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# Modeling the Effects of Viscosity and Thermal Conduction on Acoustic Propagation in Rigid Tubes with Various Cross-Sectional Shapes

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## Summary

When modeling acoustics with viscothermal effects included, typically of importance for narrow tubes and slits, one can often use the so-called low reduced frequency model. With this model a characteristic length is assumed for which the sound pressure is constant. For example for a circular cylindrical tube the characteristic length is the radius. A triangular cross-section does not have a characteristic length, but as will be shown in this paper the model can in fact be used as long as 1) the cross-sectional pressure is constant and 2) a characteristic impedance and propagation wavenumber can be established for the geometry. These parameters can be found for a tube with a triangular cross-section and an implementation of the low reduced frequency which can handle tubes with both circular, rectangular triangular cross-sections has been made in COMSOL Multiphysics. For the circular and the rectangular tube results found using this implementation have been compared to results from an analytical model, a so-called full Navier-Stokes implementation in COMSOL Multiphysics and the commercial package FFT ACTRAN which also uses the low reduced frequency model. The triangular tube implementation has been compared to the analytical case as well as the full Navier-Stokes implementation.

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## 1. Introduction

The propagation of sound in cylindrical tubes with thermal and viscous losses is a topic which has received much attention for more than a century. Kirchhoff [1] was the first to give a solution for a viscous and thermally conducting fluid in a rigid tube with a circular cross-section. The solution, however, was in the form of a transcendental solution which was not readily suited for acoustic calculations. Simplifications were proposed by Rayleigh [2] giving approximate expressions for different cases depending on the ratio between radius and boundary layer thickness. Zwikker and Kosten [3] introduced an approximate approach for which the inertial and viscous effects were included by the introduction of a complex density, and the compressional and thermal effects by a complex compressibility. The applicability of this approach was later examined by Tidjeman [4] by comparison to numerical solutions of Kirchhoff's expression and it was found that Zwikker and Kosten's approach is valid for a broad range of frequencies and radii, excluding cases where either the radius or the wavelength approaches the molecular mean-free path (approximately  $10^{-7}$  m).

Tidjeman's work was essential for the development of the so-called low reduced frequency model which was examined in great detail by Beltman. [5, 6, 7, 8] With the low reduced frequency model it is possible to include viscous and thermal losses in an acoustic model without having to solve explicitly for velocity and temperature. The model can be applied to both tubes and layers when a characteristic length exists in which the pressure is constant. Four cases are usually associated with the model: The circular tube; the rectangular tube, the circular layer and the rectangular layer. In the following only tubes are considered.

As the shape of the cross-section affects the acoustic properties of a tube, one needs to be able to establish a set of parameters which describe these properties. Different approximative expressions have been established for some common shapes throughout the last century. Cummings [9] found such expressions using a variational technique for the parallel slit, the rectangular cross-section tube, the equilateral triangular cross-section and other geometries. The results are, whenever possible, compared to exact solutions, and discrepancies are found due to the trial functions used in the variational technique not being exact. The case of an equilateral triangular cross-section has also been investigated by Stinson and Champoux [10] who give a complex density function and a complex compressibility function which describe the acoustic behavior of the tube for a large range of frequencies and radii (with the radius

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$a$  and frequency  $f$  the range is given by Stinson [11] as  $a > 10^{-3}$  cm and  $af^{3/2} < 10^6$  cm s $^{-3/2}$ ). The triangular case is not considered in the low reduced frequency model, but as will be demonstrated in the following the triangular case can in fact be included in such an implementation. This requires establishing a relationship between the Zwikker and Kosten parameters, i.e. the complex density and the complex compressibility, and the low reduced frequency model parameters. The triangular case is then described by the parameters normally used in the low reduced frequency model and so it can be viewed as an extension to existing implementations (commercial or otherwise).

The low reduced frequency model has been implemented in the Finite Element package COMSOL Multiphysics. Using this implementation tubes with either circular, rectangular or equilateral triangular cross-section can be handled. The commercial software FFT ACTRAN which also uses the low reduced frequency implementation has been used for comparison (circular and rectangular cross-section) and a so-called full Navier-Stokes implementation (see appendix A1) provided by COMSOL was used for comparison as well. With the latter implementation the governing equations are solved simultaneously and any geometry can in principle be used in the calculation. However, calculation time is generally long compared to the low reduced frequency model and one has to make sure that the mesh is refined enough in the regions where the viscothermal losses are important in order to capture the physics sufficiently well.

## 2. Theory

First, the quantities that are important for the acoustic behavior in the fluid are defined. The total pressure  $\bar{p}$  is comprised of an ambient pressure  $p_0$  and the sound pressure  $p$  which varies harmonically with the angular frequency  $\omega$  as  $e^{i\omega t}$  so that

$$\bar{p} = p_0 + pe^{i\omega t}. \quad (1)$$

Similarly, the total density  $\bar{\rho}$  consists of the static density  $\rho_0$  and the excess density  $\delta$ .

$$\bar{\rho} = \rho_0 + \delta e^{i\omega t}. \quad (2)$$

The total temperature  $\bar{T}$  consists of an ambient temperature  $T_0$  and a temperature variation  $T$  as

$$\bar{T} = T_0 + Te^{i\omega t}. \quad (3)$$

Finally, assuming no mean flow, the total velocity  $\bar{\mathbf{v}}$  is comprised of the velocity  $\mathbf{v}$  and a harmonic variation term

$$\bar{\mathbf{v}} = \mathbf{v}e^{i\omega t}. \quad (4)$$

In the implementations the following values have been used:  $p_0 = 10^5$  Pa,  $\rho_0 = 1.2$  kg/m $^3$  and  $T_0 = 300$  K.

