sigma}+\frac{\gamma-1}{\sqrt{\sigma}}\right)\Bigg]. \end{aligned} \tag{105}$$

The attenuation rate obtained from the results of equation (105) is compared with the experimental data taken from the reference [15] for the gas argon in Figure 1. The lower curve is obtained from the Kirchhoff tube absorption, equation (102) and the middle curve is obtained from

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equation (104). The upper curve is obtained from equation (105).

The expressions found in equations (101), (102) and (104) are again rewritten in dimensionless form as

$$\begin{aligned}\left(\frac{a}{\omega}m_1\right) = -\Bigg[&\frac{\sqrt{\nu}}{R\sqrt{\omega}}\left(\frac{1}{\sqrt{2}}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}\right)\right)\\ &+\frac{\nu}{R^2\omega}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}-\frac{\gamma(\gamma-1)}{2\sigma}\right)\\ &+\frac{\nu}{R^2\omega}\sqrt{\frac{\omega\nu}{a^2}}\sqrt{\frac{\pi\gamma}{4}}\left((\gamma-1)\frac{2\gamma}{\sigma(\gamma+1)}\right.\\ &\left.\cdot\left(\frac{\gamma}{\sqrt{\sigma}}-1\right)\left(\frac{2-\beta}{\beta}\right)\right)\\ &-\frac{\nu}{R^2\omega}\sqrt{\frac{\omega\nu}{a^2}}\sqrt{\frac{\pi\gamma}{4}}\left(\left(2+\frac{\gamma-1}{\sqrt{\sigma}}\right)\left(\frac{2-\alpha_1}{\sigma_1}\right)\right)\\ &+\frac{1}{2}\frac{\omega\nu}{a^2}\left(\frac{3}{2}+\alpha+\frac{\gamma-1}{\sigma}+\frac{(\gamma-1)^2}{2\sigma}+\frac{\gamma-1}{\sqrt{\sigma}}\right)\Bigg],\end{aligned} \tag{106}$$

$$\left(\frac{a}{\omega}\alpha_t\right) = -\frac{\sqrt{\nu}}{R\sqrt{\omega}}\frac{1}{\sqrt{2}}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}\right), \tag{107}$$

$$\begin{aligned}\left(\frac{a}{\omega}\alpha_{\text{total}}\right) = -\Bigg[&\frac{\sqrt{\nu}}{R\sqrt{\omega}}\left(\frac{1}{\sqrt{2}}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}\right)\right)\\ &+\frac{\omega\nu}{a^2}\frac{1}{2}\left(1+\alpha+\frac{\gamma-1}{\sigma}\right)\Bigg].\end{aligned} \tag{108}$$

The dimensionless reduced absorption coefficient am_1/ω is related to the dimensionless frequency variable $a^2/\omega\nu$ that has the nature of a Reynolds number. In comparing the results with those of reference [16], the numerical values for the accommodation coefficients α_1 and β are the same as assumed in that reference. The dimensionless reduced absorption coefficient calculated from the analytical results of equation (106) is compared with the experimental values taken from the reference [16] for Ar in Figure 2. The lower curve in Figure 2 is obtained from the Kirchhoff tube absorption, equation (107) and the middle curve is obtained from equation (108). The upper curve is drawn by assuming α_1 of 1.0 and β of 0.9 in the expressions given in equation (106). Figure 3 shows the dimensionless reduced absorption in N_2. The lower curve is obtained from the Kirchhoff tube absorption, equation (107) and the middle curve is obtained from equation (108). The upper curve is drawn by assuming α_1 of 1.0 and β of 0.9 in the expressions given in equation (106). The physical properties of the gases are taken from Barron's book [19]. In the experimental results shown in Figures 2 and 3, the numerical value of the parameter $|\varepsilon|$ changes from 0.038 to 0.18 as the pressure is varied from 3 mm to 0.1 mm Hg. Hence, our assumption that $|\varepsilon| \ll 1$ is ensured. Typical values of mean free path for argon are 0.01769 mm at 3 mm Hg and 0.53086 mm at 0.1 mm Hg. As the tube radius is 1.27 cm, the condition $\lambda \ll R$ is also fulfilled.

