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Note on Acoustic Pseudopotentials: Energy Viewpoint, Real Scatterers and Linear Arrays

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Summary
A study of scalar acoustic pseudopotentials and their connection to actual spherical scatterers is presented. The pseudopotential concept and the governing scattering equations are established through an alternative energy viewpoint from which the operator-based definition follows. Limiting formulas are derived for spherical acoustic scatterers with subwavelength size and it is shown that pseudopotentials can approximate only certain scattering types: impedance or soft impenetrable ones or penetrable ones having equal density but different compressibility from the surrounding medium. The accuracy of the approximation is investigated numerically and mapped versus the scatterer size and distance of the observation point. As an application, the infinite linear periodic array of small spheres is considered and it is shown that the frequency dependence of a scatterer's responsivity to the incident field forbids the existence of traveling waves except in the cases of positive impedance scatterers or penetrable scatterers having the same density but higher compressibility than the surrounding medium.

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1. Introduction

In wave physics the pseudopotential is a mathematical concept that serves to replace an actual scatterer of subwavelength size by a point with the same macroscopic scattering behavior. Its conception dates back to 1936 and is due to E. Fermi who introduced a Dirac's δ potential into Schrödinger's equation to model the scattering of a neutron by a hydrogen nucleus. A concept similar to the pseudopotential –termed pseudodipole– has also been introduced for the case of Maxwell equations [1], to represent infinitesimal dipoles [2] that radiate under the influence of an external field and has been used for the analysis of resonances in antenna arrays [1, 3]. On the other hand, (scalar) pseudopotentials are appropriate to model acoustic (pressure) wave scattering by infinitesimal objects.

Perusing the literature it turns out that there exist very few papers on the concept of pseudopotentials in electromagnetic or acoustic wave scattering. Among them, [1, 4] are the most salient, yet very mathematical oriented. However, even though the term pseudopotential is hardly used, wave phenomena connected with subwavelength sized objects, often referred to as point scattering, have always attracted significant attention [5, 6, 7]. Apart from the physical importance, i.e. in scattering by particles [8], point scattering has also been found to be a useful mathematical tool, e.g. in the context of discrete-dipole-approximation methods, where a continuous scattering body is approximated by a cluster of discrete point scatterers [9, 10]. Furthermore and particularly in the photonics field, considerable effort has been targeted over the past decade at the light transfer properties of chains of metallic nanoparticles with diameters in the order of $\lambda/10$ [11, 12, 13], which revives the classic antenna array problem and the pseudopotential concept in the optical domain. The scattering properties of deep subwavelength ($\lambda/200$) artificial objects are also under research for biomedical and other applications [14, 15].

The purpose of this paper is to present a thorough and tutorial treatment of scalar pseudopotenials in the context of acoustic waves and clarify the connection with scattering by actual subwavelength sized spheres. Instead of the operator-based definition given in [1, 4] and aiming for a deeper physical understanding, an alternative path is followed by resorting to simple energy conservation arguments. Specifically the paper is organized as follows: Initially (section 2) the concept of scalar acoustic pseudopotentials is revisited through an energy perspective. Equating the radiated and incident powers leads to the basic scattering equations for single and multiple pseudopotentials revealing, at the same time, the locus of the complex responsivity factor (the polarizability for pseudodipoles). Based on these results, the equivalent definition as an additional operator to the Helmholtz wave equation is derived heuristically with the aid of the radiation reaction field concept. In section 3 we consider actual, finite-sized spherical scatterers with hard, soft, impedance or penetrable boundary conditions. We derive limiting formulas in the

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deep sub-wavelength size regime and deduce the boundary conditions that allow one to approximate the scatterer by a pseudopotential. The accuracy of the approximation is investigated numerically versus the radius of the scatterer and the proximity of the observation point. Finally, in section 4 we revisit the infinite linear periodic array (ILPA) of pseudopotentials and derive the dispersion equation of Bloch modes. Using the results of section 3, we reach the so far unreported conclusion that, due to the frequency dependence of the responsivity factor, traveling waves in an ILPA of small spherical scatterers do not exist except in the case of a positive surface impedance or for penetrable scatterers having the same density but being more compressible than the surrounding medium.

2. Energy view of acoustic pseudopotentials

2.1. Energy conservation in scattering

To study the concept of scalar pseudopotentials, we consider $\exp(\mathrm{j}\omega t)$ time-harmonic acoustic waves produced in an inviscid fluid by sources with spatial distribution $f(\boldsymbol{r})$. The resulting pressure and velocity fields p_f, v_f, satisfy the linearized continuity and force (Euler's) equations [16]

$$\nabla\cdot\boldsymbol{v}_f(\boldsymbol{r}) = -\mathrm{j}k_0 Z_0^{-1} p_f(\boldsymbol{r}) + f(\boldsymbol{r}), \tag{1}$$

$$\nabla p_f(\boldsymbol{r}) = -\mathrm{j}k_0 Z_0 \boldsymbol{v}_f(\boldsymbol{r}), \tag{2}$$

where $k_0 = \omega/c_0$ is the acoustic wavenumber and $Z_0 = \rho_0 c_0$ is the acoustic impedance with ρ_0 being the equilibrium density and c_0 the speed of sound in the fluid. The source $f(\boldsymbol{r})$ has units 1/s and is interpreted as the rate of fractional volume change produced by the acoustic source or, alternatively, as ρ_0^{-1} times the rate of virtual mass injection per unit volume. Substitution of equation (2) into equation (1) yields the inhomogeneous Helmholtz equation

$$\left(\nabla^2 + k_0^2\right) p_f(\boldsymbol{r}) = -\mathrm{j}k_0 Z_0 f(\boldsymbol{r}), \tag{3}$$

which has the well-known solution

$$p_f(\boldsymbol{r}) = \mathrm{j}k_0 Z_0 \iiint f(\boldsymbol{r}')G_0(\boldsymbol{r},\boldsymbol{r}')\,\mathrm{d}V', \tag{4}$$

where

$$G_0(\boldsymbol{r},\boldsymbol{r}') = \frac{\mathrm{e}^{-\mathrm{j}k_0|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{5}$$

is Green's function and vectors $\boldsymbol{r},\boldsymbol{r}'$ are observation and source points respectively. The total time-averaged power radiated by the sources is given by the integral of power intensity over an enclosing surface S [16]

$$P_{\mathrm{rad},f} = \oiint_S \frac{1}{2}\Re e\left\{p_f(\boldsymbol{r})\boldsymbol{v}_f^*(\boldsymbol{r})\right\}\cdot\mathrm{d}S, \tag{6}$$

where the asterisk stands for complex conjugation, $\Re e\{..\}$ for the real part and the surface element $\mathrm{d}S$ is directed outwards. Using Gauss's divergence theorem (Appendix A1) this power can equivalently be written as

$$P_{\mathrm{rad},f} = \frac{1}{2}\Re e\iiint_{V(S)} p_f(\boldsymbol{r}')f^*(\boldsymbol{r}')\,\mathrm{d}V', \tag{7}$$

where $V(S)$ is the volume enclosed by S. The latter equation states that the radiated power is produced by the pulsation of the sources against the pressure field p_f at a rate of f Hz and is consistent with energy conservation given that the medium is lossless.