The sound propagation in tubes of various cross-sections is considered. For narrow tubes the so-called viscothermal losses become of great importance when modeling the acoustics inside the tube. Due to viscosity the fluid particles near the wall of the tube will tend to stick to the surface, effectively lowering the particle velocity. This effect is most predominant in a so-called boundary layer near the wall whereas further away into the bulk the effects are of less importance. Similarly, due to the thermal properties of the tube wall being significantly different from those of the fluid, thermal difference between regions in the fluid with high pressure and those with low pressure will be equalized near the tube wall. Both the viscous and thermal effects will convert acoustic energy into thermal energy, and so an acoustic loss has been introduced. This loss must be modeled correctly and this task is addressed in the following.

The velocity, density and temperature generally vary across the cross-section of a tube. Additionally, the velocity has both a longitudinal and a transversal component, but the transversal component is generally several orders of magnitude smaller than the longitudinal component [11], and thus it can be ignored in the model. The pressure also varies along the cross-section, although very slightly [11]. In many applications these variations are not of great importance, and only the mean values, i.e. values found by integration across the cross-section, need to be considered. If the pressure is assumed constant across the cross-section and a volume velocity  $q$  can be found as the product of the mean velocity  $\langle v_z \rangle$  in the longitudinal direction (here the  $z$ -direction) and the cross-sectional area  $S$ , a transmission line model will suffice in describing the sound field.

### 2.1. Transmission line model

The sound field in a tube with constant cross-sectional pressure and length  $l$  can be described by a transmission matrix relating the input pressure and volume velocity to the output pressure and volume velocity [12]

$$\begin{pmatrix} p_i \\ q_i \end{pmatrix} = \begin{pmatrix} \cosh(\Gamma l) & Z_c \sinh(\Gamma l) \\ Z_c^{-1} \sinh(\Gamma l) & \cosh(\Gamma l) \end{pmatrix} \begin{pmatrix} p_o \\ q_o \end{pmatrix}, \quad (5)$$

where subscripts  $i$  and  $o$  denote input and output, respectively. The characteristic impedance  $Z_c$  and the propagation wavenumber  $\Gamma$  of the transmission line are found as [12]

$$Z_c = \sqrt{\frac{Z'}{Y'}} \quad (6)$$

and

$$\Gamma = \sqrt{Z'Y'}. \quad (7)$$

with the series impedance per unit length  $Z'$  and the shunt admittance per unit length  $Y'$  depending on the shape and area of the cross-section as well as the fluid characteristics. The series impedance relates the pressure gradient along

an infinitesimal length of the tube to the volume velocity [11]

$$-\frac{dp}{dz} = Z'q. \quad (8)$$

The shunt admittance relates the volume velocity gradient along the same length to the pressure [11]

$$-\frac{dq}{dz} = Y'p. \quad (9)$$

The matrix elements can be established as long as the series impedance and the shunt admittance can be calculated. The series impedance can be described in terms of a complex density  $\rho(\omega)$  [11]

$$Z' = i\omega\rho(\omega)/S \quad (10)$$

which takes into account both the inertial and the viscous effects and the shunt admittance can be described via a complex compressibility  $C(\omega)$  [11]

$$Y' = i\omega C(\omega)S \quad (11)$$

which takes into account both the compressional and the thermal effects.

Stinson [11] gives the following general complex density and compressibility, respectively, for a tube with arbitrarily shaped cross-section:

$$\rho(\omega) = \rho_0/F(\nu) \quad (12)$$

and

$$C(\omega) = (1/\gamma p_0) (\gamma - (\gamma - 1)F(\nu')), \quad (13)$$

with  $\gamma p_0 = \rho_0 c_0^2$ . Here,  $\gamma$  is the ratio of specific heats (set to 1.4 in the implementations) and  $c_0$  is the adiabatic sound speed (340 m/s). The function  $F$  is the same in the two expressions, only the input argument differs, as  $\nu = \mu/\rho_0$  and  $\nu' = \lambda/(\rho_0 C_p)$ , where  $\mu$  is the shear viscosity ( $18.6 \cdot 10^{-6}$  Pa s),  $C_p$  is the specific heat at constant temperature ( $1,010 \text{ J kg}^{-1} \text{ K}^{-1}$ ) and  $\lambda$  is the thermal conductivity ( $0.026 \text{ W m}^{-1} \text{ K}^{-1}$ ). Stinson [11] gives the function  $F(\eta)$ , where the input argument  $\eta$  is either equal to  $\nu$  or  $\nu'$ , for different geometries, e.g. for the case of a circular (subscript  $c$ ) cross-section with a radius of  $a$  as shown in Figure 1,

$$F_c(\eta) = 1 - \frac{2}{a\sqrt{-i\omega/\eta}} \frac{J_1(a\sqrt{-i\omega/\eta})}{J_0(a\sqrt{-i\omega/\eta})}, \quad (14)$$

where  $J_\alpha(x)$  is a Bessel function of the first kind and order  $\alpha$  with input argument  $x$ . Alternatively, by using the Bessel relation  $2\alpha/x J_\alpha(x) = J_{\alpha-1}(x) + J_{\alpha+1}(x)$ , the function can be written as

$$F_c(\eta) = -\frac{J_2(a\sqrt{-i\omega/\eta})}{J_0(a\sqrt{-i\omega/\eta})}. \quad (15)$$

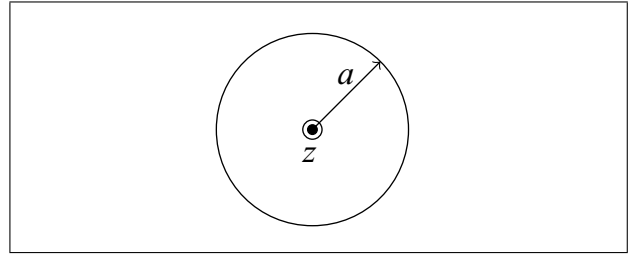


Figure 1. The circular geometry with radius  $a$ .

