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From a Model of Lossy Flared Pipes to a General Framework for Simulation of Waveguides

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Summary

This paper deals with the theory and application of waveguide modeling of lossy flared acoustic pipes, relying on the Webster-Lokshin equation. This model describes the propagation of longitudinal waves in axisymmetric acoustic pipes with a varying cross-section, visco-thermal losses at the walls, and without assuming planar waves or spherical waves. Solving this model for a piece of pipe leads to a two-port system made of four transfer functions which mimic the global acoustic effects. Moreover, introducing some relevant physical interpretations makes it possible to separate elementary effects due to the geometry of the piece of pipe (section, slope, and curvature) and isolate corresponding elementary transfer functions. From this decomposition a framework is obtained which allows to recover some digital waveguide models introduced earlier in the literature. This work contributes to the standardization of some different waveguide models, and brings a higher level of refinement that is visco-thermal losses combined with curvature effects.

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1. Introduction

Whereas signal processing approaches describe sounds, physical modeling describes internal acoustic or mechanic phenomena of the modeled system. For sound synthesis (of musical instruments or speech production, for example), it faithfully reproduces the behavior of the system, especially during transient states. Moreover, parametric models allow to obtain new virtual systems and explore new sounds, together with physical validation. However, digital time simulations sometimes require intensive computation from computers. That is why special care must be taken on the algorithmic complexity to perform real-time sound synthesis.

The aim of the present work is to build the model and the simulation of a varying cross-section pipe using the standard "Digital Waveguide Network" approach (see eg. [1, 2]). With this technique, a whole virtual resonator is built by connecting several systems which mimic the acoustic of pieces of pipe. The difficulty and the novelty is to include subtle, though perceptible, phenomena due to visco-thermal losses at the wall and continuously varying cross-sections (see eg. [3, 4, 5, 6, 7, 8, 9]).

In [10], a resonator is represented by the connection of pieces of lossless cylinder to model vocal tract. For low-cost digital simulation, a scattering network (the *Kelly-Lochbaum* framework) is derived. This framework contains delays for wave propagation and coefficients for reflexion at junctions of cylinders. In [11], the *Kelly-Lochbaum* framework is derived for a refined model of connection of lossless cones (with continuity of radius at junctions). In [12] and [14], visco-thermal losses are taken into account for connections of cylinders and cones respectively. In [16], connection of lossy cones with discontinuity of section is considered. In [17] the same *Kelly-Lochbaum* framework is recovered by considering a more refined model of connection of lossy flared pipes, using the *Webster-Lokshin* equation. It is a 1D-acoustic model, which describes the propagation of longitudinal waves in axisymmetric acoustic pipes, involving visco-thermal losses at the wall and a varying cross-section.

In the present work, starting from the *Webster-Lokshin* model for a piece of pipe, solutions in terms of traveling waves lead to a new framework in which each effect of the geometry of the pipe can be isolated. In spite of the apparent complexity of this framework at first glance, all elements are already known and well understood. Thanks to the remarkable symmetry of this framework, connecting pieces of pipe with a particular order of discontinuity (of section, slope, or curvature) makes it possible to recover all the above-mentioned waveguide models (cf. eg. [10]–[18]).

This paper is organized as follows. In section 2, the acoustic model is presented and the analytic solution for the acoustic pressure is given in the Laplace domain. Then,

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pieces of pipe are modeled, deriving two-port systems in admittance and impedance convention, and finally for traveling waves. In section 3, a physically inspired decomposition of two-port systems is performed. It allows the isolation of the "elementary" acoustic effects of the pipe on traveling waves, according to the chosen acoustic model. Then, a general framework for simulation of digital waveguides is obtained. In section 4, connecting two-ports systems, the model of a whole pipe can be built with a refined model. Moreover this model allows to recover former models of waveguides (concatenation of cylinders or cones). In section 5 are briefly presented the building of the discrete-time system, and some comparisons of digital simulations. Finally, section 6 concludes this paper and opens some perspectives.

2. Propagation in a lossy flared acoustic pipe

This section presents the *Webster-Lokshin* model of a pipe, from which scattering matrices and equivalent two-port systems are derived. Two-ports are input/output systems related to the standard acoustic state (acoustic pressure and volume flow) or to traveling waves.

2.1. Webster-Lokshin model

2.1.1. Equations for acoustic pressure and volume flow

The *Webster-Lokshin* model is a mono-dimensional model which characterizes linear longitudinal waves propagation in axisymmetric pipes, assuming the quasi-sphericity of isobars near the inner wall (cf. [8, 19]), and taking into account visco-thermal losses (cf. [5, 3, 4]) at the wall. The acoustic pressure P and the volume flow U are governed by the following equations, given in the Laplace domain:

$$\left(\left[\left(\frac{s}{c_0}\right)^2 + 2\varepsilon(\ell)\left(\frac{s}{c_0}\right)^{\frac{3}{2}} + \mathrm{Y}(\ell)\right] - \partial_\ell^2\right) \left\{r(\ell)P(\ell,s)\right\} = 0, \quad (1)$$

$$\rho_0 s\frac{U(\ell,s)}{\mathrm{A}(\ell)} + \partial_\ell P(\ell,s) = 0, \quad (2)$$

where $s \in \mathbb{C}$ is the Laplace variable ($\Im\mathrm{m}(s) = \omega$ is the angular frequency), ℓ is the space variable measuring the arclength of the wall, $r(\ell)$ is the radius of the pipe, $\mathrm{A}(\ell) := \pi r(\ell)^2$ is the section area, $\varepsilon(\ell) := \kappa_0\sqrt{1 - r'(\ell)^2}/r(\ell)$ quantifies the visco-thermal losses and $\mathrm{Y}(\ell) := r''(\ell)/r(\ell)$ denotes the curvature of the wall. Equation (1) is called *Webster-Lokshin* equation, and (2) is *Euler* equation satisfied outside the visco-thermal boundary layer.