When a pseudopotential is introduced at the source-free point $\boldsymbol{r}_s$, it acts as a secondary wave source with complex amplitude A that is proportional to the incident pressure

$$A = \frac{4\pi}{k_0^2 Z_0}\tau p_f(\boldsymbol{r}_s), \tag{8}$$

where $4\pi/(k_0^2 Z_0)$ has been introduced to make the proportionality (or responsivity) factor τ dimensionless. As a result of radiation from the pseudopotential, the total pressure and velocity fields are disturbed to the values

$$p_{\mathrm{tot}}(\boldsymbol{r}) = p_f(\boldsymbol{r}) + p_s(\boldsymbol{r}), \tag{9}$$

$$\boldsymbol{v}_{\mathrm{tot}}(\boldsymbol{r}) = \boldsymbol{v}_f(\boldsymbol{r}) + \boldsymbol{v}_s(\boldsymbol{r}), \tag{10}$$

where the scattered fields satisfy analogous to (1), (2) differential equations

$$\nabla\cdot\boldsymbol{v}_s(\boldsymbol{r}) = -\mathrm{j}k_0 Z_0^{-1} p_s(\boldsymbol{r}) + A\delta(\boldsymbol{r}-\boldsymbol{r}_s), \tag{11}$$

$$\nabla p_s(\boldsymbol{r}) = -\mathrm{j}k_0 Z_0 \boldsymbol{v}_s(\boldsymbol{r}). \tag{12}$$

Combining the latter, we obtain the inhomogeneous Helmholtz equation for the scattered pressure

$$\left(\nabla^2 + k_0^2\right) p_s(\boldsymbol{r}) = -\mathrm{j}k_0 Z_0 A\delta(\boldsymbol{r}-\boldsymbol{r}_s), \tag{13}$$

whose solution is proportional to Green's function

$$p_s(\boldsymbol{r}) = \mathrm{j}k_0 Z_0 A G_0(\boldsymbol{r},\boldsymbol{r}_s). \tag{14}$$

Now let us consider the integral expressing the time-averaged total power flowing out of a surface that encloses the pseudopotential but leaves the sources out,

$$P_{S_s} = \oiint_{S_s} \frac{1}{2}\Re e\left\{p_{\mathrm{tot}}(\boldsymbol{r})\boldsymbol{v}_{\mathrm{tot}}^*(\boldsymbol{r})\right\}\cdot\mathrm{d}S. \tag{15}$$

Since a pseudopotential is a passive and lossless scatterer this integral must be exactly zero because energy is neither created nor dissipated inside S_s. This approach was also used in [10] for continuous dielectric media. Substituting equations (9), (10) into equation (15) we obtain

$$\begin{aligned} P_{S_s} = 0 &= \frac{1}{2}\Re e\oiint_{S_s}\left\{p_f(\boldsymbol{r})\boldsymbol{v}_f^*(\boldsymbol{r})\right\}\cdot\mathrm{d}S \\ &+ \frac{1}{2}\Re e\oiint_{S_s}\left\{p_s(\boldsymbol{r})\boldsymbol{v}_s^*(\boldsymbol{r})\right\}\cdot\mathrm{d}S \\ &+ \frac{1}{2}\Re e\oiint_{S_s}\left\{p_f(\boldsymbol{r})\boldsymbol{v}_s^*(\boldsymbol{r}) + p_s(\boldsymbol{r})\boldsymbol{v}_f^*(\boldsymbol{r})\right\}\cdot\mathrm{d}S. \end{aligned} \tag{16}$$

The first integral is the power flowing out of S_s due to the incident field and is exactly zero because its sources are outside S_s and the medium is lossless. This is also rigorously deduced if equation (7) is applied for any surface inside which $f(\boldsymbol{r}) = 0$. The second integral is the power $P_{\mathrm{rad},s}$ radiated by the pseudopotential through its scattered field. For a lossless medium, $P_{\mathrm{rad},s}$ is independent of the shape of S_s thus it is convenient to assume a sphere centered at $\boldsymbol{r}_s$ and use equations (12), (14) to find

$$P_{\mathrm{rad},s} = \frac{k_0^2 Z_0}{8\pi} \left| A \right|^2. \tag{17}$$

The third integral expresses power flowing out of S_s due to interaction between the incident and scattered fields. In Appendix A2 this integral is shown to be expressed as

$$\frac{1}{2}\Re e \oiint_{S_s} \left\{ p_f(\boldsymbol{r}) \boldsymbol{v}_s^*(\boldsymbol{r}) + p_s(\boldsymbol{r}) \boldsymbol{v}_f^*(\boldsymbol{r}) \right\} \mathrm{d}S = \frac{1}{2}\Re e\left\{ p_f(\boldsymbol{r}_s) A^* \right\}, \tag{18}$$

which is substituted together with equation (17) into equation (16) to obtain

$$\frac{1}{2}\Re e\left\{ -p_f(\boldsymbol{r}_s) A^* \right\} = P_{\mathrm{rad},s} = \frac{k_0^2 Z_0}{8\pi} \left| A \right|^2. \tag{19}$$

The latter is an elegant expression of energy conservation: the power radiated by the pseudopotential through its scattered field is equal to the power offered by the incident field to the pseudopotential which is expressed by $\frac{1}{2}\Re e\{-p_f A^*\}$. Notice that the minus sign in the latter product is consistent with its definition as power absorbed by the source $A\delta(\boldsymbol{r}-\boldsymbol{r}_s)$, in comparison to the positive sign in equation (7) which defines power offered by the source $f(\boldsymbol{r})$.

2.2. Radiation reaction field

Energy conservation imposes a constraint on the proportionality factor τ, i.e. the responsivity of the pseudopotential to the incident field. Indeed by substituting equation (8) into equation (19) we obtain

$$\left| \tau \right|^2 = -\Re e\{\tau\} \quad \Leftrightarrow \quad \tau = \frac{\mathrm{j}\alpha}{1 - \mathrm{j}\alpha}, \tag{20}$$

where $\alpha = -\cot(\varphi_\tau)$, φ_τ being the argument of τ. Therefore the locus of τ on the complex plane is the circle with radius $\frac{1}{2}$ centered at $(-\frac{1}{2}, 0)$ while α can take any real value. Moreover, equation (20) allows an interesting explanation of the excitation of the pseudopotential. Introducing it into equation (8) we obtain

$$A = \mathrm{j}\alpha \frac{4\pi}{k_0^2 Z_0} \left[p_f(\boldsymbol{r}) + p_{rr} \right], \tag{21}$$

where

$$p_{rr} = \frac{k_0^2 Z_0}{4\pi} A. \tag{22}$$

It turns out that the pseudopotential amplitude A is proportional and 90° out of phase to the sum of the incident field and an additional field term p_{rr} which is proportional to A and represents a self-interaction. From equations (17), (22) we also find

$$\frac{1}{2}\Re e\left\{ p_{rr} A^* \right\} = P_{\mathrm{rad},s}, \tag{23}$$

which implies that p_{rr} is the *radiation reaction* field of the pseudopotential, i.e. the effective pressure field against which the pseudopotential works at rate $A\delta(\boldsymbol{r} - \boldsymbol{r}_s)$ Hz to produce the radiated power $P_{\mathrm{rad},s}$. In that sense, equation (21) can be interpreted as an alternative to equation (8), showing that the radiation reaction field is part of the total field that excites the pseudopotential and to which the pseudopotential responds with a pure imaginary proportionality factor $\mathrm{j}\alpha$.