The good agreement between the theoretical predictions and the experimental values suggests that the Kirchhoff solutions need corrections for acoustic wave propagation in rarefied gases in cylindrical tubes. The analytical approxi-

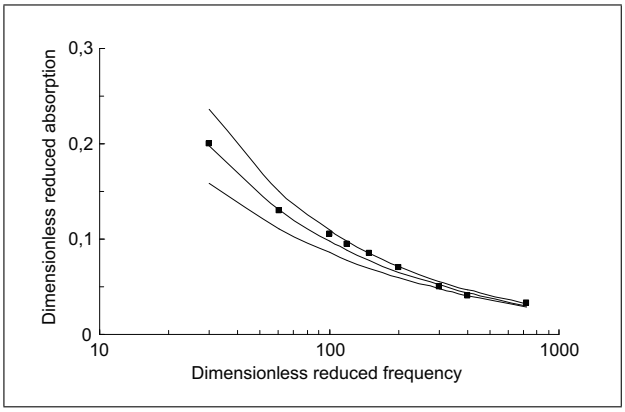


Figure 2. Sound absorption in argon. The measurements shown by filled squares were taken from reference [16]. The lower curve represents equation (107). The middle curve represents equation (108). The upper curve is obtained from equation (106).

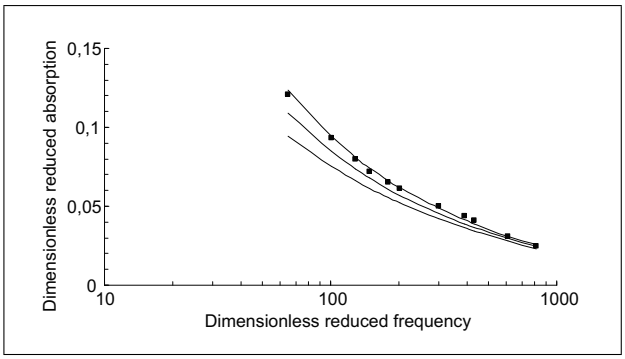


Figure 3. Sound absorption in nitrogen. The measurements shown by filled squares were taken from reference [16]. The lower curve represents equation (107). The middle curve represents equation (108). The upper curve is obtained from equation (106).

mations with the leading terms of the zero-, first- and second order of the expansion scheme provide a quick method of checking each result with the experimental values. In handling with the data found in references [15] and [16], conservative use of the experimental values has been made so that our results would not be in any way exaggerated.

10.4. Comparison with other models and theoretical results

As stated in the introduction, the results obtained in the present work are identical with the analytical expressions derived in reference [10] for both the wide and narrow tubes. Of course, we have retained three terms in the asymptotic expansions of the Bessel's functions for the wide and narrow tubes in the present method and in the method detailed in reference [10]. The expressions for the attenuation coefficient are furnished below for the wide tube case from the results of equations (91), (92) and (93)

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for making a comparison with other theoretical results:

$$
\begin{aligned}
m_1 = &-\frac{\omega}{a}\left[\frac{\sqrt{\nu}}{R\sqrt{\omega}}\left(\frac{1}{\sqrt{2}}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}\right)\right)\right.\\
&+\frac{\nu}{R^2\omega}\left(1+\frac{\gamma-1}{\sqrt{\sigma}}-\frac{\gamma(\gamma-1)}{2\sigma}\right)\\
&-\frac{\nu\sqrt{\nu}}{R^2\sqrt{\omega}a}\sqrt{\frac{\pi\gamma}{4}}\left(\left(2+\frac{\gamma-1}{\sqrt{\sigma}}\right)\left(\frac{2-\alpha_1}{\sigma_1}\right)\right)\\
&+\frac{\nu\sqrt{\nu}}{R^2\sqrt{\omega}a}\sqrt{\frac{\pi\gamma}{4}}\left((\gamma-1)\frac{2\gamma}{\sigma(\gamma+1)}\left(\frac{\gamma}{\sqrt{\sigma}}-1\right)\left(\frac{2-\beta}{\beta}\right)\right)\\
&+\frac{1}{2}\frac{\omega\nu}{a^2}\left(1+\alpha+\frac{\gamma-1}{\sigma}\right)\\
&\left.+\frac{1}{2}\frac{\omega\nu}{a^2}\left(\frac{1}{2}+\frac{\gamma-1}{\sqrt{\sigma}}+\frac{(\gamma-1)^2}{2\sigma}\right)\right].
\end{aligned}
$$

The first term is the Kirchhoff tube absorption and the second term tallies with the results reported in reference [20] (cf. their equation 17). The fifth term, which represents the Kirchhoff's classical attenuation, and the sixth term, which arises because the sound energy is not uniformly distributed over the tube, are in agreement with the result obtained by Weston (cf. his equation 60).