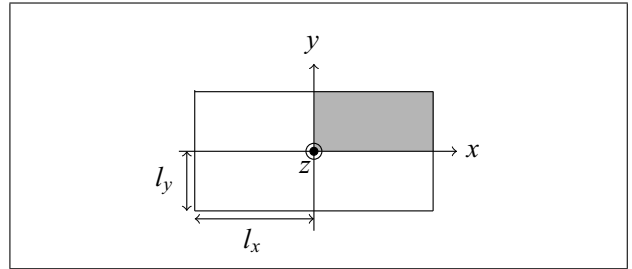


Figure 2. The rectangular geometry with side lengths  $2l_x$  and  $2l_y$ . The shaded area indicates the region to be meshed when using the full Navier-Stokes implementation, if symmetry planes are considered.

Another geometry is the rectangular (subscript  $r$ ) cross-section with side lengths  $2l_x$  and  $2l_y$  as in Figure 2 where the function  $F_r(\eta)$  is given as

$$F_r(\eta) = \frac{4i\omega}{\eta l_x^2 l_y^2} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (\alpha_k^2 \beta_n^2 (\alpha_k^2 + \beta_n^2 + i\omega/\eta))^{-1}, \quad (16)$$

where  $\alpha_k = (k + \frac{1}{2})\pi/l_x$  and  $\beta_n = (n + \frac{1}{2})\pi/l_y$ . This series solution of course has to be truncated in a numerical implementation.

Finally for the case of an equilateral triangular (subscript  $t$ ) cross-section with side lengths equal to  $d$  as shown in Figure 3 the function  $F(\eta)$  is given by Stinson and Champoux [10] as

$$F_t(\eta) = (\epsilon^2 - 3\epsilon \coth \epsilon + 3)/\epsilon^2, \quad (17)$$

where

$$\epsilon = d \sqrt{\frac{3}{16} \frac{i\omega}{\eta}}. \quad (18)$$

For any geometry for which the function  $F(\eta)$  can be calculated a transmission line model can be established.

## 2.2. The low reduced frequency model

The low reduced frequency model provides means for incorporating inertial, compressional, viscous and thermal effects in a numerical model without increasing the number of degrees of freedom compared to a dissipation free model. The low frequency model is described extensively in a number of papers by Beltman [5, 6, 7, 8] with Tidjeman [4] being listed as the originator of the model. Two parameters are introduced, one being the so-called

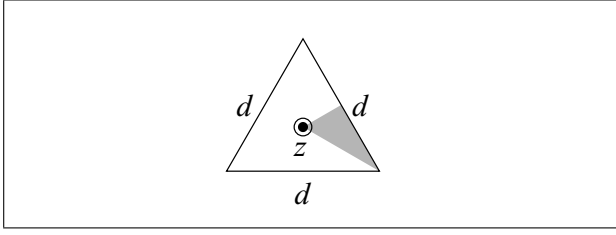


Figure 3. The triangular geometry with side lengths  $d$ . The shaded area indicates the region to be meshed when using the full Navier-Stokes implementation, if symmetry planes are considered.

shear wavenumber  $s$  which is dependent on a characteristic length  $l_c$  as [7]

$$s = l_c \sqrt{\frac{\rho_0 \omega}{\mu}}. \quad (19)$$

The lower the  $s$ , the more dominant will the viscous effects be over inertial effects. A second consequence is that thermal effects will dominate over compressional effects. The other parameter is the so-called reduced frequency  $\hat{k}$  given as [7]

$$\hat{k} = \frac{\omega l_c}{c_0}. \quad (20)$$

With the low reduced frequency model, the acoustic wavelength is assumed larger than the characteristic length which can be expressed via the reduced frequency as

$$\hat{k} \ll 1. \quad (21)$$

Also, the acoustic wavelength is assumed larger than the boundary layer thickness, described as

$$\frac{\hat{k}}{s} = \frac{\omega l_c}{c_0 s} = \sqrt{\frac{\mu \omega}{\rho_0 c_0^2}} \ll 1. \quad (22)$$

With these assumptions the linearized governing equations (see Appendix) can be combined into a Helmholtz equation for the propagational direction (superscript  $pd$ ) when structural coupling is omitted can be written as [7]

$$\Delta^{pd} p - \Gamma^2 p = 0, \quad (23)$$

with the propagation wavenumber

$$\Gamma = \frac{\omega}{c_0} \sqrt{\frac{\gamma}{n(\sigma s) B(s)}} \quad (24)$$

and

$$n(\sigma s) = \left( 1 + \left( \frac{\gamma - 1}{\gamma} \right) D(\sigma s) \right)^{-1}. \quad (25)$$

The function  $B(s)$  in equation (24) is related to the mean velocity across the cross-sectional area. It is both frequency dependent and dependent on the both the shape

of the cross-section and the characteristic length. The velocity in the propagational direction, here the  $z$ -direction, is given as

$$v_z(s, z, \mathbf{x}^{cd}) = -A(s, \mathbf{x}^{cd}) \frac{i}{\omega \rho_0} \frac{dp(\mathbf{x}^{cd})}{dz}, \quad (26)$$

where  $A(s, \mathbf{x}^{cd})$  describes the velocity variation across the cross-section ( $cd$  is the cross-sectional direction). With  $B(s)$  being the integration of  $A(s, \mathbf{x}^{cd})$  over the cross-sectional area  $S$ ,

$$B(s) = \frac{1}{S} \int_S A(s, \mathbf{x}^{cd}) dS, \quad (27)$$

the mean velocity  $\langle v_z \rangle$  is found as

$$\begin{aligned} \langle v_z \rangle &= \frac{1}{S} \int_S -A(s, z) \frac{i}{\omega \rho_0} \frac{dp}{dz} dS \\ &= -B(s) \frac{i}{\omega \rho_0} \frac{d}{dz}. \end{aligned} \quad (28)$$