Through the concept of radiation reaction field one can reach the operator-based definition of pseudopotentials given in [1, 4]. Rewriting equation (22) as

$$p_{rr} = \left(\frac{\mathrm{j}k_0 Z_0}{4\pi} A \right) \left(-\mathrm{j}k_0 \right) = \left(\frac{\mathrm{j}k_0 Z_0}{4\pi} A \right) \lim_{R_s \to 0} \left[\frac{\partial}{\partial R_s} R_s \frac{\mathrm{e}^{-\mathrm{j}k_0 R_s}}{R_s} \right],$$

where $R_s = |\boldsymbol{r} - \boldsymbol{r}_s|$, and using equation (14), the radiation reaction field is given in terms of the scattered field as

$$p_{rr} = \lim_{R_s \to 0} \frac{\partial}{\partial R_s} \left[R_s p_s(R_s) \right]. \tag{24}$$

Substituting the latter into equation (21)

$$A = \mathrm{j}\alpha \frac{4\pi}{k_0^2 Z_0} \lim_{R_s \to 0} \frac{\partial}{\partial R_s} \left[R_s p_{\mathrm{tot}}(\boldsymbol{r}) \right], \tag{25}$$

where p_{tot} is the total pressure field of equation (9). If this expression is introduced into the Helmholtz equation for the total field, namely the sum of equations (3) and (13), we obtain

$$\left[\nabla^2 + k_0^2 - \delta(\boldsymbol{r} - \boldsymbol{r}_s) \frac{4\pi\alpha}{k_0} \frac{\partial}{\partial R_s} R_s \right] p_{\mathrm{tot}}(\boldsymbol{r}) = -\mathrm{j}k_0 Z_0 f(\boldsymbol{r}), \tag{26}$$

which is exactly the operator given in equation (2.2) of [1], noticing however the different notation in that our dimensionless real α corresponds to $k_0\alpha$ of [1].

2.3. Extension to multiple scattering

Now assume a set of N pseudopotential scatterers located at the source-free points $\boldsymbol{r}_n$, $n = 1, 2, \ldots, N$ and radiating with amplitudes A_n under the excitation of the fields produced by the sources f but also under their mutual interaction. Then in equation (8) the field which excites the m-th pseudopotential must be augmented by the sum of the fields radiated from the other pseudopotentials. Explicitly

$$A_m \left(1 - \mathrm{j}\alpha_m \right) + \frac{4\pi\alpha_m}{k_0} \sum_{n \neq m} A_n G_0(\boldsymbol{r}_m, \boldsymbol{r}_n) = \frac{\mathrm{j}4\pi\alpha_m}{k_0^2 Z_0} p_f(\boldsymbol{r}). \tag{27}$$

Repeating (27) for $m = 1, 2, \ldots, N$ we obtain a $N \times N$ system of coupled linear equations which are essentially the Foldy-Lax multiple-scattering equations [17] of the interacting objects. Notice that the subscript in the responsivity factor α_m indicates that the pseudopotentials are not necessarily identical. In the absence of sources ($p_f = 0$) equation (26) is exactly the equation obtained in [1] for arrays of pseudopotentials through an operator-based definition of the pseudopotential, noting however that our $-\mathrm{j}$ follows from the assumed $\exp(+\mathrm{j}\omega t)$ time dependence. It is also instructive to write the augmented Helmholtz equations satisfied by the total field in the presence of multiple pseudopotentials. In analogy to equation (26) we have

$$\Big[\nabla^2 + k_0^2 - \sum_n \delta(\boldsymbol{r} - \boldsymbol{r}_n)\frac{4\pi\alpha_n}{k_0}\frac{\partial}{\partial R_n}R_n\Big]p_{\mathrm{tot}}(\boldsymbol{r}) = -\mathrm{j}k_0 Z_0 f(\boldsymbol{r}), \qquad (28)$$

where $R_n = |\boldsymbol{r} - \boldsymbol{r}_n|$.

3. Actual spherical scatterers

3.1. Impenetrable scatterers

Here we proceed to investigate the asymptotic equivalence of small scatterers with pseudopotentials. Consider an acoustically impenetrable spherical scatterer centered at the origin ($r = 0$), having radius d and an impedance (Robin) boundary condition on its surface

$$\left(k_0 Z_0 p_{\mathrm{tot}} + Z_s\frac{\partial p_{\mathrm{tot}}}{\partial r}\right)\Big|_{r=d} = 0, \qquad (29)$$

where $\mathrm{j}Z_s$ is the surface impedance which is assumed to be imaginary so that the scatterer does not consume or produce power. Without loss of generality, the incident field (p_f) is taken to be that of a point source with amplitude F located at $\boldsymbol{r}_f = (r_f, \theta_f, \phi_f)$, i.e. $f(\boldsymbol{r}) = F\delta(\boldsymbol{r} - \boldsymbol{r}_f)$. Thus from equation (4) $p_f(\boldsymbol{r}) = \mathrm{j}k_0 Z_0 F G_0(\boldsymbol{r}, \boldsymbol{r}_f)$. Noting that $G_0(\boldsymbol{r}, \boldsymbol{r}_f) = -\mathrm{j}k_0 h_0^{(2)}(k_0|\boldsymbol{r} - \boldsymbol{r}_f|)/(4\pi)$, where $h_n^{(2)}$ is the second-kind spherical Hankel function [18], p_f can be expanded in spherical waves by virtue of the addition theorem for $h_0^{(2)}$ [19]

$$\begin{aligned} p_f(\boldsymbol{r}) &= \frac{k_0^2 Z_0 F}{4\pi} h_0^{(2)}\big(|\boldsymbol{r} - \boldsymbol{r}_f|\big) \qquad (30)\\ &= \frac{k_0^2 Z_0 F}{4\pi}\sum_{n=0}^{\infty}(2n+1)h_n^{(2)}\big(k_0 r_f\big) j_n\big(k_0 r\big) P_n(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}_f), \end{aligned}$$

where j_n is the spherical Bessel function, P_n is the Legendre polynomial and $\hat{\boldsymbol{r}} = \boldsymbol{r}/r$, $\hat{\boldsymbol{r}}_f = \boldsymbol{r}_f/r_f$ are unit vectors. This formula holds for $r < r_f$. Under the influence of the incident field, the sphere produces a scattered field that can be similarly expanded in outgoing spherical waves

$$p_s(\boldsymbol{r}) = \sum_{n=0}^{\infty} s_n h_n^{(2)}\big(k_0 r\big) P_n(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}_f). \qquad (31)$$