Note that the standard horn equation (cf. [6]) corresponds to (1) with no losses ($\varepsilon(\ell) = 0$) and assuming planar waves (ℓ is replaced by the axis coordinate z). Straight and conical pipes are characterized by $\mathrm{Y}(\ell) = 0$. In the following, Y is assumed to be non-negative, corresponding to straight, conical or flared pipes. The negative case would require special treatments for deriving stable versions of waveguides (cf. [7]), which are out of the scope of this paper.

Denoting $\underline{r}(z)$ the radius of the pipe for the z-ordinate, the arclength of the wall from 0 to z is

$$\ell(z) = \int_0^z \sqrt{1 + \underline{r}'(z)^2}\,\mathrm{d}z \quad (3)$$

and $\underline{r}(z) := r(\ell(z))$. In consequence, $|r'(\ell)| < 1$ and $r'(\ell) \to 1$ when $\underline{r}'(z) \to +\infty$.

For numerical computations, the physical constants are chosen for standard conditions. They are: the mass density $\rho_0 = 1.2\mathrm{kg/m}^3$, the speed of sound $c0 = 344$ m/s, the coefficient $\kappa_0 = \sqrt{l'_v} + (\gamma - 1)\sqrt{l_h} \approx 3.5\ 10^{-4}\mathrm{m}^{-1/2}$.

2.1.2. Limitations of the model

The *Webster-Lokshin* model only considers axisymmetric pipes. Consequently, the possible bends (of a trombone slide for example) have to be virtually straightened. Moreover, two other limitations appear because of the visco-thermal losses model and because only longitudinal propagations are considered.

With l'_v and l_h the characteristic lengths of viscous ($l'_v \approx 4\,10^{-8}$ m) and thermal ($l_h \approx 6\,10^{-8}$ m) effects (cf. [20]), this model of losses is valid for "large tubes" given by $r \gg \sqrt{2c_0 l'_v/\omega}$ and $r \gg \sqrt{2c_0 l_h/\omega}$. For example, with $r = 5$ mm, the model is valid for $f = \omega/(2\pi) \gg 0.25$ Hz.

For a lossless cylinder with radius r, the first transversal mode appears at $f = 1.84c_0/(2\pi r)$. In the case of a varying cross-section pipe, transversal modes appear with frequencies higher than $1.84c_0/(2\pi r_m)$, where r_m is the maximal radius of the pipe. For example, with $r_m = 10$ cm, the *Webster* model remains valid for $f < 1000$ Hz. Note that, multi-modal approaches allow to simulate transversal modes (cf. eg. [21]).

2.1.3. Equations for traveling waves

In section 2.1.1 the acoustic pressure $P(\ell, s)$ and volume flow $U(\ell, s)$ are used to describe acoustics. Another description, more adapted to the waveguide simulations, consists in using the traveling waves defined by

$$\begin{bmatrix}\phi^+(\ell,s)\\ \phi^-(\ell,s)\end{bmatrix} := \frac{r(\ell)}{2}\begin{bmatrix}1 & Z(\ell)\\ 1 & -Z(\ell)\end{bmatrix}\begin{bmatrix}P(\ell,s)\\ U(\ell,s)\end{bmatrix}, \quad (4)$$

where $Z(\ell) := 1/Y(\ell) := \rho_0 c_0/\mathrm{A}(\ell)$ is the equivalent of acoustic impedance for planar waves traveling inside cylinder of (constant) section area $\mathrm{A} = \mathrm{A}(\ell)$.

According to (1), (2) and (4), these traveling waves are governed by the coupled system

$$\left(\frac{s}{c_0} \pm \partial_\ell\right)\phi^\pm(\ell,s) = \pm\zeta(\ell)\phi^\mp(\ell,s) - \varepsilon(\ell)\left(\frac{s}{c_0}\right)^{1/2}\phi(\ell,s), \quad (5)$$

where $\phi := \phi^+ + \phi^- = rP$, and $\zeta := r'/r$. The operator $\left(s/c_0 \pm \partial_\ell\right)$ is the traveling operator and the right-hand side describes the coupling due to $\zeta(\ell)$ and losses.

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For planar waves traveling inside a lossless straight pipe ($\varepsilon(\ell) = 0$ and $\zeta(\ell) = 0$), the acoustic state $\phi^{\pm}$ corresponds to decoupled progressive waves. For the Wester-Lokshin model, these waves are still progressive insofar as they preserve the causality principle.

2.2. Piece of pipe and general solution

In this paper, a pipe with varying cross-section is approximated by a concatenation of pieces of pipe with constant parameters. So, a piece of pipe is defined as a finite pipe with length L, and with constant curvature and losses parameters (i.e $\Upsilon(\ell) = \Upsilon$ and $\varepsilon(\ell) = \varepsilon$). This allows to preserve the continuity of radius $r(\ell)$ and slope $r'(\ell)$ that is $\mathcal{C}^1$-regular shapes. Junctions are characterized by discontinuities of curvature.

Remark that, except for a cylinder, $\Upsilon(\ell)$ and $\varepsilon(\ell)$ cannot be simultaneously constant. In pratice, pieces of pipe with constant curvature are considered, and ε is chosen as the mean of $\varepsilon(\ell)$. This approximation makes sense, at least for physically realistic lengths and curvatures for wind instruments.