Zwikker and Kosten [11] derived expressions for the propagation constant under extreme conditions of 'low' and 'high' frequencies. An expression for the intermediate frequency range was also given and the equation (following the notation of [11]) is as

$$
\gamma = \frac{\mathrm{j}\omega}{c_0}\sqrt{\frac{1+\frac{2}{B\mu\sqrt{-\mathrm{j}}}(\kappa-1)\frac{J_1(B\mu\sqrt{-\mathrm{j}})}{J_0(B\mu\sqrt{-\mathrm{j}})}}{1-\frac{2}{\mu\sqrt{-\mathrm{j}}}\frac{J_1(\mu\sqrt{-\mathrm{j}})}{J_0(\mu\sqrt{-\mathrm{j}})}}}.
$$

The expression derived for the propagation constant by Stinson [12] (cf. his equation (43) is

$$
\begin{aligned}
m^2 = &-\frac{\omega^2}{c^2}\left[1+2(\gamma-1)(-\mathrm{i}\omega\gamma/\nu^1)^{-1/2}\right.\\
&\left.\cdot J_1\left(r_w(-\mathrm{i}\omega\gamma/\nu^1)^{1/2}\right)\Big/J_0\left(r_w(-\mathrm{i}\omega\gamma/\nu^1)^{1/2}\right)/r_w\right]\\
&\cdot\left[1-2(-\mathrm{i}\omega/\nu)^{-1/2}J_1\left(r_w(-\mathrm{i}\omega/\nu)^{1/2}\right)\right.\\
&\left.\Big/J_0\left(r_w(-\mathrm{i}\omega/\nu)^{1/2}\right)/r_w\right]^{-1}.
\end{aligned}
$$

The above two expressions tally with the zero order solution of the present expansion scheme given by the equation (64), i.e.

$$
M_0 = \sqrt{\frac{1+\frac{(\gamma-1)2J_1(\mathrm{i}\sqrt{\sigma}\eta_w)}{\mathrm{i}\sqrt{\sigma}\eta_w J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}}{1-\frac{2J_1(\mathrm{i}\eta_w)}{\mathrm{i}\eta_w J_0(\mathrm{i}\eta_w)}}}.
$$

As a test case, the last equation is numerically solved using MAT LAB tools for obtaining the imaginary(representing attenuation) and real(representing phase shift) parts of the propagation constant M_0 for the general case(i.e. $|\eta_w|$ or $R\sqrt{\omega/\nu}$ ranging from 0.2 to 100). The results so obtained are shown in Table II and compared with the exact numerical results of Tijdeman [13] for the long wave solution, i.e. $R \ll a/\omega$ or $R\omega/a \ll 1$. In addition, the imaginary and real parts of the propagation constants M_0 and M_2 are obtained by solving the equations (64) and (82) numerically using MATLAB tools for a few cases when the exact numerical results of Tijdeman are available in the reference [13]. The results are shown in Table III. We have not included the corrections due to the velocity slip and temperature jump in the above cases. A good agreement is obtained between the results of reference [13] and the present results. The accuracy of present results depends on the smallness of the parameter ε which in our case is the ratio of $(R\omega/a)/|\eta_w|$. Further study will be concentrated on the numerical evaluation of real and imaginary parts of equations (64), (70) and (82) for other cases and a detailed comparison with the results of Tijdeman [13] and Shields *et al.* [15] are planned for the future work.

Table II. Comparison of numerical results from reference [13] and results of present method for Attenuation Γ' and Phase shift Γ'' for long wave solution, $R \ll a/\omega$ or $R\omega/a \ll 1$.