By applying the boundary condition (29) for the total field on $r = d$ and utilizing the orthogonality of Legendre polynomials, the coefficients s_n are found

$$s_n = t_n \frac{k_0^2 Z_0 F}{4\pi}(2n+1)h_n^{(2)}\big(k_0 r_f\big), \qquad (32)$$

where

$$t_n = \frac{\mathrm{j}w_n}{1 - \mathrm{j}w_n}, \qquad (33)$$
$$w_n = -\frac{Z_0 j_n(k_0 d) + Z_s j_n'(k_0 d)}{Z_0 y_n(k_0 d) + Z_s y_n'(k_0 d)}$$

is the scattering coefficient of the corresponding spherical harmonic, y_n is the spherical Neumann function, while the prime denotes the first derivative with respect to the argument.

It is very interesting to notice the similarity of the expression for t_n in equation (33) with that of τ in equation (20) and the consequent connection between w_n and α. This is a result of energy conservation: If the wave $j_n(k_0 r)P_n(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}_f)$ is incident on the sphere, spherical symmetry and orthogonality of Legendre polynomials dictate that the scattered wave will be $t_n h_n^{(2)}(k_0 r)P_n(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}_f)$. Since the scatterer does not consume or produce power, the outward power flow integral (equation 15) on $r = d$ must be zero. If this integral is performed analytically one obtains for $|t_n|^2 = -\Re e(t_n)$, i.e. the condition of equation (20).

Letting the radius of the scatterer become arbitrarily small compared to the wavelength ($k_0 d \to 0$) and using the small-argument approximations of spherical Bessel and Hankel functions [18], we find the limiting formulas for w_n

$$w_n \sim \begin{cases} -\frac{n(k_0 d)^{2n+1}}{(n+1)(2n-1)!!(2n+1)!!}, & n \geq 1,\\ -\frac{Z_0}{Z_s}\big(k_0 d\big)^2, & n = 0, \end{cases} \qquad (34)$$

which are valid for $(k_0 d) \ll |Z_s|/Z_0 \ll (k_0 d)^{-1}$. In the special case of an acoustically soft ($Z_s = 0$) or rigid ($Z_s = \infty$) scatterer these become respectively

$$w_n \sim \frac{(k_0 d)^{2n+1}}{(2n-1)!!(2n+1)!!}, \quad n \geq 0, \quad Z_s = 0 \qquad (35)$$

(by definition $(-1)!! = 0$) and

$$w_n \sim \begin{cases} -\frac{n(k_0 d)^{2n+1}}{(n+1)(2n-1)!!(2n+1)!!}, & n \geq 1,\\ \frac{1}{3}\big(k_0 d\big)^3, & n = 0, \end{cases} \quad Z_s = \infty. \qquad (36)$$

We now turn to the scattered field at a point of observation $\boldsymbol{r}$ which is at an arbitrary but finite distance from the origin. From equation (31) and with the help of equations (34), (35), (36), it is concluded that, in the limit of a small scatterer, p_s is expressed as a Taylor series for $k_0 d$. For an impedance or a soft scatterer the prevalent term is $n = 0$ which is an isotropically scattered term. From equations (31), (32) this zero-order approximation for the scattered field is written

$$p_s(\boldsymbol{r}) \approx \mathrm{j}\frac{4\pi t_0}{k_0} p_f(\boldsymbol{0}) G_0(\boldsymbol{r}, \boldsymbol{0}), \qquad (37)$$

where, by the first equality of equation (30), we have replaced $k_0^2 Z_0 F\, h_0^{(2)}(r_f)/(4\pi)$ with $p_f(\mathbf{0})$, i.e. the incident field at the center of the scatterer. Comparing equations (14) and (37) and with the aid of equation (8) it turns out that the field of equation (37) is identical with the field of a pseudopotential with $\tau = t_0$ or, equivalently, with $\alpha = w_0$.

Summarizing, an infinitesimally small impedance or soft acoustic scatterer behaves as a pseudopotential with responsivity factor

$$\alpha = w_0 = \begin{cases} -\frac{Z_0}{Z_s}\left(k_0 d\right)^2, & Z_s \neq 0, \\ k_0 d, & Z_s = 0. \end{cases} \tag{38}$$

The above conclusion is not true for an acoustically rigid (hard) scatterer. In that case, equation (36) shows that the prevalent terms in the scattered field Taylor series are $n = 0, 1$, both being proportional to $(k_0 d)^3$. The scattered field contains a comparable dipole term $h_1^{(2)}(k_0 r)\cos(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}_f)$ in addition to the isotropic term $h_0^{(2)}(k_0 r)$ and therefore an infinitesimally small and acoustically rigid scatterer cannot be replaced by a pseudopotential.

3.2. Penetrable scatterers

Now we consider the case of an acoustically penetrable sphere whose medium is characterized by density ρ_1 and acoustic speed c_1. We also assume the incident field of equation (30). By expanding the pressure inside the sphere in nonsingular spherical waves

$$p_{\text{tot}}(\boldsymbol{r}) = \sum_{n=0}^{\infty} \sigma_n j_n\left(k_1 r\right) P_n(\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{r}}_f), \tag{39}$$

where $k_1 = \omega/c_1$, applying the continuity boundary conditions for pressure, $p_{\text{tot}}(d^+) = p_{\text{tot}}(d^-)$, and normal velocity, $\rho_0^{-1} p'_{\text{tot}}(d^+) = \rho_1^{-1} p'_{\text{tot}}(d^-)$, and eliminating coefficients σ_n, we find that the coefficients of the scattered field are again given by equation (32) where

$$t_n = \frac{\mathrm{j} w_n}{1 - \mathrm{j} w_n}, \tag{40}$$
$$w_n = -\frac{Z_0 j_n(k_0 d) j_n'(k_1 d) - Z_1 j_n(k_1 d) j_n'(k_0 d)}{Z_0 y_n(k_0 d) j_n'(k_1 d) - Z_1 j_n(k_1 d) y_n'(k_0 d)},$$

which is the same with equation (32) with the impedance Z_s replaced by $-Z_1 j_n(k_1 d)/j_n'(k_1 d)$, $Z_1 = \rho_1 c_1$ being the acoustic impedance of the scatterer's medium. Letting $k_1 d \to 0$, $k_0 d \to 0$ the limiting formulas for w_n are found