In a such piece of pipe (Υ and ε contant), the general solution of the acoustic pressure $P(\ell, s)$ for (1) is

$$P(\ell, s) = \mathcal{A}(s)\frac{e^{\Gamma(s)\ell}}{r(\ell)} + \mathcal{B}(s)\frac{e^{-\Gamma(s)\ell}}{r(\ell)}, \tag{6}$$

where $\mathcal{A}(s)$ and $\mathcal{B}(s)$ depend on boundary conditions, and $\Gamma(i\omega) = ik(\omega)$ for which $k(\omega)$ is the usual complex wavenumber. Γ is given in the Laplace domain by

$$\Gamma(s) = \sqrt{\left(\frac{s}{c_0}\right)^2 + 2\varepsilon\left(\frac{s}{c_0}\right)^{3/2} + \Upsilon}. \tag{7}$$

In (7) $\sqrt{\cdot}$ denotes an analytical continuation of the positive square root of $\mathbb{R}^+$ on a domain compatible with the one-sided Laplace transform, namely $\mathbb{C}_0^+ = \{s \in \mathbb{C} / \Re e(s) > 0\}$ (see [9] for more details). In $\mathbb{C}_0^+$, Γ is proved to be analytical and $\Re e(\Gamma(s)) \geq 0$ if $\varepsilon \geq 0$.

If $\Upsilon \geq 0$ and $\varepsilon = 0$, the unique positive solution $\omega_c = \sqrt{\Upsilon}/c_0$ of $\Gamma(i\omega)^2 = 0$, corresponds to the cutoff angular frequency of a lossless pipe with positive constant curvature. Indeed, if $\omega > \omega_c$, $\Gamma(i\omega) \in i\mathbb{R}$, it corresponds to propagating waves, and if ω is such that $0 \leq \omega < \omega_c$, $\Gamma(i\omega) \in \mathbb{R}^+$, it corresponds to evanescent waves.

Now, if $\Upsilon \geq 0$ and $\varepsilon > 0$, a similar behavior is obtained for $\omega_c = \Im m(s_c) > 0$ where $(s_c, \overline{s_c})$ are the unique solutions of $\Gamma(s)^2 = 0$; they satisfy $\Re e(s_c) < 0$ (cf. [9]).

More details of the effects of visco-thermal losses and curvature on transfer functions are given in Appendix A3.

2.3. Scattering matrices and equivalent two-port systems

In the following, P_0 and P_L are defined as the acoustic pressure at the left end ($\ell = 0$) and at the right end ($\ell = L$) of a piece of pipe with $\Upsilon(\ell) = \Upsilon$ and $\varepsilon(\ell) = \varepsilon$. In the same way U_0, U_L, are defined as the volume flow at the left and the right ends, respectively.

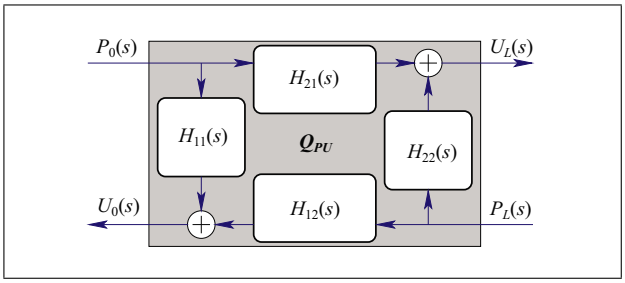


Figure 1. Two-port system $\mathbf{Q}_{PU}$, with admittance representation.

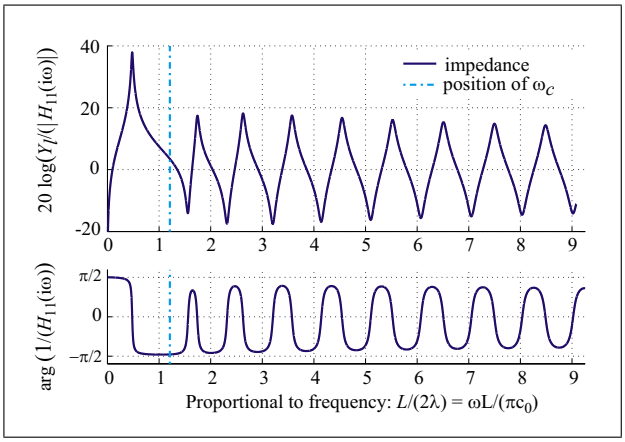


Figure 2. Impedance of open piece of pipe, H_{11}^{-1}. $r_l = 5$ mm, $r_r = 1$ cm, $L = 1$ m, $\Upsilon = 15$ m^{-2}. $Y_l = \mathrm{A}_l/(\rho_0 c_0)$.

2.3.1. Scattering matrix with admittance representation

Writing $[U_0, U_L]$ as functions of $[P_0, P_L]$, the piece of pipe is modeled by the 2×2 scattering matrix $\mathcal{H}(s)$. Figure 1 represents the corresponding two-port system $\mathbf{Q}_{PU}$.

$$\begin{bmatrix} U_0 \\ U_L \end{bmatrix} = \mathcal{H}\begin{bmatrix} P_0 \\ P_L \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}\begin{bmatrix} P_0 \\ P_L \end{bmatrix}. \tag{8}$$