$\lvert\eta_w\rvert$	Γ'	Present	Γ''	Present
0.2	11.7988	11.7990	11.8657	11.8660
0.4	5.8497	5.8497	5.9835	5.9835
0.6	3.8456	3.8456	4.0461	4.0461
0.8	2.8288	2.8288	3.0958	3.0958
1.0	2.2083	2.2083	2.5411	2.5411
1.2	1.7875	1.7875	2.1849	2.1849
1.4	1.4824	1.4824	1.9423	1.9423
1.6	1.2512	1.2513	1.7708	1.7708
1.8	1.0709	1.0709	1.6463	1.6463
2.0	0.9272	0.92723	1.5540	1.5540
2.2	0.8112	0.81115	1.4843	1.4843
2.4	0.7163	0.71627	1.4308	1.4308
2.6	0.6380	0.63798	1.3889	1.3889
2.8	0.5728	0.57281	1.3556	1.3556
3.0	0.5181	0.51813	1.3287	1.3286
3.2	0.4719	0.47194	1.3064	1.3064
3.4	0.4326	0.43264	1.2878	1.2878
3.6	0.3990	0.39901	1.2720	1.2720
3.8	0.3701	0.37006	1.2583	1.2583
4.0	0.3450	0.34497	1.2462	1.2463
5.0	0.2583	0.25830	1.2012	1.2012
6.0	0.2073	0.20729	1.1699	1.1699
7.0	0.1732	0.17322	1.1466	1.1466
8.0	0.1487	0.14872	1.1288	1.1288
9.0	0.1302	0.13024	1.1148	1.1148
10.0	0.1158	0.11584	1.1035	1.1035
12.0	0.0948	0.094837	1.0865	1.0864
14.0	0.0803	0.080270	1.0742	1.0742
16.0	0.0696	0.069577	1.0650	1.0650
18.0	0.0614	0.061395	1.0578	1.0578
20.0	0.0549	0.054934	1.0520	1.0520
40.0	0.0268	0.026756	1.0261	1.0261
60.0	0.0177	0.017683	1.0174	1.0174
80.0	0.0132	0.013205	1.0130	1.0130
100.0	0.0105	0.010537	1.0104	1.0104

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Table III. Comparison of results of Tijdeman, T, [13] and results of present method, p.

$\lvert\eta_w\rvert$	$R\omega/a \ll 1$	$R\omega/a = 0.05\pi$		$R\omega/a = 0.15\pi$		$R\omega/a = 0.25\pi$		$R\omega/a = 0.35\pi$	
		T	p	T	p	T	p	T	p
(a) Attenuation									
2.6069	0.6355	0.6425	0.6425	0.6959	0.6984	0.7867	0.8100	0.8735	0.9776
2.9794	0.5233	0.5285	0.5285	0.5686	0.5697	0.6414	0.6522	0.7251	0.7760
3.4759	0.4193	0.4229	0.4230	0.4519	0.4524	0.5068	0.5112	0.5782	0.5995
4.1711	0.3261	0.3286	0.3286	0.3483	0.3485	0.3868	0.3883	0.4407	0.4480
5.2139	0.2453	0.2469	0.2469	0.2592	0.2592	0.2837	0.2840	0.3194	0.3211
6.9519	0.1746	0.1755	0.1754	0.1822	0.1822	0.1957	0.1957	0.2158	0.2159
10.4278	0.1106	0.1110	0.1110	0.1139	0.1139	0.1197	0.1197	0.1284	0.1284
20.8556	0.0526	0.0527	0.0527	0.0534	0.0534	0.0548	0.0548	0.0569	0.0569
(b) Phase shift									
2.6069	1.3877	1.3845	1.3846	1.3551	1.3596	1.2782	1.3098	1.1419	1.2351
2.9794	1.3312	1.3292	1.3293	1.3112	1.3139	1.2640	1.2831	1.1743	1.2369
3.4759	1.2815	1.2804	1.2804	1.2701	1.2716	1.2433	1.2538	1.1902	1.2272
4.1711	1.2370	1.2364	1.2364	1.2310	1.2317	1.2172	1.2223	1.1893	1.2081
5.2139	1.1937	1.1933	1.1934	1.1909	1.1911	1.1846	1.1867	1.1720	1.1799
6.9519	1.1476	1.1475	1.1475	1.1466	1.1466	1.1442	1.1449	1.1397	1.1423
10.4278	1.0993	1.0993	1.0993	1.0990	1.0990	1.0984	1.0986	1.0973	1.0979
20.8556	1.0499	1.0499	1.0499	1.0499	1.0499	1.0498	1.0498	1.0497	1.0498

11. Conclusion

The problem of the damping of acoustic waves excited in a semi-infinite cylindrical tube by a piston acting at one end has been solved by employing an expansion scheme in powers of a small parameter which may be interpreted as either the ratio of Stoke's layer thickness to the wave length or the square root of the ratio of frequency of oscillations to that of collisions of gas molecules. The corrective terms due to wall temperature fluctuations and the slip and temperature jump have also been included. Explicit results for wide and narrow tubes are obtained and compared, when possible, with those obtained by other authors who use Kirchhoff's exact solution as the starting point. The main simplification of the present expansion scheme is that the terms containing the radial pressure gradient and the axial dissipation do not appear till the second order of the scheme.