$$w_n \sim \begin{cases} \frac{n(\rho_0 - \rho_1)(k_0 d)^{2n+1}}{[\rho_0 n + \rho_1(n+1)](2n-1)!!(2n+1)!!}, & n \geq 1, \rho_0 \neq \rho_1, \\ \frac{(1 - c_0^2/c_1^2)(k_0 d)^{2n+3}}{(2n+1)(2n+3)(2n-1)!!(2n+1)!!}, & n \geq 1, \rho_0 = \rho_1, \\ \left(1 - \frac{\kappa_1}{\kappa_0}\right)\frac{(k_0 d)^3}{3} & n = 0, \kappa_0 \neq \kappa_1, \\ \left(1 - \frac{c_0^2}{c_1^2}\right)\frac{(k_0 d)^5}{45} & n = 0, \kappa_0 = \kappa_1, \end{cases} \tag{41}$$

where $\kappa_i = 1/(\rho_i c_i^2)$, $i = 0, 1$, is the compressibility of the corresponding medium (units of inverse pressure). The formulas for $n = 0$ are valid for $Z_1/Z_0 \gg k_0 k_1 d^2$. From equations (40) follows that, in general, when $\rho_0 \neq \rho_1$ and $\kappa_0 \neq \kappa_1$, the prevalent $(k_0 d)^3$ powers are these of polar harmonics $n = 0, 1$; thus the radiated field is not isotropic. However, in the special case $\rho_0 = \rho_1$ and $\kappa_0 \neq \kappa_1$, namely when the scatterer has the same density but different compressibility from the surrounding medium, the prevalent cubic power is that of term $n = 0$ and thus the scatterer is asymptotically equivalent to a pseudopotential. It is also interesting to note that in the special case $\rho_0 \neq \rho_1$ and $\kappa_0 = \kappa_1$, the prevalent cubic power belongs to the term $n = 1$ while the isotropic term is proportional to $(k_0 d)^5$ and is negligible.

3.3. Accuracy of the pseudopotential model

In the above analysis, we let the radius of the sphere become deep subwavelength in order to check the asymptotic equivalence to a pseudopotential. The distance at which the scattered field was observed was kept fixed and finite. However, in case of arrays (or lattices) of interacting scatterers, the inter-element distance can also be very small compared to the wavelength which may compromise the validity of the pseudopotential approximation. Consequently, the important question is raised about how close two scatterers can be before the pseudopotential approximation becomes inaccurate. To answer it, we return to the scattering problem and consider the scattered field at the location of the source that produces it, i.e. their interaction field within the zero order approximation. Setting $\boldsymbol{r} = \boldsymbol{r}_f$ in equation (31) and combining it with equation (32) we get

$$p_s(\boldsymbol{r}_f) = \frac{k_0^2 Z_0 F}{4\pi} \sum_{n=0}^{\infty} t_n (2n+1) \left[h_n^{(2)}(k_0 \Lambda)\right]^2, \tag{42}$$

where we have used $P_n(1) = 1$ and replaced r_f by Λ, a frequently used symbol for arrays' inter-element spacing. This series, which is convergent for any $\Lambda > d$, can be used to check the accuracy of the pseudopotential approximation: the scatterer will behave as a pseudopotential, i.e. radiate isotropically, if the series can be replaced by its leading ($n = 0$) term. The accuracy is quantified by the criterion

$$C \triangleq \left| \frac{t_0 \left[h_0^{(2)}(k_0 \Lambda)\right]^2 - \sum_{n=0}^{\infty} t_n (2n+1) \left[h_n^{(2)}(k_0 \Lambda)\right]^2}{\sum_{n=0}^{\infty} t_n (2n+1) \left[h_n^{(2)}(k_0 \Lambda)\right]^2} \right| < \varepsilon, \tag{43}$$

where $\varepsilon \ll 1$, t_0 is given by equation (32) and w_0 is given by the limiting formulas (37).

Figure 1 depicts the isolines of $\log_{10}(C)$ versus $\log_{10}(k_0 d)$, $\log_{10}(k_0 \Lambda)$ for small ($k_0 d < 1$) and acoustically soft ($Z_s = 0$) spheres. The region $\Lambda > 2d$ corresponds to the physical constraint for the distance between the centers of two scatterers with radius d. It is seen that for given values of $k_0 d$ and C, there exists a minimum $(k_0 \Lambda)_{\min}$, such that the accuracy is better than C for all $k_0 \Lambda > (k_0 \Lambda)_{\min}$. For sufficiently small $k_0 d$, the relationship between $(k_0 \Lambda)_{\min}$

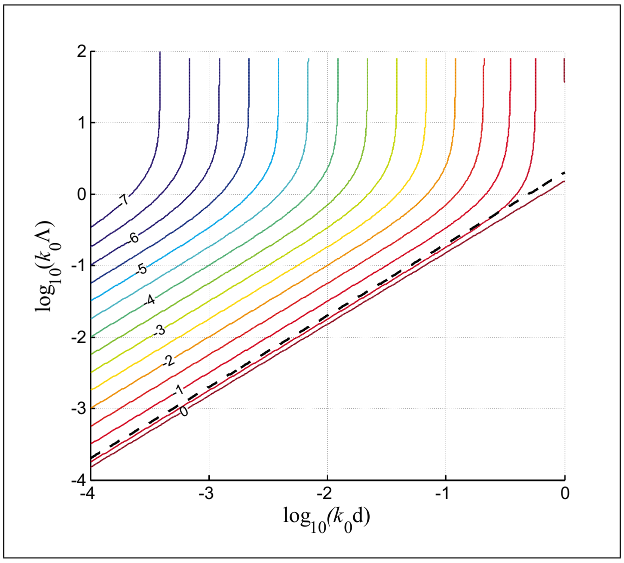

Figure 1. Isolines of $\log_{10}(C)$ (step 0.5), i.e. the accuracy of the pseudopotential approximation, versus $\log_{10}(k_0 d)$, $\log_{10}(k_0\Lambda)$ for acoustically soft ($Z_s = 0$) spheres with $k_0 d < 1$. The dotted line corresponds to the physical constraint $\Lambda = 2d$.

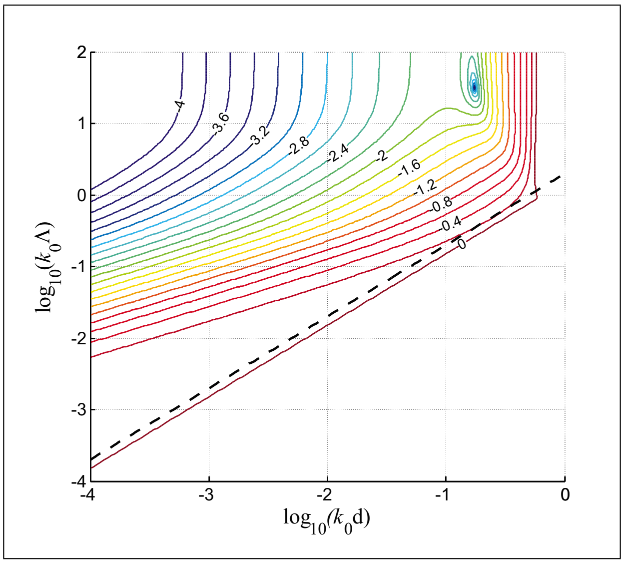

Figure 2. Isolines of $\log_{10}(C)$ (step 0.2), versus $\log_{10}(k_0 d)$, $\log_{10}(k_0\Lambda)$ for impenetrable spheres with $k_0 d < 1$ and surface impedance $Z_s = Z_0$. The dotted line corresponds to the physical constraint $\Lambda = 2d$.

