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Emergence of Acoustic Green's Functions from Time Averages of Ambient Noise

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Summary
The emergence of deterministic Green's functions from cross-correlation of ambient noise is the physical basis for a number of passive remote sensing techniques. The data-accumulation time necessary for the Green's function emergence is of critical significance for the feasibility and implementation of the passive techniques. This paper re-examines calculation of the necessary data-accumulation times. Previously published estimates are extended to more general ambient noise fields and source-receiver geometries. The emergence time calculation is performed for arbitrary anisotropy of the noise field generated by random acoustic noise sources located on a surface or on a curve. The results apply to a general case of an inhomogeneous, moving or motionless medium with time-independent parameters. It is shown that in certain situations the emergence time estimates are influenced by the geometry (dimensionality) of the noise sources' layout, and not necessarily by the dimensionality of the propagation medium. A simpler technique using basic statistical principles is suggested for calculating the emergence time under rather general assumptions.

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1. Introduction

Passive remote sensing techniques known as noise interferometry have become increasingly popular in recent years after Weaver and Lobkis [1, 2] obtained confirmation of the practical significance of the property of an ambient noise cross-correlation function (CCF) to reproduce the shape of a time-domain Green's function describing propagation between two points in space. A number of publications appeared in diverse research fields, including ultrasonics, geophysics, helioseismology, and acoustical oceanography, covering various applications of these techniques and their theoretical aspects. A good bibliography may be found in [3] and in other recent publications [4, 5, 6, 7, 8, 9].

Experimental results and theoretical estimates show that, in many practical situations, peaks of the cross-correlation function have small amplitudes compared to the noise dispersion. This means that relatively long time series of data are required to reveal the structure of the cross-correlation function. Calculation of the required accumulation time is a central problem for many applications of the noise interferometry. This issue has been addressed by several authors [10, 11, 12, 13, 14]. As summarized in [3], the accumulation time depends on the ratio R/λ of the distance R between the two points and the wavelength λ, and on the dimensionality of the problem. However, the derivations mentioned rely on certain assumptions such as uniformity of the noise sources' distribution, which do not necessarily hold for acoustic noise in the ocean and atmosphere.

In this paper, we re-examine the emergence of acoustic Green's functions from time averages of ambient noise with the goal of improving and extending the existing estimates of the required data-accumulation time. Our theoretical approach in section 3 is similar to that of [12]. But our results are obtained under more general conditions, and some new physics is highlighted in section 4. In particular, unlike [11] and [12], we do not assume that noise sources are uniformly distributed in space. In section 5, we suggest an alternative, simpler technique to obtain the accumulation time estimates. Section 6 summarizes our conclusions.

2. Basic assumptions and relationships

The main idea of noise interferometry is that the ambient noise passing through two points in space, a and b, pro-

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duces a cross-correlation function the shape of which is similar to the shape of the corresponding Green's function describing propagation between a and b.

We will assume that random noise sources $S(\vec{r},t)$ are stationary in the statistical sense and delta correlated in space (but not necessarily in time, thus allowing noise signals with an arbitrary spectrum shape). Then, their second-order statistical moment can be written as

$$\left\langle S(\vec{r}_1,t_1)\,S(\vec{r}_2,t_2)\right\rangle = \varepsilon(\vec{r}_1,t_1-t_2)\,\delta(\vec{r}_1-\vec{r}_2), \tag{1}$$

where the function $\varepsilon(\vec{r}_1,t)$ describes the power density of the noise sources, and the angular brackets denote averaging over the full statistical ensemble of realizations of the noise sources. The acoustic pressure field created at the point a by all noise sources is

$$P(\vec{r}_a,t_1) = \int \mathrm{d}\vec{r}_1 \int \mathrm{d}t_1 S(\vec{r}_1,t_1) G(\vec{r}_1,\vec{r}_a,t-t_1), \tag{2}$$

where the spatial integration is performed over the region of appropriate dimensionality occupied by the noise sources, and $G(\vec{r}_1,\vec{r}_a,t-t_1)$ is the time-domain Green's function, which, according to the causality principle, possesses the following property: $G(\vec{r}_1,\vec{r}_a,t-t_1) = 0$ when $t_1 > t$. If the medium is time-independent (stationary), a practical estimate of the cross-correlation function between the two points, a and b, may be obtained through temporal averaging,

$$C_{a,b;T_r}(\tau) = T_r^{-1}\int_{-T_r/2}^{T_r/2} \mathrm{d}t\, P(\vec{r}_a,t) P(\vec{r}_b,t-\tau). \tag{3}$$

Here, integration over time corresponds to the conditions of a specific experiment, where data accumulation may take place only over finite recording time T_r. Obtaining the exact values of CCF $C_{a,b}(\tau)$, however, requires averaging over the full ensemble of realizations of the noise sources,

$$C_{a,b}(\tau) = \left\langle P(\vec{r}_a,t)\,P(\vec{r}_b,t+\tau)\right\rangle = \left\langle C_{a,b;T_r}(\tau)\right\rangle. \tag{4}$$

The time average (3) is only an approximation to the exact CCF (4), the quality of which depends on the recording length T_r. The CCF estimate obtained from a specific recording includes random variations around its theoretical values, which can also be described statistically. In fact, the condition of smallness of these variations compared to the exact CCF determines the required duration of the data-accumulation time T_r.

Combining equations (1)–(3), we obtain

$$C_{a,b}(\tau) = T_r^{-1}\int_{-T_r/2}^{T_r/2} \mathrm{d}t \int \mathrm{d}\vec{r} \int_{-\infty}^{\infty} \mathrm{d}t_1\, G(\vec{r},\vec{r}_a,t-t_1) \cdot \int_{-\infty}^{\infty} \mathrm{d}t_2\, G(\vec{r},\vec{r}_b,t+\tau-t_2)\,\varepsilon(\vec{r},t_1-t_2). \tag{5}$$

Making the transition to spectral representations of the Green's functions and of the noise intensity,

$$G(\vec{r},\vec{r}_a,t-t_1) = \int_{-\infty}^{\infty} G(\vec{r},\vec{r}_a,\omega_1)\,\mathrm{e}^{-\mathrm{i}\omega_1(t-t_1)}\mathrm{d}\omega_1, \tag{6}$$

$$G(\vec{r},\vec{r}_b,t+\tau-t_2) = \int_{-\infty}^{\infty} G(\vec{r},\vec{r}_b,\omega_2)\,\mathrm{e}^{-\mathrm{i}\omega_2(t+\tau-t_2)}\mathrm{d}\omega_2, \tag{7}$$

$$\varepsilon(\vec{r},t_1-t_2) = \int_{-\infty}^{\infty} \varepsilon(\vec{r},\omega_3)\,\mathrm{e}^{-\mathrm{i}\omega_3(t_1-t_2)}\mathrm{d}\omega_3, \tag{8}$$

$$C_{a,b}(\tau) = \int_{-\infty}^{\infty} C_{a,b}(\omega)\,\mathrm{e}^{-\mathrm{i}\omega\tau}\mathrm{d}\omega, \tag{9}$$

and introducing new variables $u = t - t_1$, $v = t + \tau - t_2$, $v-u-\tau = t_1-t_2$, one can obtain the spectral representation of the CCF in the following form:

$$C_{a,b}(\omega) = (2\pi)^2 \int \mathrm{d}\vec{r}\,\varepsilon(\vec{r},\omega)\,G^*(\vec{r},\vec{r}_a,\omega)\,G(\vec{r},\vec{r}_b,\omega). \tag{10}$$

As has been shown in a number of works, the shape of the CCF approximates the shape of the Green's function describing propagation between points a and b. Usually, the correspondence between the CCF and the Green's function consists of accurate reproduction of travel times of various arrivals, with the amplitudes of the arrivals reproduced with accuracy to some proportionality coefficient. For example, as was shown in [15], for quite general conditions in a stationary moving inhomogeneous medium with noise sources located on a surface S, the temporal spectra of the two are related by

$$G_s(\vec{r}_b,\vec{r}_a,\omega) + G_s^*(\vec{r}_a,\vec{r}_b,\omega) \sim -C_{a,b}(\omega)_s \vec{N}(\vec{r}_s)\cdot\vec{q}(\vec{r}_s)/2\pi^2\bar{\rho}(\vec{r}_s)\,c(\vec{r}_s)\,\varepsilon(\vec{r}_s,\omega), \tag{11}$$

where index s is a reminder that this relationship holds for a specific ray path connecting points $\vec{r}_a$, $\vec{r}_b$, and a stationary point $\vec{r}_s$ of the integral (10) belongs to the surface S. $\vec{N}$ is a unit normal vector to the surface S, $\rho(\vec{r})$ is the mass density, $c(\vec{r})$ is the sound speed, $\vec{q} = (\omega^2/ch)(\vec{h}/h + \vec{u}/c)$, $\vec{h}(\vec{r})$ is the wave vector, $\omega = 2\pi f$, f is the wave frequency, and $\vec{u}(\vec{r})$ is the flow velocity.