$B(s)$  ensures that both inertial and viscous effects are taken into account in the model. If  $B(s)$  is set to -1 viscous effects are neglected.  $D(\sigma s)$  is related to the mean temperature variation across the cross-sectional area. Just as  $B(s)$  it is both frequency and shape dependent. The temperature variation is given as

$$T(\sigma s, z, \mathbf{x}^{cd}) = - \left( \frac{\gamma - 1}{\gamma} \right) \frac{p T_0}{p_0} C(\sigma s, \mathbf{x}^{cd}) \quad (29)$$

where  $C(s, \mathbf{x}^{cd})$  describes the temperature variation across the cross-section. With  $D(\sigma s)$  being the integration of  $C(\sigma s, \mathbf{x}^{cd})$  over the cross-sectional area  $S$

$$D(\sigma s) = \frac{1}{S} \int_S C(\sigma s, \mathbf{x}^{cd}) dS \quad (30)$$

the mean temperature variation  $\langle T \rangle$  is found as

$$\langle T \rangle = \frac{1}{S} \int_S - \left( \frac{\gamma - 1}{\gamma} \right) \frac{p T_0}{p_0} C(\sigma s, \mathbf{x}^{cd}) dS \quad (31)$$

or using equation (30)

$$\langle T \rangle = -D(s) \left( \frac{\gamma - 1}{\gamma} \right) \frac{p T_0}{p_0}. \quad (32)$$

$D(\sigma s)$  takes into account both compressional and thermal effects. If  $D(\sigma s)$  is set to -1 thermal effects are neglected, equivalent to adiabatic boundary conditions on the walls of the tube. For isothermal boundary conditions  $D(\sigma s) = B(\sigma s)$  which means that  $D$  takes the value of  $B$  with the input argument  $\sigma s$ . The excess density is also a function of  $C(s, \mathbf{x}^{cd})$  in that

$$\delta(\sigma s, z, \mathbf{x}^{cd}) = \frac{p \rho_0}{p_0} \left( 1 + \left( \frac{\gamma - 1}{\gamma} \right) C(\sigma s, \mathbf{x}^{cd}) \right), \quad (33)$$

and the mean excess density is thus

$$\langle \delta \rangle = \frac{p \rho_0}{p_0} \left( 1 + \left( \frac{\gamma - 1}{\gamma} \right) D(\sigma s) \right). \quad (34)$$

With  $B(s)$  and  $D(\sigma s)$  established the Helmholtz equation can be written with a complex sound speed  $c_e$

$$\Delta^{pd} p + \frac{\omega^2}{c_e^2} p = 0, \quad (35)$$

where

$$c_e = c_0 \sqrt{\frac{-n(\sigma s)}{\gamma B(s)}} = c_0 \sqrt{\frac{-B(s)}{\gamma + (\gamma - 1)D(\sigma s)}}. \quad (36)$$

In the tube domains where viscothermal losses is of importance this sound speed is used, see section 3 Implementation.

For the circular geometry depicted in Figure 1 the function  $B(s)$  is found as

$$B_c(s) = \frac{J_2(\text{si}\sqrt{i})}{J_0(\text{si}\sqrt{i})}. \quad (37)$$

In this case the characteristic length  $l_c$  which enters into the shear wave  $s$  is equal to the radius  $a$ .

For the rectangular geometry in Figure 2 the function  $B(s)$  is

$$B_r(s) = Q_1 \sum_{q=1,3,\dots}^{\infty} \frac{(-1)^{q-1}}{q^2 Q_2^2} \left( \frac{\tanh(l_x Q_2 / l_y)}{l_x Q_2 / l_y} - 1 \right), \quad (38)$$

with

$$Q_1 = \frac{i l_x^2 s^2 8^2}{l_y^2 \pi^2} \quad (39)$$

and

$$Q_2 = \frac{l_x}{l_y} \sqrt{\left(\frac{q\pi}{2}\right)^2 + i s^2}. \quad (40)$$

The characteristic length in this case is equal to  $l_y$ .

### 2.3. Model comparison

A relationship between the parameters used in the transmission line model and the low reduced frequency model can be established at this point. Rearranging equation (28) leads to

$$-\frac{dp}{dz} = i\omega\rho_0\langle v_z \rangle (-B(s))^{-1}. \quad (41)$$

Comparing this to equations (8), (10) and (12) and with the volume velocity being related to the area and the mean velocity as  $q = S\langle v_z \rangle$  leads to the simple relationship

$$B(s) = -F(\nu). \quad (42)$$

As described previously both  $B(s)$  and  $F(\nu)$  describe the inertial and viscous effects. A similar relationship is found for the parameters which take into account compressional

and thermal effects, namely  $D(\sigma s)$  and  $F(\nu')$ : First the complex compressibility  $C(\omega)$  is described as [11]

$$C(\omega) = \frac{\langle \delta \rangle}{\rho_0 p} \quad (43)$$

and combined with equation (34) this gives

$$C(\omega) = \frac{1}{\gamma p_0} (\gamma + (\gamma - 1)D(\sigma s)). \quad (44)$$