and $k_0 d$ is approximately linear, as implied by the constant unity slope of the isolines (the axes are scaled logarithmically). As $k_0 d$ increases, the required $(k_0\Lambda)_{\min}$ increases dramatically with the isoline becoming almost vertical, implying that there exists a maximum $(k_0 d)_{\max}$, beyond which an accuracy C cannot be achieved no matter how large $k_0\Lambda$ gets. Overall, the map proves the intuitively expected fact that the pseudopotential approximation is accurate at sufficiently large distances from sufficiently small scatterers. For example, Figure 1 states that the accuracy of the pseudopotential approximation is better than 1% ($C = 0.01$) for soft spheres with size $k_0 d < 0.12$ and

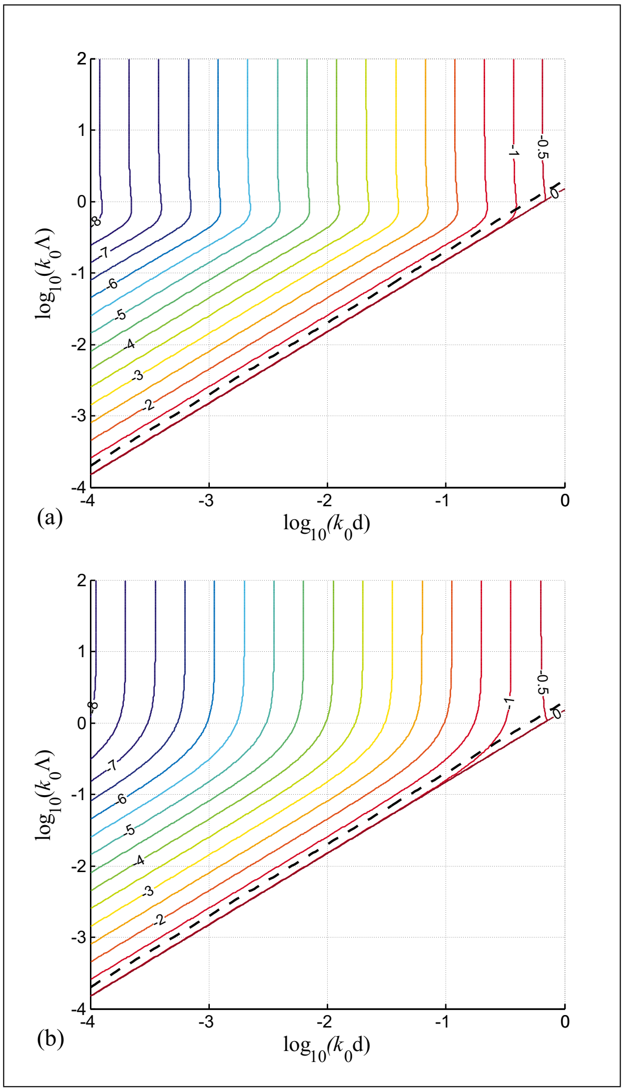

Figure 3. Isolines of $\log_{10}(C)$ (step 0.2), versus $\log_{10}(k_0 d)$, $\log_{10}(k_0\Lambda)$ for penetrable spheres with $k_0 d < 1$, $\rho_1 = \rho_0$ and a) $c_1 = 2c_0$, b) $c_1 = c_0/2$. The dotted line corresponds to the physical constraint $\Lambda = 2d$.

at distances $\Lambda > 10d$, which can be remembered as a rule of thumb.

An analogous map can be drawn for impedance spheres. An example is shown in Figure 2 with $Z_s = Z_0$. Now the slope of the isolines, for small $k_0 d$, is smaller than unity, implying a fractional power dependence between $(k_0\Lambda)_{\min}$ and $k_0 d$. A region of increased accuracy appears for moderate $k_0 d > 0.1$ and $k_0\Lambda > 10$. Roughly speaking, the accuracy is better than 1% for $k_0 d < 0.1$ at distances $k_0\Lambda > 8.5\sqrt{k_0 d}$.

Finally Figure 3a shows the accuracy of the pseudopotential approximation for penetrable spheres with the same density with the surrounding medium ($\rho_0 = \rho_1$) but with acoustic speed $c_1 = 2c_0$ (equivalently with ¼ the compressibility of the surrounding medium). Accuracy 1% requires $k_0 d < 0.12$ and at distances $\Lambda > 4.5d$, which is a larger range compared to $\Lambda > 10d$ required for soft spheres (Figure 1). The inverse case $c_1 = c_0/2$ is shown in Figure 3b and is quite similar with roughly the same condition for 1% accuracy. Despite the similarity, the two cases are very different when the spheres are arranged in an ILPA, as we will see in the next Section.

4. Infinite linear periodic arrays

The dispersion equation of modes of a pseudopotentials ILPA has been derived in [1]. Here we revisit the subject and conclude in the existence or nonexistence of traveling waves in ILPAs of small spheres that are approximable by pseudopotentials. The problem is interesting as a fundamental one in the context of studies of acoustic wave propagation in periodic media [20, 21] (and references therein), [22].

Modes propagating along a linear Λ-periodic array of identical ($\alpha_n = \alpha$ for all n) pseudopotentials satisfy the Bloch (or Floquet) condition which is expressed in terms of the pseudopotential amplitudes

$$A_n = A_0 \mathrm{e}^{-\mathrm{j}nK\Lambda}, \tag{44}$$

where K is the Bloch wavevector and A_0 is the complex amplitude of the $n = 0$ element. Substituting into equation (27) with suppressed incident field, we obtain the necessary condition for nonzero A_0,

$$1 - \mathrm{j}\alpha + \frac{\alpha}{k_0\Lambda} \sum_{n\neq 0} \frac{\mathrm{e}^{-\mathrm{j}(|n|k_0\Lambda + nK\Lambda)}}{|n|} = 0. \tag{45}$$

Splitting the sum into two sums, one for the negative and one for the positive values of n, the condition becomes

$$1 - \mathrm{j}\alpha + \frac{\alpha}{k_0\Lambda}\left[T^*\left(\frac{k_0\Lambda - K\Lambda}{2\pi}\right) + T^*\left(\frac{k_0\Lambda + K\Lambda}{2\pi}\right)\right] = 0, \tag{46}$$

where

$$T(x) = \sum_{n=1}^{\infty} \frac{\mathrm{e}^{\mathrm{j}2\pi n x}}{n} = -\ln\left[2|\sin(\pi x)|\right] - \mathrm{j}\pi\left(x - \lfloor x \rfloor - \frac{1}{2}\right) \tag{47}$$

is the trigonometric series, whose formula is found in the Appendix of [3]. Use of the property $T(-x) = T^*(x)$ has also been made. Substituting equation (47) into (46) the condition is further simplified to

$$1 - \frac{\alpha}{k_0\Lambda} \ln\left[2|\cos(K\Lambda) - \cos(k_0\Lambda)|\right] - \mathrm{j}\frac{\pi\alpha}{k_0\Lambda}\left(1 + \left\lfloor \frac{k_0\Lambda - K\Lambda}{2\pi} \right\rfloor + \left\lfloor \frac{k_0\Lambda + K\Lambda}{2\pi} \right\rfloor\right) = 0. \tag{48}$$