Appendix

The second order solutions for U_2, ρ_2 and V_2 are as follows:

$$
\begin{aligned}
U_2 = \frac{\mathrm{i}}{a\rho_m}\Bigg\{ & \frac{\mathrm{d}P_2}{\mathrm{d}\xi}\left[1-\frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right] \\
& + \frac{\mathrm{d}P_1}{\mathrm{d}\xi}\left[E\frac{J_1(\mathrm{i}\eta_w)}{J_0(\mathrm{i}\eta_w)}\frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right] \\
& - \frac{\mathrm{d}P_0}{\mathrm{d}\xi}\left[\frac{1}{4}\left(\eta^2-\eta_w^2\frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right)+(1+\alpha)\left(1-\frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right)\right. \\
& \left. + \frac{\gamma-1}{\vartheta\sigma}\left(\frac{J_0(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}-\frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right)-\frac{E^2\eta_w^2}{4}f^2\frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right] \\
& + \frac{\mathrm{d}^3P_0}{\mathrm{d}\xi^3}\left[\frac{\eta_w^2}{4}\frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}-\frac{\mathrm{i}}{2}\left(\frac{\eta J_1(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right.\right. \\
& \left.\left. - \frac{\eta_w J_1(\mathrm{i}\eta_w)}{J_0(\mathrm{i}\eta_w)}\frac{J_0(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right)-\frac{\eta^2}{4}\right]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
\rho_2 = \frac{1}{a^2}\Bigg\{ & P_2\left[1+\frac{\gamma-1}{\vartheta}\frac{J_0(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right] \\
& - P_1\left[\mathrm{i}F\frac{\gamma-1}{\vartheta^2}\frac{\sigma\eta_w}{2}f^*\frac{J_0(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right] \\
& + P_0\left[\frac{\gamma-1}{\sigma}-\frac{\eta^2}{4}-\frac{\mathrm{i}\sqrt{\sigma}}{2\vartheta}(\gamma-1)^2\left(1+\alpha-\frac{1}{\sigma}\right)\frac{\eta J_1(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right. \\
& \quad - \frac{\gamma-1}{\vartheta}\left\langle\frac{1}{\sigma}+\frac{\eta_w^2}{4}+\left(\frac{F\sigma\eta_w f^*}{2\vartheta}\right)^2\right. \\
& \quad \left.\left. + \left(1+\alpha-\frac{1}{\sigma}\right)\left(\frac{\sigma(\gamma-1)}{\vartheta}\frac{\eta_w^2}{4}f^*-\frac{\gamma}{\vartheta}\right)\right\rangle\frac{J_0(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)} \right. \\
& \quad - \frac{K(\gamma-1)}{\vartheta\sqrt{\sigma}}\frac{\eta_w}{2}\left\langle 1+\frac{\sigma}{\vartheta}\left(1+\alpha-\frac{1}{\sigma}\right)\right. \\
& \qquad \left.\left.\cdot\left(\gamma-1-\gamma f^*\right)\right\rangle\frac{J_0(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right] \\
& - \frac{\mathrm{d}^2P_0}{\mathrm{d}\xi^2}\left[\frac{\eta^2}{4}-\frac{(\gamma-1)}{2\vartheta}\left(\frac{\mathrm{i}\eta}{\sqrt{\sigma}}\frac{J_1(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}-\left\langle\frac{\eta_w^2}{2}\left(1-\frac{f^*}{\vartheta}\right)\right.\right.\right. \\
& \qquad \left.\left.\left. + \frac{\eta_w K}{\sqrt{\sigma}}\left(1-\frac{1}{\vartheta}\right)\right\rangle\frac{J_0(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right)\right]\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
V_2 = -\frac{\mathrm{i}}{a\rho_m}\Bigg\{ & \frac{\mathrm{d}^2P_2}{\mathrm{d}\xi^2}\left[\frac{\eta}{2}-\frac{J_1(\mathrm{i}\eta)}{\mathrm{i}J_0(\mathrm{i}\eta_w)}\right] \\
& + P_2\left[\frac{\eta}{2}+\frac{\gamma-1}{\vartheta}\frac{J_1(\mathrm{i}\sqrt{\sigma}\eta)}{\mathrm{i}\sqrt{\sigma}J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right]
\end{aligned}
$$