If there are several ray paths connecting points a and b, the CCF may have several corresponding maxima in the time domain. We assume that the ray arrivals in the Green's function are resolved in time in the frequency band considered. It is the relation between specific CCF peak amplitude and its variance that determines the necessary data-accumulation time to reveal the peak position and/or shape and to use this information for remote sensing of the propagation medium. The time necessary to accumulate enough data to reveal the maxima of the CCF is critical for this method. The task of evaluating this time has been addressed in several publications [10, 11, 12, 13, 14], and the results were summarized in [3] with the following simple statement

$$T_r\,\delta f \gg (R/\lambda)^{d-1}, \tag{12}$$

where R is the distance between points a and b, λ is the wavelength, δf is the bandwidth of the measuring

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device (receiver) that may not exceed the frequency of the signal's discretization, and d is the dimensionality of the problem ($d = 1$, 2, or 3). In practical applications, $R \gg \lambda$. That is why the dimensionality of the problem is one of the most critical parameters. Most media support three-dimensional wave propagation (e.g., body waves in the Earth's interior, atmosphere, ocean, etc.). There are also important examples of two-dimensional propagation (surface waves in seismology, normal modes in underwater acoustic waveguides, etc.). We will show in section 4 that, in addition to the dimensionality of the propagation medium, the dimensionality of the spatial distribution of the noise sources may impact T_r. In the following section we discuss possible generalizations of the condition (12).

3. Variance of the cross-correlation function

The variance of the cross-correlation function is

$$\mathrm{var}\left[C_{a,b;T_r}(\tau)\right] = \left\langle C^2_{a,b;T_r}(\tau)\right\rangle - C^2_{a,b}(\tau), \tag{13}$$

where $C_{a,b}(\tau)$ is given by equations (9) and (10). By using equations (3) and (2), the following expression can be obtained for $C^2_{a,b;T_r}(\tau)$:

$$\begin{aligned} C^2_{a,b;T_r}(\tau) = T_r^{-2} & \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde{t} \int \mathrm{d}\vec{r}_1 \int \mathrm{d}\vec{r}_2 \int \mathrm{d}\vec{r}_3 \int \mathrm{d}\vec{r}_4 \\ & \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}t_3 \int_{-\infty}^{\infty} \mathrm{d}t_4 \\ & \cdot S(\vec{r}_1, t_1) S(\vec{r}_2, t_2) S(\vec{r}_3, t_3) S(\vec{r}_4, t_4) \\ & \cdot G(\vec{r}_1, \vec{r}_a, t - t_1) G(\vec{r}_2, \vec{r}_b, t + \tau - t_2) \\ & \cdot G(\vec{r}_3, \vec{r}_a, \tilde{t} - t_3) G(\vec{r}_4, \vec{r}_b, \tilde{t} + \tau - t_4). \end{aligned} \tag{14}$$

If the noise sources obey Gaussian statistics, the fourth-order moment of their amplitudes reduces to

$$\begin{aligned} & \left\langle S(\vec{r}_1, t_1) S(\vec{r}_2, t_2) S(\vec{r}_3, t_3) S(\vec{r}_4, t_4)\right\rangle = \\ & \quad \left\langle S(\vec{r}_1, t_1) S(\vec{r}_2, t_2)\right\rangle \left\langle S(\vec{r}_3, t_3) S(\vec{r}_4, t_4)\right\rangle \\ & \quad + \left\langle S(\vec{r}_1, t_1) S(\vec{r}_3, t_3)\right\rangle \left\langle S(\vec{r}_2, t_2) S(\vec{r}_4, t_4)\right\rangle \\ & \quad + \left\langle S(\vec{r}_1, t_1) S(\vec{r}_4, t_4)\right\rangle \left\langle S(\vec{r}_2, t_2) S(\vec{r}_3, t_3)\right\rangle, \end{aligned} \tag{15}$$

thus allowing representation for $\langle C^2_{a,b;T_r}(\tau)\rangle$ as a sum of three terms,

$$\left\langle C^2_{a,b;T_r}(\tau)\right\rangle = I(\tau) + J(\tau) + K(\tau). \tag{16}$$

As was shown in [12], for the noise sources uncorrelated both in space and in time the first term in equation (16), $I(\tau)$, always equals $C^2_{a,b}(\tau)$ and is cancelled in the right-hand side of equation (13) by the second term, while $J(\tau)$ always dominates over $K(\tau)$. Let us show how the same result can be obtained without the assumption of delta-correlation in time. For $I(\tau)$ we have

$$\begin{aligned} I(\tau) = T_r^{-2} & \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde{t} \int \mathrm{d}\vec{r}_1 \int \mathrm{d}\vec{r}_2 \int \mathrm{d}\vec{r}_3 \int \mathrm{d}\vec{r}_4 \\ & \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}t_3 \int_{-\infty}^{\infty} \mathrm{d}t_4 \\ & \cdot \left\langle S(\vec{r}_1, t_1) S(\vec{r}_2, t_2)\right\rangle \left\langle S(\vec{r}_3, t_3) S(\vec{r}_4, t_4)\right\rangle \\ & \cdot G(\vec{r}_1, \vec{r}_a, t - t_1)\, G(\vec{r}_2, \vec{r}_b, t + \tau - t_2) \\ & \cdot G(\vec{r}_3, \vec{r}_a, \tilde{t} - t_3)\, G(\vec{r}_4, \vec{r}_b, \tilde{t} + \tau - t_4) \\ = T_r^{-2} & \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde{t} \int \mathrm{d}\vec{r}_1 \int \mathrm{d}\vec{r}_2 \int \mathrm{d}\vec{r}_3 \int \mathrm{d}\vec{r}_4 \\ & \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}t_3 \int_{-\infty}^{\infty} \mathrm{d}t_4 \\ & \cdot \varepsilon(\vec{r}_1, t_1 - t_2)\, \delta(\vec{r}_1 - \vec{r}_2) \\ & \cdot \varepsilon(\vec{r}_3, t_3 - t_4)\, \delta(\vec{r}_3 - \vec{r}_4) \\ & \cdot G(\vec{r}_1, \vec{r}_a, t - t_1)\, G(\vec{r}_2, \vec{r}_b, t + \tau - t_2) \\ & \cdot G(\vec{r}_3, \vec{r}_a, \tilde{t} - t_3)\, G(\vec{r}_4, \vec{r}_b, \tilde{t} + \tau - t_4) \\ = T_r^{-2} & \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde{t} \int \mathrm{d}\vec{r}_1 \int \mathrm{d}\vec{r}_3 \\ & \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}t_3 \int_{-\infty}^{\infty} \mathrm{d}t_4 \\ & \cdot \varepsilon(\vec{r}_1, t_1 - t_2)\, \varepsilon(\vec{r}_3, t_3 - t_4) \\ & \cdot G(\vec{r}_1, \vec{r}_a, t - t_1)\, G(\vec{r}_1, \vec{r}_b, t + \tau - t_2) \\ & \cdot G(\vec{r}_3, \vec{r}_a, \tilde{t} - t_3)\, G(\vec{r}_3, \vec{r}_b, \tilde{t} + \tau - t_4) \\ = T_r^{-2} & \left\{ \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int \mathrm{d}\vec{r}_1 \int_{-\infty}^{\infty} \mathrm{d}t_1\, G(\vec{r}_1, \vec{r}_a, t - t_1) \right. \\ & \left. \cdot \int_{-\infty}^{\infty} \mathrm{d}t_2\, G(\vec{r}_1, \vec{r}_b, t + \tau - t_2)\, \varepsilon(\vec{r}_1, t_1 - t_2) \right\} \\ & \cdot \left\{ \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde{t} \int \mathrm{d}\vec{r}_3 \int_{-\infty}^{\infty} \mathrm{d}t_3\, G(\vec{r}_3, \vec{r}_a, \tilde{t} - t_3) \right. \\ & \left. \cdot \int_{-\infty}^{\infty} \mathrm{d}t_4\, G(\vec{r}_3, \vec{r}_b, \tilde{t} + \tau - t_4) \varepsilon(\vec{r}_3, t_3 - t_4) \right\} \\ = C^2_{a,b}&(\tau). \end{aligned} \tag{17}$$