Solving (1) and (2) leads to the following closed forms of elements of $\mathcal{H}$ (cf. [19]):

$$H_{11}(s) = \frac{\mathrm{A}_l}{\rho_0}\frac{\Gamma(s)\cosh(\Gamma(s)L) + \zeta_l \sinh(\Gamma(s)L)}{s\sinh(\Gamma(s)L)}, \tag{9}$$

$$H_{22}(s) = \frac{\mathrm{A}_r}{\rho_0}\frac{\Gamma(s)\cosh(\Gamma(s)L) - \zeta_r \sinh(\Gamma(s)L)}{s\sinh(\Gamma(s)L)}, \tag{10}$$

$$H_{12}(s) = -\frac{r_l}{r_r}\frac{\mathrm{A}_l}{\rho_0}\frac{\Gamma(s)}{s\sinh(\Gamma(s)L)}, \tag{11}$$

$$H_{21}(s) = -\frac{r_r}{r_l}\frac{\mathrm{A}_r}{\rho_0}\frac{\Gamma(s)}{s\sinh(\Gamma(s)L)}, \tag{12}$$

where $\Gamma(s)$ is defined by (7), and the following notations are introduced (with indexes $i = l$ and $i = r$, which mean *left* and *right*): $r_i = r(\ell_i)$ (radius), $\mathrm{A}_i = \pi r_i^2$ (section area), $r_i' = r'(\ell_i)$ (slope), $\zeta_i = r_i'/r_i$, with $\ell_l = 0$ and $\ell_r = L$.

Remark: If $P_L(s) = 0$, equation (8) gives $U_0 = H_{11}P_0$. Hence, H_{11} can be interpreted as the left input admittance of a lossy flared pipe with constant curvature which is ideally open at the right end. Conversely, the impedance of such a pipe is given by $1/H_{11}(s)$ (cf. Figure 2). Its comparisons for some values of Υ are given in Appendix A3.

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2.3.2. Scattering matrix with impedance representation

Similarly to section 2.3.1, now the piece of pipe is modeled by the scattering matrix $\mathcal{G}(s)$ where $[U_0(s), U_L(s)]^T$ is the input and $[P_0(s), P_L(s)]^T$ is the output.

$$\begin{bmatrix} P_0 \\ P_L \end{bmatrix} = \mathcal{G} \begin{bmatrix} U_0 \\ U_L \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} U_0 \\ U_L \end{bmatrix}. \tag{13}$$

It results in the following relation between the scattering matrices: $\mathcal{G} = \mathcal{H}^{-1}$. The corresponding two-port system (not represented) is named $\mathbf{Q}_{UP}$.

Remark: $U_L(s) = 0$ leads to $P_0 = G_{11}U_0$, so that G_{11} can be interpreted as the left input impedance of a closed lossy flared pipe with constant curvature (cf. Figure 3).

2.3.3. Scattering matrix for the traveling waves (ϕ^+, ϕ^-)

Now traveling waves $\phi^{\pm}$ defined by (4) are considered. The piece of pipe is modeled by a system, where inputs are $\phi_0^+(s) := \phi^+(\ell = 0, s)$ and $\phi_L^-(s) := \phi^-(\ell = L, s)$ (incoming waves at $\ell = 0$ and $\ell = L$), and outputs are $\phi_0^-(s)$ and $\phi_L^+(s)$ (outgoing waves).

This system is represented by the scattering matrix $\mathcal{M}_\phi$ and the two-port $\mathbf{Q}_\phi$ of Figure 4.

$$\begin{bmatrix} \phi_0^- \\ \phi_L^+ \end{bmatrix} = \mathcal{M}_\phi \begin{bmatrix} \phi_0^+ \\ \phi_L^- \end{bmatrix} = \begin{bmatrix} R_g^l & T_g^- \\ T_g^+ & R_g^r \end{bmatrix} \begin{bmatrix} \phi_0^+ \\ \phi_L^- \end{bmatrix}. \tag{14}$$

The closed form expressions of elements of $\mathcal{M}_\phi$ are not detailed in this paper, but can be found eg. in [19].

Remark: Transfer functions R_g^l and R_g^r represent left (index l) and right (index r) *global* (index g) reflexions of the pipe (with length L) on traveling waves. "Global" means that every internal effects of the pipe are mixed in these transfer functions. An analysis of internal effects is detailed in section 3. T_g^+ and T_g^- represent global transmissions of traveling waves through the pipe.

2.3.4. Conversion two-port systems

The three different two-ports represented in sections 2.3.1, 2.3.2 and 2.3.3 can be intertwined using conversion two-ports given in Figure 5, which are deduced from (4). These conversion two-ports are denoted $\mathcal{C}_{a,b}$, where a is the left acoustic state (PU, UP or ϕ) and b is the right state.

For example to deduce the two-port $\mathbf{Q}_\phi$ from $\mathbf{Q}_{PU}$, the two-port $\mathcal{C}_{\phi,PU}$ has to be connected on the left side of $\mathbf{Q}_{PU}$ and $\mathcal{C}_{UP,\phi}$ on the right side.

That means: $\mathbf{Q}_\phi \equiv \mathcal{C}_{\phi,PU} \odot \mathbf{Q}_{PU} \odot \mathcal{C}_{UP,\phi}$, where $\equiv$ is the equivalent relation and $\odot$ is the connection operator. See Appendix A1 where global interconnection laws of 2 two-ports are given, or see [22] for a state-space representation.

3. Physically inspired decomposition of the two-port

The scattering matrices $\mathcal{H}$, $\mathcal{G}$ and $\mathcal{M}_\phi$ derived from the *Webster-Lokshin* model characterize the global acoustic behavior of a piece of pipe. In this section, their decomposition into elementary two-port systems is performed.