Comparing with equation (13) it is clear that

$$D(\sigma s) = -F(\nu'). \quad (45)$$

For the circular tube  $F(\nu)$  and  $B(s)$  are compared to each other. The expressions, equation (15) with the input argument  $\eta = \nu = \mu/\rho_0$ , are

$$F_c(\nu) = -\frac{J_2(a\sqrt{-i\omega\rho_0/\mu})}{J_0(a\sqrt{-i\omega\rho_0/\mu})}, \quad (46)$$

and equation 37 with  $s = a\sqrt{\rho_0\omega/\mu}$  gives

$$B_c(s) = \frac{J_2(ai\sqrt{i\omega\rho_0/\mu})}{J_0(ai\sqrt{i\omega\rho_0/\mu})}. \quad (47)$$

Clearly, equation (46) is equal to minus equation (47), in accordance with equation (42). The same correspondence is found when comparing  $F_c(\nu')$  and  $D_c(\sigma s)$ .

For the rectangular case a similar comparison is done between  $F_r(\nu)$ , i.e. equation (16) with the input argument  $\eta = \nu = \mu/\rho_0$ , and  $B_r(s)$  in equation (38). The two expressions seem quite dissimilar at first glance but a numerical comparison reveals that at least for some side length ratios they actually tend to the same value when enough summations are considered as well as the sign in equation (42). In Figure 4 the difference between the two expressions as a function of frequency is shown when summing 16, 64 and 256 terms, respectively. In the actual implementation several thousand summations were performed for each frequency. It seems to be an inherent restriction in the low reduced frequency model that the side length  $l_x$  be larger than  $l_y$ , which is not the case for the transmission line expression.

The equilateral triangular cross-section is considered in the transmission line model, but not in the low reduced frequency model as it has no characteristic length. However, since the characteristic length is not of great importance when considering only the expressions related to mean values such as  $F(\nu)$ ,  $F(\nu')$ ,  $B(s)$  and  $D(\sigma s)$  the low reduced frequency model can be applied as long as these mean values can be established such as is the case with the triangular tube.

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when considering only the expressions related to mean values such as  $F(\nu)$ ,  $F(\nu')$ ,  $B(s)$  and  $D(\sigma s)$  the low reduced frequency model can be applied as long as these mean values can be established such as is the case with the triangular tube. From equations (17) and (42) an expression for  $B$  in the case of an equilateral triangular cross-section is found as

$$B_t(\nu) = -(\epsilon^2 - 3\epsilon \coth \epsilon + 3)/\epsilon^2, \quad (48)$$

again with

$$\epsilon = d \sqrt{\frac{3}{16} \frac{i\omega}{\nu}}. \quad (49)$$

The input argument  $\nu$  is now used instead of the shear wavenumber  $s$  as  $s$  depends on a characteristic length. Similarly, the low reduced frequency parameter  $D$  for the equilateral triangular cross-section is found as  $D_t(\sigma\nu) = D_t(\nu') = B_t(\nu')$  for the isothermal case. With  $B_t(\nu)$  and  $D_t(\nu')$  established, the triangular case can now be handled in a low reduced frequency implementation.

### 3. Implementation

The low reduced frequency model was implemented in COMSOL Multiphysics. Having entered  $B(s)$  and  $D(\sigma s)$  for the case in question, i.e. either a circular, a rectangular or an equilateral triangular cross-section, an effective sound speed was calculated. When considering just a single tube, the tube was meshed using one-dimensional elements and boundary conditions were applied at either end of the mesh. When the tube served as a connection between two volumes special care had to be taken at the interfaces between the tube meshed with one-dimensional elements and the volumes mesh using three-dimensional elements to ensure continuity of pressure and volume velocity. The sound pressure is assumed constant in the cross-section of the tube and so this must also be the case for the part of volume which makes up the area of the tube (see Figure 5). Similarly, the volume velocity at the interface, given by the mean velocity in equation (28) times the area, must be continuous from the end node of the tube to the area of the volume. In COMSOL Multiphysics this can be done using a so-called weak constraint boundary condition which in a weak or integral sense ensures continuity in one or more variables between two boundaries.

For the rectangular case  $B(s)$  consists of a number of summations. This summations cannot be calculated directly in COMSOL Multiphysics, but instead the calculations were done in MATLAB, utilizing the option in COMSOL Multiphysics to interface with MATLAB.

### 4. Test cases

Several test cases were used to validate the implementation of the low reduced frequency. First, single tubes with

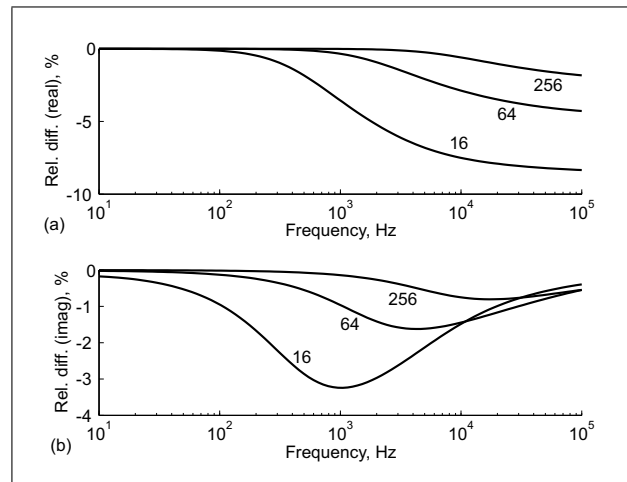


Figure 4. The relative difference in percent between  $B(s)$  and  $-F(\nu)$  for the real (a) and the imaginary part (b) compared to the absolute value of  $B(s)$  for a tube with a cross-section of 1 mm by 1 mm for 16, 64 and 256 summations, respectively.