As is also obvious from equation (45), this complex equation is even and 2π-periodic with respect to $K\Lambda$, implying that it is adequate to search for its solution in the interval $0 \leq K\Lambda \leq \pi$, namely the positive half of the first Brillouin zone. Within this zone, the imaginary part of equation (48) is zero for all $K\Lambda > k_0\lambda$ and, under this constraint, the real part is zero for

$$K\Lambda = \cos^{-1}\left[\cos(k_0\Lambda) - \frac{1}{2}\mathrm{e}^{k_0\Lambda/\alpha}\right], \tag{49}$$

which is the analytical dispersion equation of the Bloch modes. This equation with a varying but frequency independent α was plotted in [22] for an ILPA of electroacoustic transducers. Notice the difference in notation in that our α is $-\tan\psi_e$, ψ_e being the phase of the transducer's scattering coefficient.

We now assume an ILPA of small and acoustically soft spheres with a ratio Λ/d that assures an acceptable accuracy for the pseudopotential approximation. After substituting $\alpha \approx k_0 d$ from equation (38) into equation (49), the exponential factor becomes the constant $(1/2)\mathrm{e}^{\Lambda/d} > 3.694$ ($\Lambda > 2d$) and equation (49) has no real solution for $K\Lambda$. Therefore, as a result of the linear dependence of α with frequency, there are no traveling waves in an ILPA of small and soft spheres. This should be collated with [22] where there is a real $K\Lambda$ curve for any value of the frequency-independent ψ_e.

The case of impedance spheres is quite different. Substituting $\alpha \approx -(Z_0/Z_s)(k_0 d)^2$ from equation (38) into equation (49) we obtain

$$K\Lambda = \cos^{-1}\left[\cos(k_0\Lambda) - \frac{1}{2}\exp\left(-\frac{Z_s\Lambda}{Z_0 d}\frac{1}{k_0 d}\right)\right]. \tag{50}$$

From the validity range of equations (34), we have $|Z_s|/Z_0 \gg k_0 d$. Therefore, for $Z_s > 0$, the exponential factor in equation (49) is very small and the solution becomes very close to free-space propagation $K\Lambda \approx k_0\Lambda$ and particularly for decreasing $k_0 d$. On the other hand, for $Z_s < 0$ the exponential factor is very large and there exists no solution.

Finally consider the case of penetrable spheres having the same density with the surrounding medium. From equation (41) and for $\rho_0 = \rho_1$ we have $\alpha = (1 - c_0^2/c_1^2)(k_0 d)^3/3$ and the dispersion equation (49) becomes

$$K\Lambda = \cos^{-1}\left[\cos(k_0\Lambda) - \frac{1}{2}\exp\left(\frac{3(\Lambda/d)^3}{(1 - c_0^2/c_1^2)}\frac{1}{(k_0\Lambda)^2}\right)\right]. \tag{51}$$

For $c_1 > c_0$ the exponential term is always greater than $0.5\mathrm{e}^6 \approx 202$ and equation (51) has no real solution. Contrary, for $c_1 < c_0$ this term is smaller than $1/2$ and there always exists a solution. Within the range of 1% accuracy of the pseudopotential approximation this solution is always very close to $K\Lambda \approx k_0\Lambda$. Here one should notice that, even when c_0 is sufficiently greater than c_1, the required $k_0\Lambda$ to assure the 1% accuracy is smaller; thus $K\Lambda$ cannot deviate significantly from the free-space line $K\Lambda = k_0\Lambda$, within the pseudopotential approximation.

In conclusion we have shown that, within the subwavelength size regime of the pseudopotential approximation, traveling waves are supported only by an ILPA of impenetrable spheres with a positive surface impedance or by penetrable spheres with the same density but higher compressibility (lower acoustic speed) than the surrounding medium. Most importantly, our conclusions emphasize the fact that the frequency dependence of factor α can render the dispersion equation (49) unsolvable.

5. Conclusion

The pseudopotential concept has been revisited through a physical energy-based approach. Combining the linear response of the pseudopotential amplitude to the incident field and energy arguments, we have derived the equations of interaction between pseudopotentials and verified its classic operator-based definition. Acoustic scattering by penetrable and impenetrable spheres was treated and it was shown analytically that only soft, impedance or penetrable spheres with the density of their surrounding behave asymptotically as pseudopotentials and with a responsivity that scales linearly, quadratically or cubicly with frequency, respectively. The problem of an ILPA of pseudopotentials was briefly revisited and the important conclusion was reached that the frequency dependence of the responsivity factor forbids the existence of traveling waves, except in the case of a positive surface impedance spheres or penetrable spheres with the same density but higher compressibility compared to the surrounding fluid.

The deep understanding of acoustic phenomena related to small scatterers and their periodic arrays is an asset for the analysis of more complicated systems involving acoustic or elastic waves in fluids or solids. The extention to elastic waves in solids is a more difficult analytical task due to the vector character of the medium vibrations and the coexistence of longitudinal and transverse waves. The current scientific trend points towards complex periodic media, termed *phononic crystals* [23], that exhibit a number of fascinating features, such as large bandgaps [24], strong dispersion [25], negative refraction [26], sound focusing [27] and others.

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Appendix

A1. Proof of equation (7)

Using Gauss’s divergence theorem, equation (6) is written

$$P_{\mathrm{rad},f} = \frac{1}{2}\Re e \iiint_{V(S)} \nabla\cdot\left[p_f(\boldsymbol{r}')\boldsymbol{v}_f^*(\boldsymbol{r}')\right]\mathrm{d}V'$$
$$= \frac{1}{2}\Re e \iiint_{V(S)} \left[\boldsymbol{v}_f^*(\boldsymbol{r}')\cdot\nabla p_f(\boldsymbol{r}') + p_f(\boldsymbol{r}')\nabla\cdot\boldsymbol{v}_f^*(\boldsymbol{r}')\right]\mathrm{d}V',$$

where $V(S)$ is the volume enclosed by surface S. Substitution of $\nabla\cdot\boldsymbol{v}_f$ and ∇p_f from equations (1) and (2), respectively, yields equation (7).