The last equality follows from a comparison with equation (5). For $J(\tau)$ we have

$$\begin{aligned} J(\tau) = T_r^{-2} & \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde{t} \int \mathrm{d}\vec{r}_1 \int \mathrm{d}\vec{r}_2 \int \mathrm{d}\vec{r}_3 \int \mathrm{d}\vec{r}_4 \\ & \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}t_3 \int_{-\infty}^{\infty} \mathrm{d}t_4 \\ & \cdot \left\langle S(\vec{r}_1, t_1) S(\vec{r}_3, t_3)\right\rangle \left\langle S(\vec{r}_2, t_2) S(\vec{r}_4, t_4)\right\rangle \\ & \cdot G(\vec{r}_1, \vec{r}_a, t - t_1)\, G(\vec{r}_2, \vec{r}_b, t + \tau - t_2) \\ & \cdot G(\vec{r}_3, \vec{r}_a, \tilde{t} - t_3)\, G(\vec{r}_4, \vec{r}_b, \tilde{t} + \tau - t_4) \\ = T_r^{-2} & \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde{t} \int \mathrm{d}\vec{r}_1 \int \mathrm{d}\vec{r}_2 \int \mathrm{d}\vec{r}_3 \int \mathrm{d}\vec{r}_4 \\ & \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}t_3 \int_{-\infty}^{\infty} \mathrm{d}t_4 \\ & \cdot \varepsilon(\vec{r}_1, t_1 - t_3)\, \delta(\vec{r}_1 - \vec{r}_3) \end{aligned}$$

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$$
\begin{aligned}
&\cdot\, \varepsilon(\vec r_2, t_2 - t_4)\, \delta(\vec r_2 - \vec r_4)\\
&\cdot\, G(\vec r_1, \vec r_a, t - t_1)\, G(\vec r_2, \vec r_b, t + \tau - t_2)\\
&\cdot\, G(\vec r_3, \vec r_a, \tilde t - t_3)\, G(\vec r_4, \vec r_b, \tilde t + \tau - t_4)\\
=\ & T_r^{-2} \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde t \int \mathrm{d}\vec r_1 \int \mathrm{d}\vec r_2\\
&\int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}t_3 \int_{-\infty}^{\infty} \mathrm{d}t_4 \qquad (18)\\
&\cdot\, \varepsilon(\vec r_1, t_1 - t_3)\, \varepsilon(\vec r_2, t_2 - t_4)\\
&\cdot\, G(\vec r_1, \vec r_a, t - t_1)\, G(\vec r_2, \vec r_b, t + \tau - t_2)\\
&\cdot\, G(\vec r_1, \vec r_a, \tilde t - t_3)\, G(\vec r_2, \vec r_b, \tilde t + \tau - t_4)\\
=\ & T_r^{-2} \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde t \Big\{ \int \mathrm{d}\vec r_1 \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_3\\
&\cdot\, G(\vec r_1, \vec r_a, t - t_1)\, G(\vec r_1, \vec r_a, \tilde t - t_3) \varepsilon(\vec r_1, t_1 - t_3) \Big\}\\
&\cdot \Big\{ \int \mathrm{d}\vec r_2 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}t_4\\
&\cdot\, G(\vec r_2, \vec r_b, t + \tau - t_2)\, G(\vec r_2, \vec r_b, \tilde t + \tau - t_4) \varepsilon(\vec r_2, t_2 - t_4) \Big\}.
\end{aligned}
$$

After the substitutions of variables $u = t - t_1$, $v = \tilde t - t_3$, $\vec r_1 = \vec r$ in the first brackets, and $u = t + \tau - t_2$, $v = \tilde t + \tau - t_4$, $\vec r_2 = \vec r$ in the second brackets, it becomes evident that the two expressions in the brackets differ only by the indices a and b, and the entire expression (18) does not depend on τ,

$$
\begin{aligned}
J(\tau) =\ & T_r^{-2} \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde t \Big\{ \int \mathrm{d}\vec r \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}v\\
&G(\vec r, \vec r_a, u)\, G(\vec r, \vec r_a, v) \varepsilon(\vec r, t - \tilde t - u + v) \Big\}\\
&\cdot \Big\{ \int \mathrm{d}\vec r \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}v \qquad (19)\\
&G(\vec r, \vec r_b, u)\, G(\vec r, \vec r_b, v) \varepsilon(\vec r, t - \tilde t - u + v) \Big\}\\
=\ & T_r^{-2} \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde t \Big\{ .. \Big\}_a \Big\{ .. \Big\}_b = \mathrm{const}(\tau).
\end{aligned}
$$

Now it is convenient to apply Fourier transforms similar to equations (6)–(8) and to compare the result with equations (9)–(10):

$$
\begin{aligned}
\Big\{ .. \Big\}_a =\ & (2\pi)^2 \int \mathrm{d}\vec r \int_{-\infty}^{\infty} \mathrm{d}\omega\\
&\cdot\, G^*(\vec r, \vec r_a, \omega)\, G(\vec r, \vec r_a, \omega) \varepsilon(\vec r, \omega)\, \mathrm{e}^{-\mathrm{i}\omega(t - \tilde t)}\\
=\ & C_{a,a}(t - \tilde t). \qquad (20)
\end{aligned}
$$

Here, $C_{a,a}(t - \tilde t)$ denotes the autocorrelation function (ACF) of the ambient noise at point a. Similarly, $\{..\}_b = C_{b,b}(t - \tilde t)$. If follows from the definition (4) that $C_{a,a}(t - \tilde t)$ and $C_{b,b}(t - \tilde t)$ are even functions of their temporal arguments as long as the propagation medium is time-independent (but not necessarily motionless) and the noise sources are stationary (in the statistical sense). Therefore, the product $f(q) = \{..\}_a\{..\}_b = C_{a,a}(q)C_{b,b}(q)$ is also an even function of its argument $q = t - \tilde t$. Making the transition to new variables q and $p = t + \tilde t$ in equation (19), one obtains

$$
\begin{aligned}
J(\tau) =\ & T_r^{-2} \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde t\, f(t - \tilde t)\\
=\ & \frac{1}{2} T_r^{-2} \Big[\int_{-T_r}^{0} \mathrm{d}q f(q) \int_{-T_r - q}^{T_r + q} \mathrm{d}p\\
&+ \int_{0}^{T_r} \mathrm{d}q f(q) \int_{q - T_r}^{T_r - q} \mathrm{d}p \Big] \qquad (21)\\
=\ & \frac{1}{T_r} \int_{-T_r}^{T_r} \mathrm{d}q f(q) - \frac{1}{2T_r^2} \int_{0}^{T_r} \mathrm{d}q\, q\, f(q)\\
\simeq\ & \frac{1}{T_r} \int_{-T_r}^{T_r} \mathrm{d}q f(q) = \frac{1}{T_r} \int_{-T_r}^{T_r} \mathrm{d}q C_{a,a}(q)\, C_{b,b}(q).
\end{aligned}
$$

The contribution of the second integral in equation (21) compared to the first one is $O(T_{\mathrm{corr}}/T_r) \ll 1$, where T_{corr} is the correlation scale of the ACF defined as

$$
T_{\mathrm{corr}}^m = \int_0^{\infty} \mathrm{d}q\, C_{m,m}(q)/C_{m,m}(0), \quad m = a, b.
$$