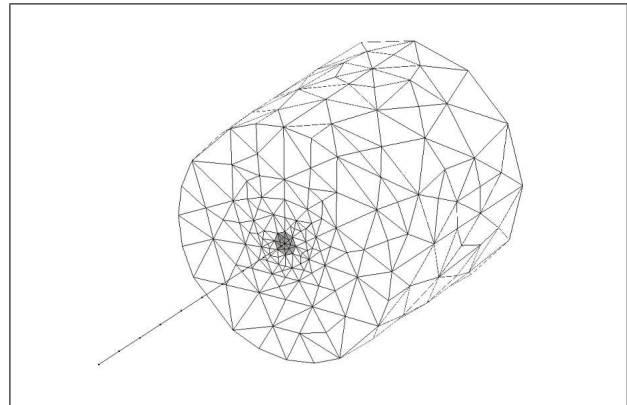


Figure 5. A tube is modeled with a one-dimensional mesh which leads up to a three-dimensional mesh. Continuity of the acoustic variables must be ensured between the end node and the shaded part of the mesh with the area of the tube, here with a circular cross-section.

circular, rectangular and triangular cross-section, respectively, were considered. Each tube was 17 cm long resulting in an axial resonance of 1 kHz with the fluid values considered here. For these tubes a velocity boundary condition of 1.4e-6 m/s at one end and a rigid, isothermal wall condition at all other boundaries were used. The sound pressure level was calculated at the opposite end of the velocity boundary condition. The meshes used for the low reduced frequency model in COMSOL Multiphysics were all one-dimensional. For all three tubes results were also found using a full Navier-Stokes implementation (see appendix A1) with the mesh used for the rectangular (or actually square) tube with  $l_x = 5$  mm and  $l_y = 5$  mm shown in Figure 6 and the mesh for the triangular case with the side length  $d = 7.6$  mm shown in Figure 7, both with mirror symmetry properties taken into account. The circular case was calculated for a radius  $a = 3$ , using both the three-dimensional implementation and a two-dimensional

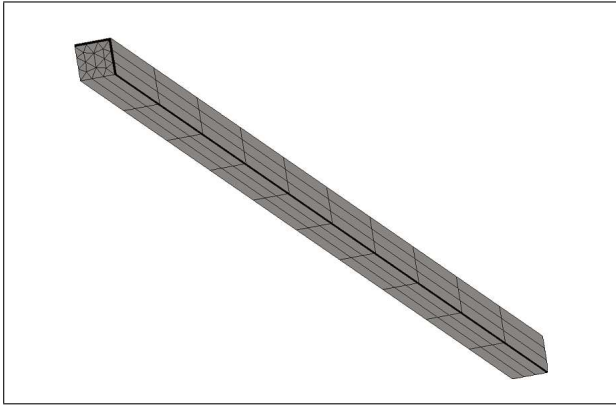


Figure 6. The mesh used for calculations on a tube with square cross-section using the full Navier-Stokes implementation. Mirror symmetry is made use of as only 1/4 of the geometry is meshed.

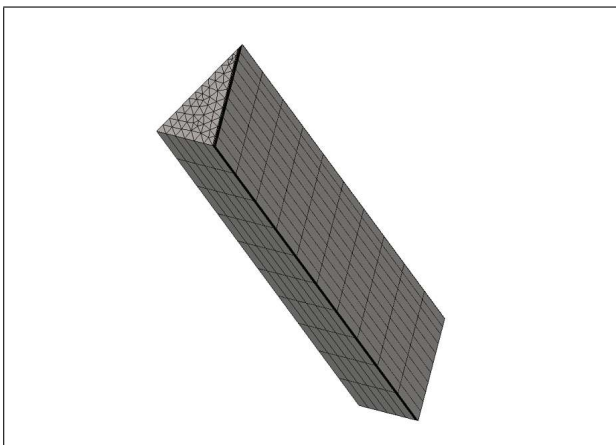


Figure 7. The mesh used for calculations on a tube with triangular cross-section using the full Navier-Stokes implementation. Mirror symmetry is made use of as only 1/6 of the geometry is meshed.

implementation with rotational symmetry taken into account. Both implementations gave practically the same results. All meshes were refined in the region near the tube wall to sufficiently resolve the boundary layer effects.

An analytical transmission line model was also used for comparison, as well as the commercial program ACTRAN which uses an implementation of the low reduced frequency with the possibility of using three-dimensional meshes. This option was chosen here, the meshes are not depicted here, as they were very simple.

An additional test case was chosen with two volumes interconnected by a narrow tube. This case is of relevance as not only must the damping - equivalent to having the correct propagation wavenumber in a transmission line model of the tube - be correctly modeled, but also the continuity at the domain intersections must be ensured, equivalent to having correct characteristic impedances in the transmission line model for each domain. Results were found for both a circular and a triangular tube with the same area as the circular tube connecting the two volumes. The case with the circular tube is illustrated in Figure 8. The area

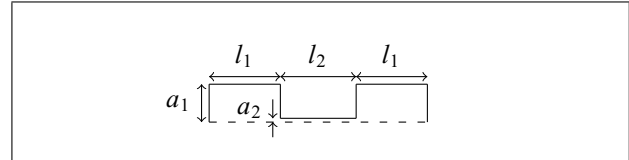


Figure 8. The geometry of the test case with a tube connecting two circular cylindrical volumes. There is rotational symmetry around the dotted axis. The values  $a_1 = 5$  mm,  $a_2 = 0.5$  mm,  $l_1 = 9.4$  mm and  $l_2 = 10$  mm were used. A similar test case with a triangular tube with an area equal to the circular tube depicted (having  $d = 1.347$  mm) was also used.

of the circular tube with a radius  $a = 0.5$  mm is equal to  $0.785 \text{ mm}^2$  and for an equilateral triangle with side lengths  $d$  the area is equal to  $d^2 \sqrt{3}/16$ , so  $d$  for the triangular tube is equal to 1.347 mm. A velocity boundary condition of 1 m/s was put on the end area of one of the volumes and the sound pressure level at the other end of the test case was calculated. The particular geometry has a Helmholtz resonance of around 700 Hz and the amplitude around that frequency is greatly affected by the dissipation in the model. For the case of a circular tube results were also found using ACTRAN.