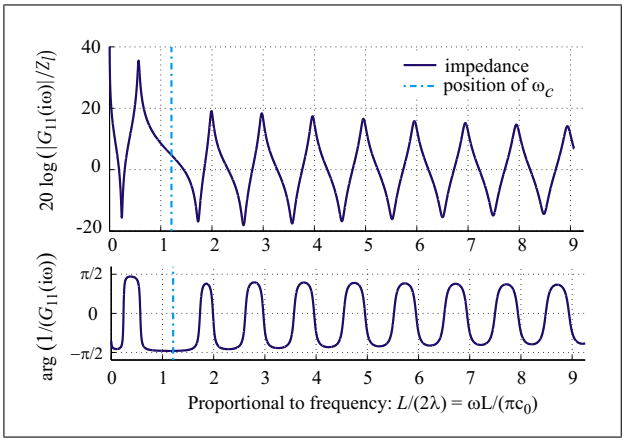


Figure 3. Impedance of closed piece of pipe, G_{11}. $r_l = 5$ mm, $r_r = 1$ cm, $L = 1$ m, $\Upsilon = 15$ m^{-2}. $Z_l = \rho_c c_0/A_l$.

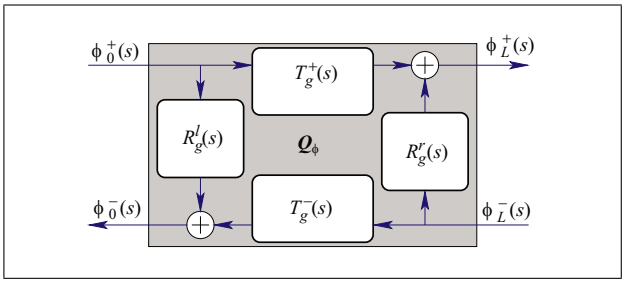


Figure 4. Two-port $\mathbf{Q}_\phi$, with traveling waves.

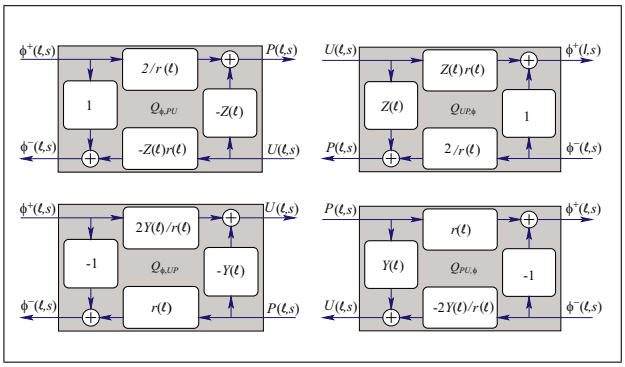


Figure 5. Conversion two-ports $\mathcal{C}_{\phi,PU}$, $\mathcal{C}_{UP,\phi}$, $\mathcal{C}_{\phi,UP}$ and $\mathcal{C}_{PU,\phi}$.

First, delays $e^{-i\omega L/c_0}$ representing an ideal propagation are isolated from other operators without internal delays. Second, in the latter operators, effects of the shape due to radius, slope and curvature can be isolated ones from the other. These decompositions are not purely algebraic manipulations: they rely on physically based reasoning and yield straightforward interpretations. Moreover, they will allow to improve the corresponding digital waveguide implementation.

3.1. Separating delays and acoustic effects

Defining

$$T(s) := e^{-\Gamma(s)L} = D(s)\, e^{-s\frac{L}{c_0}}, \tag{15}$$

with $$D(s) := e^{-\left(\Gamma(s) - \frac{s}{c_0}\right)L}, \tag{16}$$

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in [9], $D(s)$ is proved to be causal and stable, so that $T(s)$ represents the delay ($e^{-i\omega L/c_0}$) of the wave propagation through the piece of pipe, and the associated effect ($D(s)$) due to curvature and visco-themal losses.

This section is dedicated to isolate pure delay operators which are contained in $T(s) = e^{-\Gamma(s)L}$ and to deduce a framework which isolates them.

3.1.1. Decomposition of $\mathbf{Q}_\phi$

In Figure 4, the transfer functions R_g^l, R_g^r, T_g^+ and T_g^+ represent global effects of a piece of pipe on the traveling waves $\phi^\pm$. Nevertheless, some elementary effects can be isolated using the following relevant interpretations (see Figure 6): At the left end ($\ell = 0$), the incident wave ϕ_0^+ is partially reflected in the opposite direction (modeled in Figure 6 by the transfer function R_{le}) and is partially transmitted into the piece of pipe (T_{le}). Then, this transmitted part travels inside the pipe until the right end located at $\ell = L$ (T^+), before being partially reflected (R_{ri}) and partially transmitted (T_{ri}) outside the pipe. Symmetrically, the incident wave ϕ_L^- undergoes similar phenomena.

Figure 6 compiles these phenomena and includes all the contributions (the indexes i and e mean *internal* and *external*, respectively; the functions T^+ and T^- represent the forward and backward transmissions through the pipe, respectively).

This decomposition enlights two smaller two-port systems, $\mathcal{Q}_\phi^l$ and $\mathcal{Q}_\phi^r$ (cf. Figure 6), which models left and right semi-interfaces, respectively.