Now let us evaluate the third term in the right-hand side of equation (16), $K(\tau)$. After integration over $\vec r_4$ and $\vec r_3$, we obtain

$$
\begin{aligned}
K(\tau) =\ & T_r^{-2} \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde t \Big\{ \int \mathrm{d}\vec r_1 \int_{-\infty}^{\infty} \mathrm{d}t_1 \int_{-\infty}^{\infty} \mathrm{d}t_4\\
&\cdot\, G(\vec r_1, \vec r_a, t - t_1)\, G(\vec r_1, \vec r_b, \tilde t + \tau - t_4) \varepsilon(\vec r_1, t_1 - t_4) \Big\}\\
&\cdot \Big\{ \int \mathrm{d}\vec r_2 \int_{-\infty}^{\infty} \mathrm{d}t_2 \int_{-\infty}^{\infty} \mathrm{d}t_3 \qquad (22)\\
&\cdot\, G(\vec r_2, \vec r_a, \tilde t - t_3)\, G(\vec r_2, \vec r_b, t + \tau - t_2) \varepsilon(\vec r_2, t_2 - t_3) \Big\}.
\end{aligned}
$$

After the substitutions of variables $u = t - t_1$, $v = \tilde t + \tau - t_4$, $\vec r_1 = \vec r$ in the first bracket, and $u = \tilde t - t_3$, $v = t + \tau - t_2$, $\vec r_2 = \vec r$ in the second bracket, and introducing $q = t - \tilde t$, the expression for $K(\tau)$ becomes

$$
\begin{aligned}
K(\tau) =\ & T_r^{-2} \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde t \Big\{ \int \mathrm{d}\vec r \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}v\\
&\cdot\, G(\vec r, \vec r_a, u)\, G(\vec r, \vec r_b, v) \varepsilon(\vec r, q - u + v) \Big\}\\
&\cdot \Big\{ \int \mathrm{d}\vec r \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}v \qquad (23)\\
&\cdot\, G(\vec r, \vec r_a, u)\, G(\vec r, \vec r_b, v) \varepsilon(\vec r, q + u - v + \tau) \Big\}.
\end{aligned}
$$

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Now we apply the Fourier transforms similarly to what was done earlier, when deriving equation (20), to obtain

$$K(\tau) = (2\pi)^4 T_r^{-2} \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde{t} \Big\{ \int \mathrm{d}\vec{r} \int_{-\infty}^{\infty} \mathrm{d}\omega \cdot G^*(\vec{r}, \vec{r}_a, \omega)\, G(\vec{r}, \vec{r}_b, \omega) \varepsilon(\vec{r}, \omega)\, \mathrm{e}^{-\mathrm{i}\omega q} \Big\} \cdot \Big\{ \int \mathrm{d}\vec{r} \int_{-\infty}^{\infty} \mathrm{d}\omega \cdot G(\vec{r}, \vec{r}_a, \omega)\, G^*(\vec{r}, \vec{r}_b, \omega) \varepsilon(\vec{r}, \omega)\, \mathrm{e}^{-\mathrm{i}\omega(q+\tau)} \Big\} = T_r^{-2} \int_{-T_r/2}^{T_r/2} \mathrm{d}t \int_{-T_r/2}^{T_r/2} \mathrm{d}\tilde{t}\, C_{a,b}(q)\, C_{b,a}(q+\tau). \quad (24)$$

The last equality follows from comparison with equation (10). This expression can be further simplified in the same way it was done with equation (21):

$$K(\tau) = T_r^{-2} \left[T_r \int_{-T_r}^{T_r} \mathrm{d}q + \int_{-T_r}^{0} \mathrm{d}q\, q - \int_{0}^{T_r} \mathrm{d}q\, q \right] \cdot C_{a,b}(q)\, C_{b,a}(q+\tau) \quad (25)$$
$$= \frac{1}{T_r} \left[1 + O\Big(\frac{q_{\max}}{T_r} \Big) \right] \int_{-T_r}^{T_r} \mathrm{d}q\, C_{a,b}(q)\, C_{b,a}(q+\tau) \simeq \frac{1}{T_r} \int_{-T_r}^{T_r} \mathrm{d}q\, C_{a,b}(q)\, C_{b,a}(q+\tau),$$

where $q_{\max}$ is the position of a peak of the function $C_{a,b}(q)\, C_{b,a}(q+\tau)$. In all practical situations $q_{\max} \ll T_r$. Equation (25) differs from equation (21) by containing the product of the CCFs instead of the product of the ACFs. The maximum value of the integral in (25) can be expected when τ has a special value providing coincidence of the peaks of the two CCFs. However, even in this case, when the structure of the two expressions in the rightmost sides of equations (25) and (21) appears to be the same, the expected overall value of $K(\tau)$ should be negligible compared to the expected overall value of $I(\tau)$ given by equation (21), simply because the peak value of the CCF is much lower than the peak value of ACF at $R \gg \lambda$. We will consider some specific estimates confirming this reasoning a little later.

Thus, for $R \gg \lambda$ an acceptable estimate for the variance of the cross-correlation function of the ambient noise is

$$\mathrm{var}\big[C_{a,b}(\tau)\big] = T_r^{-1} \int_{-T_r}^{T_r} \mathrm{d}q\, C_{a,a}(q)\, C_{b,b}(q), \quad (26)$$

and therefore the condition determining the necessary data-accumulation time is

$$\max\big[C_{a,b}(\tau)\big] > \sqrt{T_r^{-1} \int_{-T_r}^{T_r} \mathrm{d}q\, C_{a,a}(q)\, C_{b,b}(q)}. \quad (27)$$

In the next section we will analyze this condition in detail.

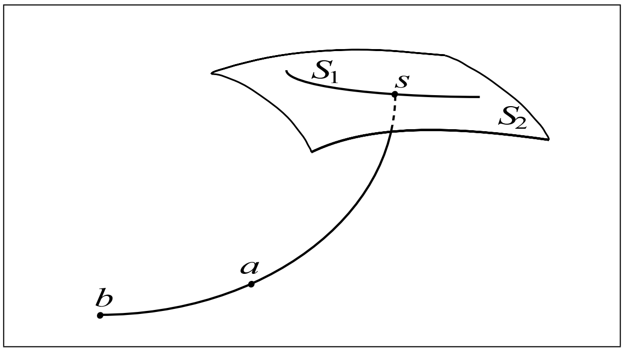

Figure 1. Two important particular cases of the geometry of the noise sources: a one-dimensional distribution of the noise sources on a line S_1 and a two-dimensional distribution of the noise sources on a surface S_2. Also shown is a ray path connecting points a and b, where the noise measurements are performed, and crossing the line S_1 or surface S_2 at a point s. The noise sources located in a vicinity of this ray path are responsible for the peaks of the noise's cross-correlation function.

4. Estimates for the data-accumulation time

For application of the condition (27), we must evaluate both the ACF and CCF of the ambient noise. Let us start with the ACF given by equation (20). If we assume that there is only one ray arrival and use the high-frequency approximation for the Frequency-Domain Green's Function (FDGF) [16]

$$G(\vec{r}, \vec{r}_a, \omega) \simeq A(\vec{r}, \vec{r}_a, \omega)\, \mathrm{e}^{\mathrm{i}\omega\varphi(\vec{r}, \vec{r}_a)}, \quad (28)$$

where $A(\vec{r}, \vec{r}_a, \omega)$ is the complex amplitude of the spectral component, and $\varphi(\vec{r}, \vec{r}_a)$ is the eikonal function, we obtain the following expression for the ACF of the ambient noise in the finite frequency band $\Delta\omega$,

$$C_{a,a}(\tau) = (2\pi)^2 \int \mathrm{d}\vec{r} \int_{\Delta\omega} \mathrm{d}\omega \cdot \varepsilon(\vec{r}, \omega) \left| A(\vec{r}, \vec{r}_a, \omega) \right|^2 \mathrm{e}^{-\mathrm{i}\omega\tau}. \quad (29)$$