## 5. Results

Results are presented in a narrow frequency range near the resonance frequencies of the test cases. The viscous and thermal effect will have the greatest influence near the resonances whereas in the rest of the frequency range the results found using the different models are more or less indistinguishable.

For the circular case several calculation methods were available: The low reduced frequency implementation made in COMSOL Multiphysics, an analytical solution, a full Navier-Stokes model implemented in COMSOL Multiphysics, using either a 3D mesh with some mirror symmetry taken into account or a 2D mesh taking advantage of the rotational symmetry, and finally the commercial program ACTRAN which uses a low reduced frequency implementation, with an option of using a three-dimensional mesh which was used in this case. The results are shown in Figure 9.

For the rectangular case the same methods are available except that the full Navier-Stokes model uses a 3D mesh where symmetry planes can be utilized to reduce the computational requirements. The results for a tube with a square cross-section are shown in Figure 10.

The triangular case is not implemented in ACTRAN since it is not traditionally a part of the possible geometries when using the low reduced frequency model, but otherwise the same modeling methods as for the rectangular case can be used, and the results for this case are shown in Figure 11.

For the tube test cases the analytical results were regarded the target values, and so the numerical results should at best coincide with these. The numerical results

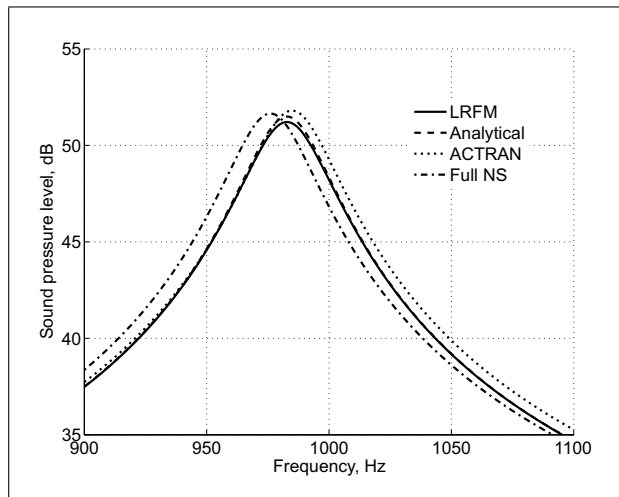


Figure 9. The sound pressure level found using the low reduced frequency model implementation (LRFM), an analytical model, ACTRAN and the full Navier-Stokes model (Full NS) for the case of a circular tube with radius  $a = 3$  mm.

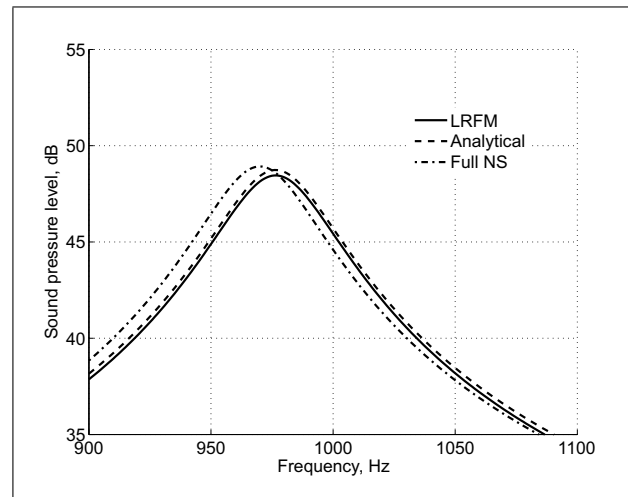


Figure 11. The sound pressure level found using the low reduced frequency model implementation (LRFM), an analytical model, and the full Navier-Stokes model (Full NS) for the case of a triangular tube with side lengths 7.6 mm.

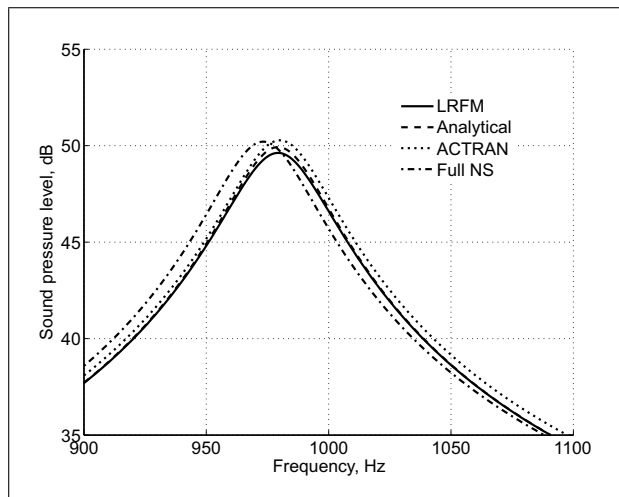


Figure 10. The sound pressure level found using the low reduced frequency model implementation (LRFM), an analytical model, ACTRAN and the full Navier-Stokes model (Full NS) for the case of a rectangular tube with  $l_x = 5$  mm and  $l_y = 5$  mm.