The two-ports described in Figure 4 and Figure 6 are equivalent if the following algebraic equations hold:

$$T_g^+ = \frac{T_{le}\,T_{ri}\,T^+}{1 - R_{ri}\,R_{li}\,T^-\,T^+}, \tag{17}$$

$$T_g^- = \frac{T_{re}\,T_{li}\,T^-}{1 - R_{ri}\,R_{li}\,T^-\,T^+}, \tag{18}$$

$$R_g^l = R_{le} + \frac{R_{ri}\,T_{le}\,T_{li}\,T^+\,T^-}{1 - R_{ri}\,R_{li}\,T^-\,T^+}, \tag{19}$$

$$R_g^r = R_{re} + \frac{R_{li}\,T_{re}\,T_{ri}\,T^+\,T^-}{1 - R_{ri}\,R_{li}\,T^-\,T^+}. \tag{20}$$

Equations (17)–(20) do not lead to a unique identification of (T_{li}, R_{li}, T_{le}, R_{le}, T^+, T^-, T_{ri}, R_{ri}, T_{re}, R_{re}) from (R_g^l, R_g^r, T_g^+ and T_g^-). Nevertheless, the following hypotheses with good sense yield a unique identification:

(H1) Pressure continuity $\phi_l^+ + \phi_l^- = q_l^+ + q_l^-$ for $l \in \{0, L\}$ is required (see Figure 6). This hypothesis implies that *every reflection r and every transmission t fed by the same input are such that $t = 1 + r$.*

(H2) The left-hand side functions (of $\mathcal{Q}_\phi^l$) depend only on the *left parameter* ζ_l. Respectively, the right-hand side functions (of $\mathcal{Q}_\phi^r$) depend on the *right parameter* ζ_r.

(H3) The propagation transfer functions $T^\pm$ include a pure delay operator: $T^\pm(s) = D^\pm(s)\,e^{-sL/c_0}$, where $D^\pm(s)$ are causal and stable.

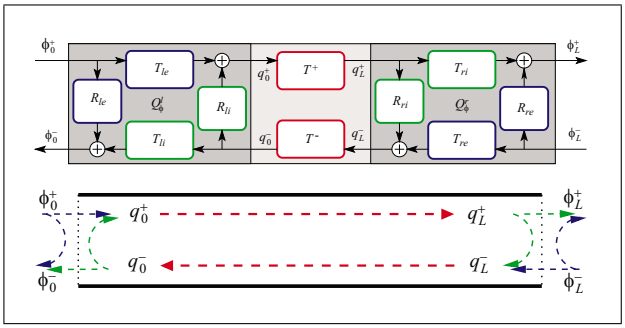


Figure 6. Decomposition of the two-port $\mathbf{Q}_\phi$.

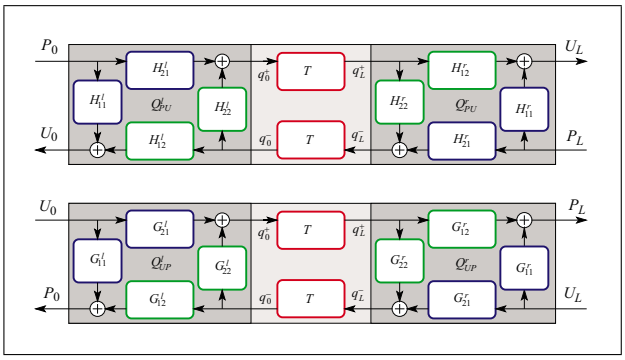


Figure 7. Decomposition of the two-port $\mathbf{Q}_{PU}$ and $\mathbf{Q}_{UP}$.

Then, the unique solution is proved to be (cf. [17])

$$T^+(s) = T^-(s) = T(s) = e^{-\Gamma(s)L}, \tag{21}$$

$$R_{re}(s) = -\frac{\Gamma(s) - \frac{s}{c_0} - \zeta_L}{\Gamma(s) + \frac{s}{c_0} - \zeta_L} = 1 - T_{re}(s), \tag{22}$$

$$R_{le}(s) = -\frac{\Gamma(s) - \frac{s}{c_0} + \zeta_0}{\Gamma(s) + \frac{s}{c_0} + \zeta_0} = 1 - T_{le}(s), \tag{23}$$

$$R_{ri}(s) = \frac{\Gamma(s) - \frac{s}{c_0} + \zeta_L}{\Gamma(s) + \frac{s}{c_0} - \zeta_L} = 1 - T_{ri}(s), \tag{24}$$

$$R_{li}(s) = \frac{\Gamma(s) - \frac{s}{c_0} - \zeta_0}{\Gamma(s) + \frac{s}{c_0} + \zeta_0} = 1 - T_{li}(s). \tag{25}$$

3.1.2. Decomposition of $\mathbf{Q}_{PU}$ and $\mathbf{Q}_{UP}$

A similar decomposition of two-ports with acoustic state (P, U) is obtained in Figure 7. For instance, $\mathcal{Q}_{PU}^l$ is obtained by connecting $\mathcal{C}_{PU,\phi}$ with $\mathcal{Q}_\phi^l$ (which means $\mathcal{Q}_{PU}^l \equiv \mathcal{C}_{PU,\phi} \odot \mathcal{Q}_\phi^l$), following the process detailed in section A1.

Transfer functions H_{ij}^x and G_{ij}^x of Figure 7 (with $x \in \{l, r\}$) are rational functions of s and $\Gamma(s)$.

Note that two-ports $\mathbf{Q}_{PU}^l$, $\mathbf{Q}_{PU}^r$, $\mathbf{Q}_{UP}^l$ and $\mathbf{Q}_{UP}^r$, of Figure 7, can be interpreted as conversion two-ports to new variables, $q^+(\ell, s)$ and $q^-(\ell, s)$, which are decoupled.