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$$
\begin{aligned}
&+ \frac{\mathrm{d}^2 P_1}{\mathrm{d}\xi^2} \frac{E J_1(\mathrm{i}\eta_w)}{J_0(\mathrm{i}\eta_w)} \frac{J_1(\mathrm{i}\eta)}{\mathrm{i}J_0(\mathrm{i}\eta_w)} \\
&- P_1 \frac{F(\gamma-1)\sqrt{\sigma}\eta_w}{2\vartheta^2} f^* \frac{J_1(\mathrm{i}\sqrt{\sigma}\eta)}{J_0(\mathrm{i}\sqrt{\sigma}\eta_w)} \\
&+ \frac{\mathrm{d}^4 P_0}{\mathrm{d}\xi^4} \left[-\frac{\eta^3}{16} + \frac{1}{2}\left(\eta + \frac{2\mathrm{i}J_1(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)}\right) \right. \\
&\quad \left. + \frac{\eta_w}{2}\left(\frac{\eta_w}{2\mathrm{i}} + \frac{J_1(\mathrm{i}\eta_w)}{J_0(\mathrm{i}\eta_w)}\right) \frac{J_1(\mathrm{i}\eta)}{J_0(\mathrm{i}\eta_w)} \right] \\
&+ \frac{\mathrm{d}^2 P_0}{\mathrm{d}\xi^2} \left[\left(\frac{E\eta_w f}{2}\right)^2 \frac{J_1(\mathrm{i}\eta)}{\mathrm{i}J_0(\mathrm{i}\eta_w)} - \frac{\eta^3}{16} \right. \\
&\quad - \frac{1+a}{2}\left(\eta - \frac{2J_1(\mathrm{i}\eta)}{\mathrm{i}J_0(\mathrm{i}\eta_w)}\right) + \frac{\eta_w^2}{4} \frac{J_1(\mathrm{i}\eta)}{\mathrm{i}J_0(\mathrm{i}\eta_w)} \\
&\quad - \frac{\gamma-1}{\vartheta\sigma}\left(\frac{\eta}{2} - \frac{J_1(\mathrm{i}\eta)}{\mathrm{i}J_0(\mathrm{i}\eta_w)}\right) - \frac{\eta^3}{16} \\
&- \left\langle \frac{\gamma-1}{\vartheta}\frac{\eta_w^2}{4}\left(1 - \frac{2}{\vartheta\eta_w} \frac{J_1(\mathrm{i}\sqrt{\sigma}\eta_w)}{\mathrm{i}\sqrt{\sigma}J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right) \right. \\
&\quad \left. \left. + \frac{K(\gamma-1)\eta_w}{2\vartheta\sqrt{\sigma}}\left(1 - \frac{1}{\vartheta}\right) \right\rangle \frac{J_1(\mathrm{i}\sqrt{\sigma}\eta)}{\mathrm{i}\sqrt{\sigma}J_0(\mathrm{i}\sqrt{\sigma}\eta_w)} \right] \\
&+ P_0 \left[\frac{\gamma-1}{\sigma}\frac{\eta}{2} - \frac{\eta^3}{16} - \frac{\gamma-1}{\vartheta}\left\langle \frac{1}{\sigma} + \frac{\eta_w^2}{4} \right. \right. \\
&\quad - \left(1 + \alpha - \frac{1}{\sigma}\right)\left(\frac{\gamma}{\vartheta} - \frac{\sigma(\gamma-1)\eta_w^2}{4\vartheta} f^*\right) \\
&\quad \left. \left. + \left(\frac{F\sigma\eta_w}{2\vartheta}\right)^2 (f^*)^2 \right\rangle \frac{J_1(\mathrm{i}\sqrt{\sigma}\eta)}{\mathrm{i}\sqrt{\sigma}J_0(\mathrm{i}\sqrt{\sigma}\eta_w)} \right. \\
&\quad - \frac{K(\gamma-1)\eta_w}{2\vartheta}\left\langle \frac{\sqrt{\sigma}}{\vartheta}\left(1+\alpha-\frac{1}{\sigma}\right)\left(\gamma-1-\gamma f^*\right) + \frac{1}{\sqrt{\sigma}} \right\rangle \\
&\qquad \cdot \frac{J_1(\mathrm{i}\sqrt{\sigma}\eta)}{\mathrm{i}\sqrt{\sigma}J_0(\mathrm{i}\sqrt{\sigma}\eta_w)} \\
&\quad \left. + \frac{(\gamma-1)^2}{2\vartheta}\left(1 + \alpha - \frac{1}{\sigma}\right)\left(\eta + \frac{2\mathrm{i}J_1(\mathrm{i}\sqrt{\sigma}\eta)}{\sqrt{\sigma}J_0(\mathrm{i}\sqrt{\sigma}\eta_w)}\right) \right] \Bigg\}
\end{aligned}
$$

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