Substituting equation (29) into the right-hand side of the inequality (27) yields the following result:

$$T_r^{-1} \int_{-T_r}^{T_r} \mathrm{d}q\, C_{a,a}(q)\, C_{b,b}(q) = (2\pi)^4 T_r^{-1} \Delta\omega \left[\int \mathrm{d}\vec{r} \varepsilon(\vec{r}, \omega_0) \left| A(\vec{r}, \vec{r}_a, \omega_0) \right|^2 \right] \left[\int \mathrm{d}\vec{r} \varepsilon(\vec{r}, \omega_0) \left| A(\vec{r}, \vec{r}_b, \omega_0) \right|^2 \right]. \quad (30)$$

Let us assume that the noise sources are located on a surface or on a contour S_n where n is the dimensionality of the source distribution ($\vec{r} \in S_n$, $n = 1, 2$, see Figure 1) and introduce the average noise power arriving in a and b,

$$\left\langle P_m \right\rangle = \frac{1}{S_n} \int_{\vec{r} \in S_n} \mathrm{d}S_n\, \varepsilon(\vec{r}, \omega_0) \left| A(\vec{r}, \vec{r}_m, \omega_0) \right|^2, \quad (31)$$

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where $m = a, b$. Then

$$\sqrt{T_r^{-1}\int_{-T_r}^{T_r} \mathrm{d}q C_{a,a}(q) C_{b,b}(q)} \tag{32}$$
$$= (2\pi)^2 \sqrt{T_r^{-1}\Delta\omega} S_n \sqrt{\langle P_a\rangle}\sqrt{\langle P_b\rangle}.$$

Now let us turn to the CCF using the same high-frequency approximation for the FDGF and the assumption of a finite-band noise spectrum:

$$\begin{aligned} C_{a,b}(\tau) &= (2\pi)^2 \int \mathrm{d}S_n \int_{-\infty}^{\infty} \mathrm{d}\omega \\ &\quad \cdot \varepsilon(\vec{r},\omega) A(\vec{r},\vec{r}_a,\omega) A^*(\vec{r},\vec{r}_b,\omega) \\ &\quad \cdot \exp\left\{\mathrm{i}\omega\left[\varphi(\vec{r},\vec{r}_a) - \varphi(\vec{r},\vec{r}_b)\right] - \mathrm{i}\omega\tau\right\} \\ &= (2\pi)^2 \int_{\Delta\omega} \mathrm{d}\omega \int_{S_n} \mathrm{d}S_n \\ &\quad \cdot \varepsilon(\vec{r},\omega) A(\vec{r},\vec{r}_a,\omega) A^*(\vec{r},\vec{r}_b,\omega) \\ &\quad \cdot \exp\left\{\mathrm{i}\omega\psi(\vec{r},\vec{r}_a,\vec{r}_b) - \mathrm{i}\omega\tau\right\}, \end{aligned} \tag{33}$$

where $\psi(\vec{r},\vec{r}_a,\vec{r}_b) = \varphi(\vec{r},\vec{r}_a) - \varphi(\vec{r},\vec{r}_b)$. The integral over S_n can be calculated using the stationary phase method. According to the stationary phase method [16], only those points $\vec{r}_s$ of the S_n contribute to the leading term of the high-frequency asymptotic expansion, where the derivatives $[\partial\psi(\vec{r},\vec{r}_a,\vec{r}_b)/\partial\vec{s}]_{\vec{r}=\vec{r}_s\in S_n}$ with respect to the curvilinear (in a general case) coordinates $\{s_j\}$ on S_n are zero. This condition is satisfied [15] for a point where a continuation of the ray path connecting points a and b crosses S_n (see Figure 1). These stationary points produce maxima of the cross-correlation function, the amplitudes of which may be evaluated by the stationary phase method in the following way,

$$\begin{aligned} &C_{a,b}\left[\tau = \psi(\vec{r}_s,\vec{r}_a,\vec{r}_b)\right] \\ &= (2\pi)^2 \Delta\omega\, \varepsilon(\vec{r}_s,\omega_0) A(\vec{r}_s,\vec{r}_a,\omega_0) A^*(\vec{r}_s,\vec{r}_b,\omega_0) \\ &\quad \left(\frac{2\pi}{\omega_0}\right)^{n/2} \left|\det\hat{\psi}_n\right|^{-1/2} \mathrm{e}^{(\mathrm{i}\pi/4)\sum_{i=1}^{n}\mathrm{sgn}q_i} \\ &\simeq (2\pi)^2 \Delta\omega \sqrt{P_{sa}}\sqrt{P_{sb}} \left(\frac{2\pi}{\omega_0}\right)^{n/2} \left|\det\hat{\psi}_n\right|^{-1/2}. \end{aligned} \tag{34}$$

Here,

$$\hat{\psi}_n = \left|\left|\frac{\partial^2\psi(\vec{r},\vec{r}_a,\vec{r}_b)}{\partial s_i \partial s_j}\right|\right|_{\vec{r}=\vec{r}_s\in S_n}, \quad i,j = 1,\ldots,n,$$

and the quantities

$$\begin{aligned} \sqrt{P_{sa}} &= \sqrt{\varepsilon(\vec{r}_s,\omega_0)} A(\vec{r}_s,\vec{r}_a,\omega_0), \\ \sqrt{P_{sb}} &= \sqrt{\varepsilon(\vec{r}_s,\omega_0)} A(\vec{r}_s,\vec{r}_b,\omega_0), \end{aligned}$$

describe the spatial density (on S_n) of the noise power arriving in the points a and b from the vicinity of the stationary point $\vec{r}_s$.

Combined use of equations (32) and (34) allows one to represent the condition (27) in the following form:

$$T_r \Delta\omega \frac{P_{sa}}{\langle P_a\rangle} \frac{P_{sb}}{\langle P_b\rangle} \left(\frac{\delta S_n}{S_n}\right)^2 > 1, \tag{35}$$

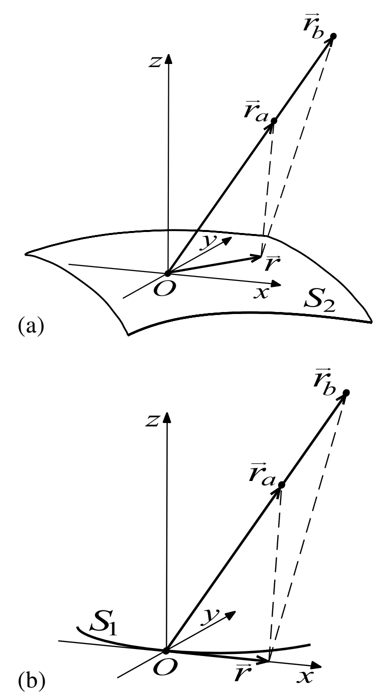

Figure 2. Coordinate system used in evaluation of a stationary point contribution into the noise cross-correlation function in a homogeneous medium. Vector $\vec{r}$ is in the plane xOy (tangential to S_2) in the case of two-dimensional distribution of the noise sources (a), or it belongs to the line Ox (tangential to S_1) in the case of one-dimensional distribution of the noise sources (b). Vectors $\vec{r}_a$ and $\vec{r}_b$ connect the stationary point, which is chosen as the coordinate origin, and the observation points a and b.

where S_n is the total area occupied by the noise sources, and $\delta S_n = (2\pi/\omega_0)^{n/2}|\det\hat{\psi}_n|^{-1/2}$ has the meaning of the area of the vicinity of the stationary point that provides the main contribution to the cross-correlation of the noise in points a and b. The anisotropy coefficients $P_{sa}/\langle P_a\rangle$ and $P_{sb}/\langle P_b\rangle$ describe another important property of this problem: there is a useful noise that propagates along the ray paths close to the ray path connecting points a and b (this noise helps to build the cross-correlation function at the time lags corresponding to propagation between points a and b) and there is a detrimental noise, coming from all other directions, that interferes with this process. Equation indicates that the more anisotropic the ambient noise and the more noise energy that belongs to the "useful" category, the shorter is the time required for the data accumulation. Another interesting fact is that the very important factor $(\delta S_n/S_n)^2$, and, therefore, the emergence time itself, are determined mainly by the eikonal (phase) properties of the noise signals; the amplitude properties of the signal are hidden within the anisotropy coefficients.