Remark: Since $\Re e(\Gamma(s)) > 0$, for all $s \in \mathbb{C}$ such that $\Re e(s) \geq 0$ (see [9]), then

$$T(s) = e^{-\Gamma(s)L} \to 0 \text{ when } L \to +\infty. \tag{26}$$

Consequently $P_0(s) \approx G_{11}^l(s)U_0(s)$ (when $L \to \infty$), and the transfer function G_{11}^l is interpreted as the left input

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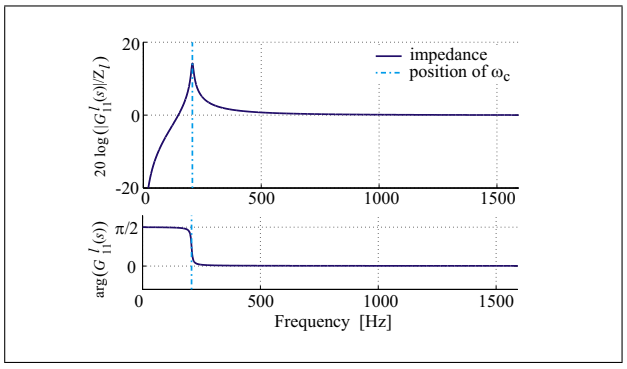


Figure 8. Impedance of semi-infinite piece of pipe, $G^l_{11}(s = 2i\pi f)$ for $r_l = 5$ mm, $\Upsilon = 15\ \mathrm{m}^{-2}$, $\varepsilon = 0.0525\ \mathrm{m}^{-\frac{1}{2}}$, $\zeta_l = 0\ \mathrm{m}^{-1}$.

impedance of a lossy semi-infinite pipe with constant curvature (cf. Figure 8). H^l_{11} is its corresponding admittance. The expressions of G^l_{11} and H^l_{11} are

$$G^l_{11}(s) = \frac{1}{H^l_{11}(s)} = \frac{Z_l}{c_0}\frac{s}{\Gamma(s)+\zeta_l}. \tag{27}$$

3.2. Separating effects of pipe geometry

This section now focuses on separating contributions on the acoustics due to the radius r, the slope r' and the curvature r''/r, and their possible discontinuities. The different orders of discontinuity of the shape are introduced:

- The 0-*order* corresponds to a discontinuity of radius r at junctions,
- The 1-*order* corresponds to a discontinuity of slope r',
- The 2-*order* corresponds to a discontinuity of curvature $\Upsilon = r''/r$.

Contrarily to connections of straight or conical pipes, the *Webster-Lokshin* model enables connections of pipes preserving C^1-regularity, which means the continuity of section and slope, and discontinuity of curvature (2-order of discontinuity).

3.2.1. Change of variables at interfaces of pipe

In order not to "hide" some effects of the geometry of the piece of pipe inside conversion two-ports, it is useful to define a change of variables independent from the pipe under interest. Thus, a *reference pipe* is introduced, which is a lossless cylinder with radius r_c, and defined variables correspond to planar traveling waves for this reference pipe. Its characteristic impedance is $Z_c = 1/Y_c = \rho_0 c_0/\mathrm{A}_c$ (with $\mathrm{A}_c = \pi r_c^2$), and planar traveling waves are, for $\varrho \in \{0, L\}$:

$$\begin{bmatrix} p^+(\varrho,s) \\ p^-(\varrho,s) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & Z_c \\ 1 & -Z_c \end{bmatrix}\begin{bmatrix} P(\varrho,s) \\ U(\varrho,s) \end{bmatrix}. \tag{28}$$

For lossy varying cross-section pipes, (p^+, p^-) are neither decoupled nor perfectly progressive inside the pipe. Nethertheless, they keep "physical meaning" at interfaces of pipe, as the decomposition detailed below enlights.

In section 4.2 will be shown that variables $p^\pm$ disappear after connection of two pieces of pipe, so that the value of r_c (or Z_c) is arbitrary and does not affect the modeling.

3.2.2. Decomposed framework

Figure 9 presents the framework where effects of geometry are separated into some two-ports: two-ports $\mathcal{Q}^l_a$ and $\mathcal{Q}^r_a$ for the effects of section at left and right ends respectively, $\mathcal{Q}^l_s$ and $\mathcal{Q}^r_s$ for slope, $\mathcal{Q}^l_{cl}$ and $\mathcal{Q}^r_{cl}$ for curvature.

With $\mathcal{C}'_{PU,p}$ the conversion two-port from (P, U) to $p^\pm$ and $\mathcal{C}'_{p,UP}$ its reciprocal conversion, this framework proves to be equivalent to $\mathbf{Q}_{PU}$ (see section 2.3.1). The proof uses interconnection laws of two-port systems (see Appendix A1).

Note that this framework naturally enlights 4 consecutive change of variables:

$p^\pm$: variables defined by (28),
$\phi^\pm$: traveling waves defined by (4),
$\psi^\pm$: symmetrical traveling waves defined in [9],
$q^\pm$: decoupled variables.

The remarkable thing of the framework of Figure 9, is that it isolates a central part for wave propagation and 6 cells of the *Kelly-Lochbaum* junction type (cf. [10]), of which 4 are well-known (effects of section and slope) and 2 are new as far as we know.