A2. Proof of equation (18)

Using Gauss’s divergence theorem we have

$$\frac{1}{2}\Re e \iiint_{V(S_s)} \nabla\cdot\left[p_f(\boldsymbol{r})\boldsymbol{v}_s^*(\boldsymbol{r}) + p_s(\boldsymbol{r})\boldsymbol{v}_f^*(\boldsymbol{r})\right]\mathrm{d}V'$$
$$= \frac{1}{2}\Re e \iiint_{V(S_s)} \left[\boldsymbol{v}_f^*(\boldsymbol{r}')\cdot\nabla p_f(\boldsymbol{r}') + \boldsymbol{v}_f^*(\boldsymbol{r}')\cdot\nabla p_s(\boldsymbol{r}')\right.$$
$$\left. + p_f(\boldsymbol{r}')\nabla\cdot\boldsymbol{v}_s^*(\boldsymbol{r}')) + p_s(\boldsymbol{r}')\nabla\cdot\boldsymbol{v}_f^*(\boldsymbol{r}')\right]\mathrm{d}V',$$

where $V(S_s)$ is the volume enclosed by S_s. The sum of the first two terms inside the integral has a zero real part because, as seen from equations (2), (12), $\nabla p_f(\boldsymbol{r}')$, $\nabla p_s(\boldsymbol{r}')$ are 90° out of phase with $\boldsymbol{v}_f$, $\boldsymbol{v}_s$, respectively. For the last two terms $\nabla\cdot\boldsymbol{v}_f(\boldsymbol{r}')$, $\nabla\cdot\boldsymbol{v}_s(\boldsymbol{r}')$ are substituted from equations (1),(11), respectively, to obtain the imaginary term $\mathrm{j}k_0 Z_0^{-1}(p_f p_s^* + p_s p_f^*)$ plus $p_f A^*\delta(.)$, since $f = 0$ in $V(S_s)$. After taking the real part we obtain equation (18).

References

[1] T. T. Wu: Fermi pseudopotentials and resonances in arrays. – In: Resonances-Models and Phenomena. Springer, Berlin, 1984, 293–306.

[2] D. K. Freeman: The pseudodipole as the limit of a resonant cylindrical dipole. J. Math. Phys. **32** (1991) 1961–1964.

[3] I. Psarros, G. Fikioris, M. Vlahoyianni: Pseudopotentials: A simplified model for certain types of array elements. IEEE Trans. Antennas and Propag. **57** (2009) 414–424.

[4] A. Grossmann, T. T. Wu: Fermi pseudopotential in higher dimensions. J. Math. Phys. **25** (1984) 1742–1745.

[5] P. De Vries, D. V. Van Coevorden, A. Lagendijk: Point scatterers for classical waves. Reviews of Modern Physics **70** (1998) 447–466.

[6] P. A. Martin: Multiple scattering, interaction of time-harmonic waves with n obstacles, chapt. 8, pp. 284-322. Cambridge University Press, 2006.

[7] C. Athanasiadis, P. A. Martin, A. Spyropoulos, I. G. Stratis: Scattering relations for point sources: Acoustic and electromagnetic waves. J. Math. Phys. **43** (2002) 5683–5697.

[8] H. C. van de Hulst: Light scattering by small particles. chapt. 18, 19, 20, 21, pp. 383-453. Dover Publications, 1981.

[9] B. T. Draine, P. J. Flatau: Discrete-dipole approximation for scattering calculations. J. Opt. Soc. Am. A: Optics and Image Science, and Vision **11** (1994) 1491–1499.

[10] M. Rusek, A. Orłowski, J. Mostowski: Localization of light in three-dimensional random dielectric media. Physical Review E **53** (1996) 4122–413.

[11] S. A. Maier, P. G. Kik, H. A. Atwater: Observation of coupled plasmon-polariton modes in Au nanoparticle chain waveguides of different lengths: Estimation of waveguide loss. Appl. Phys. Lett. **81** (2002) 1714–1716.

[12] S. A. Maier, P. G. Kik, H. A. Atwater: Optical pulse propagation in metal nanoparticle chain waveguides. Phys. Rev. B - Condensed Matter and Materials Physics **67** (2003) art. no. 205402, pp. 2054021–2054025.

[13] M. L. Brongersma, J. W. Hartman, H. A. Atwater: Electromagnetic energy transfer and switching in nanoparticle chain arrays below the diffraction limit. Phys. Rev. B - Condensed Matter and Materials Physics **62** (2000) art. no. 0R1635, pp. R16356–R16359.

[14] R. D. Averitt, S. L. Westcott, N. J. Halas: Linear optical properties of gold nanoshells. J. Opt. Soc. Am. B: Optical Physics **16** (1999) 1824–1832.

[15] C. L. Nehl, N. K. Grady, G. P. Goodrich, F. Tam, N. J. Halas, J. H. Hafner: Scattering spectra of single gold nanoshells. Nano Letters **4** (2004) 2355–2359.

[16] L. E. Kinsler, A. R. Frey, A. B. Coppens, J. V. Sanders: Fundamentals of acoustics. 4th ed., chapt. 5., pp. 113-148. Wiley, 2000.

[17] L. L. Foldy: The multiple scattering of waves. Phys. Review **67** (1945) 107–119.

[18] M. Abramowitz, I. A. Stegun: Handbook of mathematical functions with formulas, graphs, and mathematical tables. chapt. 10, pp. 437. Dover Publications, 1972.

[19] J. A. Stratton: Electromagnetic theory. chapt. 7, pp. 414. IEEE Press Series on Electromagnetic Wave Theory, 2007.

[20] A. A. Ruffa: Acoustic wave propagation through periodic bubbly liquids. J. Acoust. Soc. Am. **91** (1-12) 1992.

[21] F. Wu, H. Zhong, S. Zhong, Z. Liu, Y. Liu: Localized states of acoustic waves in three-dimensional periodic composites with point defects. European Physical Journal B3 **4** (2003) 265–268.

[22] A. D. Yaghjian: Scattering-matrix analysis of linear periodic arrays. IEEE Trans. Antennas Propag. **50** (2002) 1050–1064.

[23] I. E. Psarobas, N. Stefanou, A. Modinos: Scattering of elastic waves by periodic arrays of spherical bodies. Phys. Rev. B **62** (2000) 278–291.

[24] J. O. Vasseur, P. A. Deymier, B. Chenni, B. Djafari-Rouhani, L. Dobrzynski, D. Prevost: Experimental and theoretical evidence for the existence of absolute acoustic band gaps in two-dimensional solid phononic crystals. Phys. Rev. Lett. **86** (2001) 3012–3015.

[25] J. H. Page, S. Yang, Z. Liu, M. L. Cowan, T. C. Che, P. Sheng: Tunneling and dispersion in 3D phononic crystals. Zeitschrift fur Kristallographie **220** (2005) 859–870.

[26] X. Zhang, Z. Liu: Negative refraction of acoustic waves in two-dimensional phononic crystals. Appl. Phys. Lett. **85** (2004) 341–343.

[27] S. Yang, J. H. Page, Z. Liu, M. L. Cowan, C. T. Chan, P. Sheng: Focusing of sound in a 3D phononic crystal. Physical Review Letters **93** (2004) 024301–1.