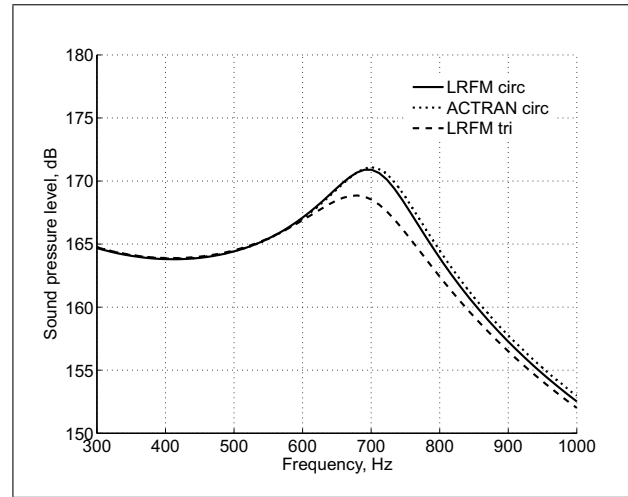


Figure 12. The sound pressure level found using the low reduced frequency model implementation (LRFM circ) and ACTRAN (ACTRAN circ) for the case depicted in Figure 8 as well as for the case of a triangular tube with the same area as the circular one interconnecting the two volumes, calculated with the low reduced frequency model implementation (LRFM tri).

all deviate from the analytical results, especially the full Navier-Stokes implementation seems to have its resonance somewhat shifted in frequency. This phenomenon was observed for other test cases too, but the reason for this could not be established. The two software packages with the low reduced frequency model implementation, i.e. COMSOL Multiphysics and ACTRAN, were evaluated at different times using different meshes for the tubes cases. However, the results found using the two implementations compare well with each other, with the COMSOL Multiphysics results being very close to the analytical results.

The case with a tube interconnecting two volumes was considered both for a circular and a triangular tube of equal cross-sectional area. The sound pressure level at one end of the geometry is found using both ACTRAN and the low reduced frequency implementation in COMSOL for

the circular tube, and the results are very similar, see Figure 12. When the tube is triangular, the loss is higher compared to a circular cross-section, as can be seen from the results found using the low reduced frequency implementation.

## 6. Conclusion

An implementation of the low reduced frequency model was done in COMSOL Multiphysics. The implementation can be used for acoustic calculations on narrow tubes where viscothermal losses are of importance. Three cross-section can be handled, the circular and the rectangular which are traditionally associated with the low reduced

frequency, and the new addition of an equilateral triangular cross-section which is not usually considered when using the model. The case is addressed in another model and once a relationship was established between the two models, low reduced frequency parameters could be found for the triangular case.

Different cases were used to test the implementation, both single tubes modeled using one-dimensional elements as well as cases where the tubes connect volumes in which viscothermal losses are not of importance. Continuity between the lossy tube and the loss-free volumes are ensured using a so-called weak constraint boundary condition, so that pressure and volume velocity are continuous from one domain to the other.

Results have been found using various implementations. The results differ slightly but the low reduced frequency implementation gives results that compare well with the other models. Using this implementation it is now possible to include the triangular cross-section in the cases that can be handled using the low reduced frequency.

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## Appendix

### A1. Full Navier-Stokes implementation

A full Navier-Stokes implementation taking into account viscothermal losses by doing calculations using a coupled system of governing equations was provided by COMSOL. The implementation will not be discussed in detail here, but a brief description is given in the following.

The linearized governing equations (the Navier-Stokes equation, the equation of continuity, the energy equation and the equation of state for an ideal gas) when small, harmonic variation and no sources are assumed are given as [7]

$$i\omega\rho_0\mathbf{v} = -\nabla p + \left(\frac{4}{3}\mu + \xi\right)\nabla(\nabla\cdot\mathbf{v}) - \mu\nabla\times(\nabla\times\mathbf{v}), \quad (\text{A1})$$

$$i\omega\delta + \rho_0\nabla\cdot\mathbf{v} = 0, \quad (\text{A2})$$

$$i\omega\rho_0 C_P T = \lambda\nabla^2 T + i\omega p \quad (\text{A3})$$

and

$$\delta = \rho_0 \left( \frac{p}{p_0} - \frac{T}{T_0} \right). \quad (\text{A4})$$

Here,  $\xi$  is the bulk viscosity, assumed equal to  $2\mu/3$ ,  $\lambda$  is the thermal conductivity and  $C_P$  is the specific heat at constant pressure. Equation (A4) can be inserted into equation (A2):

$$i\omega \left( \frac{p}{p_0} - \frac{T}{T_0} \right) + \nabla\cdot\mathbf{v} = 0 \quad (\text{A5})$$

and so the density variation is eliminated.

These coupled equations (the Navier-Stokes equation (A1), the energy equation (A3) and the equation of continuity equation A5) are implemented in a weak, or integral, form in COMSOL Multiphysics, and one can then solve the system for the unknowns, i.e. the velocity  $\mathbf{v}$ , the acoustic pressure  $p$  and the temperature variation  $T$ . A set of boundary conditions have been set up by COMSOL, namely 1) Isothermal wall, 2) Adiabatic wall, 3) Velocity, Velocity, isothermal and 5) Pressure, adiabatic, as well as mirror or rotational symmetry conditions, depending on the dimensions of the problem.

For all calculations done using this implementation the following values were used:  $C_P = 1,010 \text{ J/(kg K)}$ ,  $T_0 = 300 \text{ K}$ ,  $p_0 = 10^5 \text{ Pa}$  and  $\lambda = 0.026 \text{ W/(m K)}$ .

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