We will now obtain explicit expressions for the $(\delta S_n/S_n)^2$ factor for two cases of noise propagation in a homogeneous, motionless medium that differ only by the geometry of the spatial distribution of the noise sources. In the first case, $n = 1$ and the noise sources are located on

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a curve S_1, which is not necessarily a plane curve. In the second case, $n = 2$ and the noise sources are located on a surface S_2. In every case the wave propagation may be either three-dimensional or two-dimensional (i. e., characterized by the divergence of rays in three or two spatial dimensions, respectively), but all specifics of the amplitude variation along the ray paths are hidden within the anisotropy coefficients mentioned above. The phase of the wave behaves similarly, and for both cases we obtain

$$\psi(\vec{r}_s, \vec{r}_a, \vec{r}_b) = \left(|\vec{r} - \vec{r}_a| - |\vec{r} - \vec{r}_b|\right)/c_0, \tag{36}$$

where c_0 is the speed of sound. The differences arise only when one calculates $\det\hat{\psi}_n$. Let us show this using a Cartesian coordinate system the origin O of which coincides with the stationary point $\vec{r}_s$ (see Figure 2). Axis x in Figure 2 represents a small segment of the curve S_1 (in the case of a one-dimensional distribution of the noise sources), and an area on the xOy plane represents a small part of the S_2 surface (in the case of a two-dimensional distribution of the noise sources). It may be shown that the curvature of S_n at the stationary point $\vec{r}_s$, if it exists, does not influence the result of this derivation. In a homogeneous medium the ray connecting points O, a, and b, is a straight line. We will assume that it is located in the plane zOx and makes an angle α with the axis Ox. After a straightforward calculation, we obtain

$$\frac{\delta S_1}{S_1} \sim \frac{\sqrt{r_a r_b}}{S_1}\sqrt{\frac{\lambda_0}{R}}\frac{1}{\cos\alpha} \quad \text{if } n = 1 \tag{37}$$

and

$$\frac{\delta S_2}{S_2} \sim \frac{r_a r_b}{S_2}\frac{\lambda_0}{R}\frac{1}{\cos\alpha} \quad \text{if } n = 2. \tag{38}$$

Here R is the distance between points a and b, $r_a = |\vec{r}_a|$ and $r_b = |\vec{r}_b|$. These expressions contain "geometrical factors" $\sqrt{r_a r_b}/S_1$, $r_a r_b/S_2$, and $1/\cos\alpha$ that depend on the shape of S_1, S_2, and on the relative location of S_n and the points a and b. In many non-peculiar situations, the geometrical factors may be expected to be close to 1, at least by an order of magnitude. The main "physical" factor which, in all practical situations, will be much less than 1, is $\sqrt{\lambda_0/R}$ in the case $n = 1$ or λ_0/R in the case $n = 2$. Looking at this factor from a slightly different perspective (see Figure 3), we realize that in either geometry this factor describes a fraction of rays from the noise sources that arrives in the point b after passing point a within the first Fresnel zone (the radius of which is $\sim \sqrt{\lambda_0 R}$). This is quite a natural conclusion: only those noise signals that remain in phase when propagating over the distance R contribute constructively into the CCF.

Note that the results of this section described by equations (37) and (38), as well as illustrations used above (Figures 2 and 3), are valid only if the stationary point $\vec{r}_s$ is located on an extension of the ray path connecting points a and b. For a one-dimensional distribution of noise sources ($n = 1$), stationary points of the integral over $\vec{r}$ in equation (33) may exist that do not belong to any ray paths connecting points a and b. (Following [17], we will refer to the resulting peaks in the CCF as "spurious" arrivals.) It is interesting that the stationary points of this category provide contributions into the CCF described by a "physical" factor $\delta S_1/S_1$ very similar to (37), with the substitution $R \to ||\vec{r}_a - \vec{r}_s| - |\vec{r}_b - \vec{r}_s||$ and α defined as the angle between each of the two vectors $\vec{r}_a - \vec{r}_s$, $\vec{r}_b - \vec{r}_s$ and a tangent to the curve S_1 at the point $\vec{r}_s$. (This angle is the same for both vectors as a consequence of the stationarity condition). In a homogeneous medium, the peaks of the CCF caused by the stationary points of this type may have even larger amplitudes than those described by equation (37), because of the triangle property $R \equiv |\vec{r}_a - \vec{r}_b| > ||\vec{r}_a - \vec{r}_s| - |\vec{r}_b - \vec{r}_s||$. However, this same inequality tells us that such peaks should be expected at shorter time lags than those corresponding to the propagation between points a and b and, therefore, can be distinguished from the peaks of main interest in remote sensing applications (provided sufficient a priori information about the environment is available). In the underwater acoustics context, an experimental evidence of persistent spurious arrivals, which are likely to be of the kind considered above, is discussed in [18].

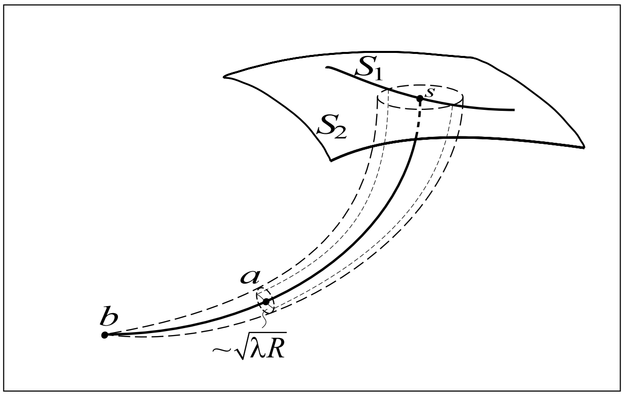

Figure 3. Location of noise sources that are responsible for peaks of the cross-correlation function. Only the ambient noise components passing within the first Fresnel zone in the vicinity of point a contribute constructively to the cross-correlation at point b. This condition determines the fraction $\delta S_n/S_n$ of the total noise power that is useful in the noise interferometry. If the dimensionality of the noise sources' layout $n = 1$ (i.e., the noise sources are distributed on a curve S_1), this fraction is estimated as the ratio of the first Fresnel zone's radius ($\sim \sqrt{\lambda_0 R}$) to the length of the circle with radius R. If the dimensionality of the noise sources' layout $n = 2$ (i.e., the noise sources are distributed on a surface S_2), this fraction is estimated as the ratio of the first Fresnel zone's area to the area of the sphere with radius R. The bundle of the ray paths crossing the first Fresnel zone near point a limits the area on S_n, where the "useful" noise sources are located; its nominal boundary is shown by a closed dashed curve in the figure.

From a formal mathematical viewpoint, the stationary points of similar nature may exist also for a surface distribution of the noise sources ($n = 2$). For instance, this is the case in a homogeneous medium, when the surface S_2 in the vicinity of a stationary point coincides, to at least the second order in displacements from the stationary point,

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with a hyperboloid having foci in points a and b. The practical significance of such stationary points is diminished by the fact that the focus points are always located at spatial regions separated both by the hyperboloid surface and by the tangent plane to it; unless the noise sources are located on an acoustically transparent surface, the geometry prohibits simultaneous propagation of the noise signals generated at such stationary points to both foci.

When the difference of the "anisotropy" and "geometrical" factors from unity may be disregarded, our estimates of the data-accumulation time reduce to those stated in [3], with the correspondence $d = n + 1$. (Now, δf in equation (12) should be understood as the width of the overlap of the frequency bands of the receivers and the noise sources.) However, the reasoning presented in the current section shows that the "dimensionality" of the problem should be treated in a more careful way. What influences the Green's function emergence time is the geometry of the noise sources layout, and not only the mode of propagation (2-D, 3-D). For example, whether the ambient acoustic noise is caused in the coastal zone by the surf at a rocky coastline (3-D ray divergence), or in deep water by a two-dimensional, long-range propagation of normal modes in an underwater waveguide from distant sources (2-D divergence of horizontal (modal) rays [16]), and there is a ray path connecting points a, b and a point on the S_1 curve, the CCF's peak emergence time will be governed by the same expression (37) corresponding to $n = 1$. As with any statement, the one that we are making has limitations. It should be applied with care, for example, in the situations where a transition occurs between resolved ray arrivals and resolved normal mode arrivals in a waveguide.

The results of this section can be generalized for the case of multiple arrivals in an inhomogeneous medium if the arrivals are separated in time in the frequency band of interest [19, 20].

5. A simpler approach to estimating the CCF variance

In noise interferometry, the necessity to accumulate data for sufficiently long time periods arises because, in practical situations, the contribution of the "detrimental" noise component into the CCF estimate (3) dominates over the contribution of the "useful" noise component at small accumulation times T_r. The ratio between the expected CCF peak and the noise dispersion (i.e., the noise ACF peak value) $\sigma_m^2 = C_{m,m}(0)$, where $m = a, b$, is described by the combination of the same "anisotropy," "geometrical," and "physical" factors introduced in the previous section. According to equation (34), the expected peak of the CCF does not depend on T_r, whereas the variance of this quantity given by equation (32) decreases with the time as $T_r^{-1/2}$. In many situations, when $R \gg \lambda_0$ and the "detrimental" noise dominates over the "useful" noise, the value of $\max\langle C_{a,b}(\tau)\rangle$ is much less than σ_m^2. For example, for a one-dimensional distribution of noise sources ($n = 1$) in a homogeneous motionless medium, for $R = 10^5$ m and $\lambda_0 = 15$ m (which corresponds to underwater sound of the frequency 100 Hz), $\max[C_{a,b}(\tau = R/c_0)] \approx \sqrt{\lambda_0/(2\pi^2 R)}\sigma_m^2 \sim 3 \cdot 10^{-3}\sigma_m^2$. It is evident that the relationship $\max[C_{a,b}(\tau = R/c_0)] \ll \sqrt{\mathrm{var}[C_{a,b}(\tau = R/c_0)]}$ persists for the most of the data-accumulation time. In such a situation, one can disregard the correlation between the noise signals at points a and b during this initial data-accumulation period when estimating the emergence time.

Practical procedures of the covariance calculation are based on averaging of long discrete time series including a finite number of terms,

$$C_{a,b}^N(\tau = R/c_0) \equiv \left\langle P(\vec{r}_a, t)\, P(\vec{r}_b, t + R/c_0)\right\rangle \equiv \frac{1}{N}\sum_{j=1}^{N} P(\vec{r}_a, t_j)\, P(\vec{r}_b, t_j + R/c_0). \tag{39}$$

This sum of the finite number of random quantities is a random quantity itself, and its dispersion can be characterized by the variance

$$\mathrm{var}\left[C_{a,b}^N(\tau = R/c_0)\right] \equiv \left\langle C_{a,b}^N(\tau = R/c_0)\, C_{a,b}^N(\tau = R/c_0)\right\rangle - \left\langle C_{a,b}^N(\tau = R/c_0)\right\rangle^2, \tag{40}$$

where the averaging is performed over an ensemble of realizations. As follows from the above considerations, for a long period of initial data accumulation, the covariance is dominated by uncorrelated noise samples. For the purpose of evaluating the variance of $C_{a,b}^N(\tau = R/c_0)$, correlation between $P(\vec{r}_a, t_j)$ and $P(\vec{r}_b, t_j + R/c_0)$, the revealing of which is the goal of the noise interferometry, can be neglected during periods of the order of the minimal data-accumulation time. Then, the variance properties are dictated by the standard rules (see, for example, [21, Ch. 2]):

$$\mathrm{var}\left[\frac{1}{N}\sum_{j=1}^{N} P(\vec{r}_a, t_j)\, P(\vec{r}_b, t_j + R/c_0)\right] = \frac{1}{N^2}\left\{\sum_{j=1}^{N}\mathrm{var}\left[P(\vec{r}_a, t_j)\, P(\vec{r}_b, t_j + R/c_0)\right] + \sum_{j\neq i}\mathrm{cov}\left[P(\vec{r}_a, t_j)\, P(\vec{r}_b, t_j + R/c_0), P(\vec{r}_a, t_i)\, P(\vec{r}_b, t_i + R/c_0)\right]\right\}, \tag{41}$$

or

$$\mathrm{var}\left[\frac{1}{N}\sum_{j=1}^{N} P(\vec{r}_a, t_j)\, P(\vec{r}_b, t_j + R/c_0)\right] = \frac{1}{N^2}\left\{\sum_{j=1}^{N}\mathrm{var}\left[P(\vec{r}_a, t_j)\right]\mathrm{var}\left[P(\vec{r}_b, t_j + R/c_0)\right] + \sum_{j\neq i}\left\langle P(\vec{r}_a, t_j)\, P(\vec{r}_a, t_i)\, P(\vec{r}_b, t_j + R/c_0), P(\vec{r}_b, t_i + R/c_0)\right\rangle\right\}, \tag{42}$$

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where angle brackets denote the mathematical expectation. Further simplification yields

$$\operatorname{var}\left[\frac{1}{N}\sum_{j=1}^{N} P(\vec r_a, t_j)\, P(\vec r_b, t_j + R/c_0)\right] = \frac{1}{N^2}\Big\{ N\sigma_a^2\sigma_b^2 + \sum_{j\neq i}\big\langle P(\vec r_a, t_j)\, P(\vec r_a, t_i)\big\rangle \big\langle P(\vec r_b, t_j)\, P(\vec r_b, t_i)\big\rangle \Big\}, \quad (43)$$

or

$$\operatorname{var}\left[\frac{1}{N}\sum_{j=1}^{N} P(\vec r_a, t_j)\, P(\vec r_b, t_j + R/c_0)\right] = \frac{1}{N^2}\left[N\sigma_a^2\sigma_b^2 + 2\sum_{j=1}^{N-1}(N-j)C_{a,a}(j\Delta t)\, C_{b,b}(j\Delta t)\right], \quad (44)$$

where $C_{a,a}(t)$, $C_{b,b}(t)$ are the autocorrelation functions of noise at respective points, and $\Delta t = 1/\Delta f = 2\pi/\Delta\omega$ is the data discretization interval.

The general expression (29) for the autocorrelation function of the finite-band ambient noise can be transformed into the following form:

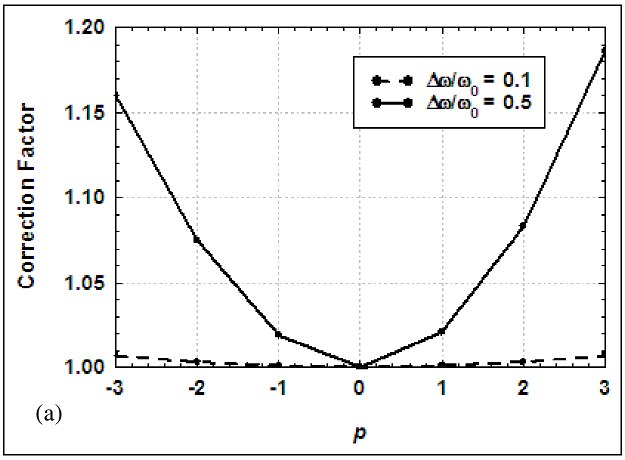
$$C_{m,m}(t) = \sigma_m^2 \frac{\int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2} \mathrm{d}\omega\, \overline{S_m(\omega)} \mathrm{e}^{-\mathrm{i}\omega t}}{\int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2} \mathrm{d}\omega\, \overline{S_m(\omega)}}, \quad (45)$$

where $\overline{S_m(\omega)}$ is the average ray intensity at point a ($m = a$) or point b ($m = b$). For white noise ($\overline{S_m(\omega)} = \text{const.}$), the contribution of the second term on the right-hand side of equation (44) is exactly zero. In other cases its contribution is negligibly small compared to the first term, because of the presence of an oscillating factor in the integrand in the numerator in the right-hand side of equation (45). For the average ray intensity of a rather general kind $\overline{S_m(\omega)} \propto \omega^p$, the correction caused by the second term has been evaluated numerically (Figure 4).

Figure 4a shows the correction factor as a function of two parameters, the exponent p and the dimensionless bandwidth $\Delta\omega/\omega_0$. Figure 4b shows the correction factor as a function of a combined parameter $(p\Delta\omega/\omega_0)/(1+p)$, which can be viewed as a measure of variability of the average ray intensity $\overline{S_m(\omega)}$. For all practical values of the parameters, the correction is not significant. Thus, under rather general assumptions, an accurate estimate for the variance of the CCF estimate is

$$\operatorname{var}\left[C_{a,b}^N(\tau = R/c_0)\right] \approx \frac{\sigma_a^2\sigma_b^2}{N}, \quad (46)$$

where N is the number of data points participating in the calculation of $C_{a,b}^N(\tau = R/c_0)$. The number of points can be expressed through the data-accumulation time T_r and the bandwidth $\Delta f = \Delta\omega/2\pi$ in the following way: $N = T_r\Delta f$. It may be shown by using the definition $\sigma_m^2 = C_{m,m}(0)$ that the result expressed by equation (46) is equivalent to the results of sections 3 and 4 (cf. equations 26, 30).



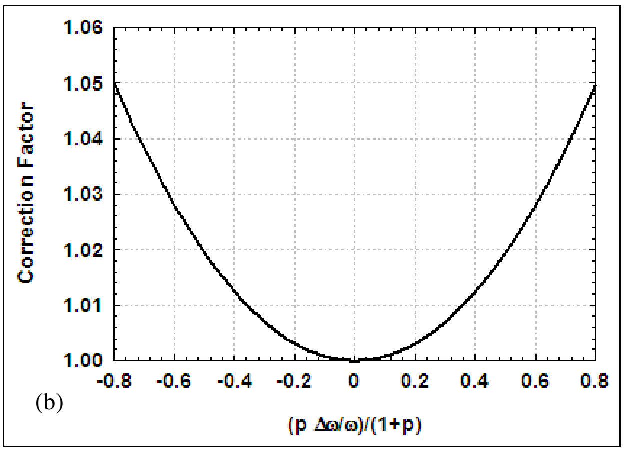


Figure 4. The difference between two-point cross-correlation functions of white and non-white ambient noise. a) The correction factor as a function of two separate parameters, the exponent p in the noise power spectrum and the dimensionless bandwidth $\Delta\omega/\omega_0$. b) The correction factor as a function of a combined parameter $(p\Delta\omega/\omega_0)/(1+p)$. Throughout the range of the parameters of practical values, the correction is not significant.

6. Conclusions

In the present paper, we have re-examined, from the first principles, the emergence time of the main peaks of the cross-correlation function of ambient noise in the context of noise interferometry as applied to passive remote sensing of the environment. The fundamental results [3, 11, 12] on the necessary data accumulation time have been found to be valid under somewhat weaker assumptions than made originally in their derivation. Equation (35) provides an extension of the results of [11] and [12] to non-isotropic noise fields generated by noise sources, which are non-uniformly distributed on a surface or on a curve. Our results apply to the general case of a moving or motionless, inhomogeneous propagation medium with time-independent parameters.

We have pointed out the possibility of the existence and evaluated emergence times of "spurious arrivals," i.e., prominent peaks of the noise CCF $C_{a,b}(t)$ that do not correspond to any rays connecting the receivers at locations a and b. The "spurious arrivals" we have considered may

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occur when noise sources are located in a narrow region around a curve, which is the case with road traffic in the atmosphere and shipping lanes in the ocean. In a homogeneous medium, the peaks of CCF caused by the stationary points of this type may have large amplitudes and occur at shorter time lags than those corresponding to wave propagation between points *a* and *b*.

Arguments have been provided that the "dimensionality" of the problem, an important parameter determining the CCF's peak emergence time, is not uniquely characterized by the mode of propagation, such as cylindrical or spherical divergence of wave fronts. In certain situations the emergence time estimates are influenced by the geometry (dimensionality) of the noise sources' layout, and not just by the mode of propagation in the medium (2-D, 3-D).

Finally, an alternative, simpler technique using basic statistical principles has been suggested for calculation of the Green's functions emergence time without loss of generality.

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References

[1] R. L. Weaver, O. I. Lobkis: Ultrasonics without a source: Thermal fluctuation correlations at MHz frequencies. Phys. Rev. Lett. **87** (2001) 134301.

[2] O. I. Lobkis, R. L. Weaver: On the emergence of the Green's function in the correlations of a diffuse field. J. Acoust. Soc. Am. **110** (2001) 3011–3017.

[3] P. Gouédard, L. Stehly, F. Brenguier, M. Campillo, Y. Colin de Verdière, E. Larose, L. Margerin, P. Roux, F. J. Sánchez-Sesma, N. M. Shapiro, R. L. Weaver: Cross-correlation of random fields: Mathematical approach and applications. Geophys. Prospecting **56** (2008) 375–393.

[4] L. A. Brooks, P. Gerstoft, D. P. Knobles: Multichannel array diagnosis using noise cross-correlation. J. Acoust. Soc. Am. **124** (2008) EL203–EL209.

[5] E. Slob, K. Wapenaar: Electromagnetic Green's functions retrieval by cross-correlation and cross-convolution in media with losses. Geophys. Res. Lett. **34** (2007) L05307. doi:10.1029/2006GL029097.

[6] O. A. Godin: Emergence of the acoustic Green's function from thermal noise. J. Acoust. Soc. Am. **121** (2007) EL96–EL102.

[7] O. A. Godin: Retrieval of Green's functions of elastic waves from thermal fluctuations of fluid-solid systems. J. Acoust. Soc. Am. **125** (2009) 1960–1970.

[8] E. Larose, A. Derode, P. R. M. Campillo: Fluctuations of correlations and Green's function reconstruction: Role of scattering. J. Appl. Phys. **103** (2008) 114907–114907.

[9] S. E. Fried, W. A. Kuperman, K. G. Sabra, P. Roux: Extracting the local Green's function on a horizontal array from ambient ocean noise. J. Acoust. Soc. Am. **124** (2008) EL183–EL188.

[10] R. L. Weaver, O. I. Lobkis: Fluctuations in diffuse field-field correlations and the emergence of the Green's function in open systems. J. Acoust. Soc. Am. **117** (2005) 3432–3439.

[11] R. L. Weaver, O. I. Lobkis: The mean and variance of diffuse field correlations in finite bodies. J. Acoust. Soc. Am. **118** (2005) 3447–3456.

[12] K. G. Sabra, P. Roux, W. A. Kuperman: Emergence rate of the time-domain Green's function from ambient noise cross-correlation function. J. Acoust. Soc Am. **118** (2005) 3524–3531.

[13] B. F. Kuryanov: On the possibility of a passive acoustic tomography in a diffuse noise field. Proceedings of the XI-th Brekhovskikh's Conference "Ocean Acoustics" joint with the XVII Session of the Russian Acoustical Society, GEOS, 2006, 179–185.

[14] V. A. Burov, S. N. Sergeev, A. S. Shurup: The use of low-frequency noise in passive tomography of the ocean. Acoustical Physics **54** (2008) 42–51.

[15] O. A. Godin: Recovering the acoustic Green's function from ambient noise cross correlation in an inhomogeneous moving medium. Phys. Rev. Lett. **97** (2006) 054301. doi: 10.1103/PhysRevLett.97.054301.

[16] L. M. Brekhovskikh, O. A. Godin: Acoustics of layered media. 2: Point sources and bounded beams. 2nd. extended edn. Springer, Berlin, Heidelberg, 1999. Chaps. 5&7 and Appendix A.

[17] R. Snieder, K. Wapenaar, K. Larner: Spurious multiples in seismic interferometry of primaries. Geophysics **71** (2006) S1111–S1124.

[18] O. A. Godin, N. A. Zabotin, V. V. Goncharov: Ocean tomography with acoustic daylight. Geophys. Res. Lett. **37** (2010) L13605. doi: 10.1029/2010GL043623.

[19] O. A. Godin: Accuracy of the deterministic travel times retrieval from cross-correlations of non-diffuse ambient noise. J. Acoust. Soc Am. **126** (2009) EL183–EL189.

[20] O. A. Godin: Cross-correlation function of acoustic fields generated by random high-frequency sources. J. Acoust. Soc Am. **128** (2010) 600–610.

[21] D. Blumenfeld: Operations research calculations handbook. CRC Press, 2001.

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