3.2.3. Interpretation of cells

- *Cells $\mathcal{Q}^l_a$ and $\mathcal{Q}^r_a$*
  $\mathcal{Q}^l_a$ and $\mathcal{Q}^r_a$, with coefficients k_l and k_r, take into account sections of pipe $\mathrm{A}_l = \pi r_l^2$ and $\mathrm{A}_r = \pi r_r^2$, at ends. The index "**a**" means "**a**rea of section". They are similar to cells of the *Kelly-Lochbaum* framework after connection with a lossless cylinder of radius r_c (cf. eg. [10, 13]).

$$k_l = \frac{\mathrm{A}_c - \mathrm{A}_l}{\mathrm{A}_c + \mathrm{A}_l} = -\frac{Z_c - Z_l}{Z_c + Z_l}, \tag{29}$$

$$k_r = \frac{\mathrm{A}_c - \mathrm{A}_r}{\mathrm{A}_c + \mathrm{A}_r} = -\frac{Z_c - Z_r}{Z_c + Z_r}. \tag{30}$$

- *Cells $\mathcal{Q}^l_s$ and $\mathcal{Q}^r_s$*
  $\mathcal{Q}^l_s$ and $\mathcal{Q}^r_s$, with transfer functions R^s_l and R^s_r, take into account slopes of pipe $r'_l = r'(\varrho = 0)$ and $r'_r = r'(\varrho = L)$ at left and right sides. The index "**s**" means "**s**lope". They are similar to cells of the framework of *Kelly-Lochbaum* after connection of cones (cf. eg. [11, 14]).

$$R^s_l(s) = \frac{\alpha_l}{s - \alpha_l}, \quad \text{with } \alpha_l = -\frac{c_0}{2}\frac{r'_l}{r_l}, \tag{31}$$

$$R^s_r(s) = \frac{\alpha_r}{s - \alpha_r}, \quad \text{with } \alpha_r = +\frac{c_0}{2}\frac{r'_r}{r_r}. \tag{32}$$

  If the slope is 0 (for cylinders for instance), these cells become identity ($\phi^\pm_\varrho = \psi^\pm_\varrho$).

- *Cells $\mathcal{Q}^l_{cl}$ and $\mathcal{Q}^r_{cl}$*
  $\mathcal{Q}^l_{cl}$ and $\mathcal{Q}^r_{cl}$, with the transfer function R, take into account the constant curvature and losses of the pipe. The indexes "**cl**" means "**c**urvature and **l**osses".

$$R(s) = \frac{\dfrac{s}{c_0} - \Gamma(s)}{\dfrac{s}{c_0} + \Gamma(s)}. \tag{33}$$

  In the case of lossless cones or cylinders ($\Upsilon = \varepsilon = 0$), $R = 0$, thus these cells are identity ($\psi^\pm_\varrho = q^\pm_\varrho$).

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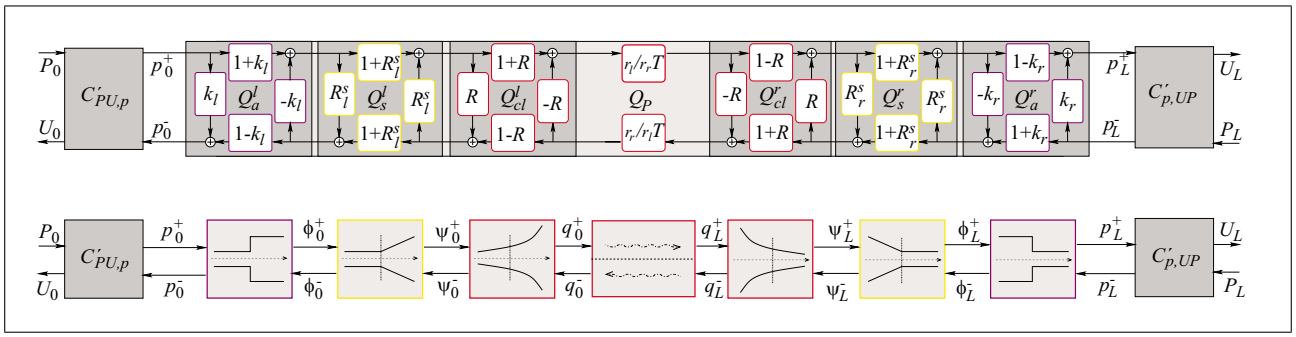


Figure 9. Decomposed framework, where effects of the pipe geometry are separated.

Using (26), in [23], the transfer function $R(s)$ is interpreted as the reflection at junction (with continuity of section and slope) between a lossless cone and an anechoic lossy flared pipe (with constant curvature). In [24], a similar interpretation has been done for R_l^s and R_r^s in the case of lossless cones.

3.2.4. Interest of this framework

For acoustic reasons, the framework of Figure 9 is interesting because the effects of the curvature and losses are isolated from the others (section and slope), and it makes their study easier. Because of the square roots of the function Γ (cf. equation 7), the study requires special treatments (see section 5.1).

For a systematic point of view, it is interesting to recover a symmetric general framework containing elementary cells of the *Kelly-Lochbaum* junction type. Moreover, as it is explained in section 4, this general framework makes possible to recover some former lossless waveguide models (cf. eg. [10, 11]), and some other lossy waveguide models by adding losses in propagation function $T(s)$ (cf. eg. [12, 14]). But in addition it takes into account both the curvature of the shape of pipe and the visco-thermal losses, according to the refined acoustic model of *Webster-Lokshin*.

4. Network of pipes

In this section two-port systems of pieces of pipe are connected in order to build the model of a complete resonator. Using variables (p^+, p^-), it is necessary to study how the connection must be done.

The right side of two-port $\mathbf{Q}_1$ is connected to left side of two-port $\mathbf{Q}_2$. The variables which are associated to $\mathbf{Q}_n$ for $n \in \{1, 2\}$, are named (p_n^+, p_n^-). Flow and pressure continuity assume that $P = p_1^+ + p_1^- = p_2^+ + p_2^-$ and $U = Y_c\left(p_1^+ - p_1^-\right) = Y_c\left(p_2^+ - p_2^-\right)$, which leads to

$$p_1^+ = p_2^+ \qquad \text{and} \qquad p_1^- = p_2^- .$$

In consequence, the connection of 2 two-ports with variables (p^+, p^-) is made by branching the output of $\mathbf{Q}_1$ (p_1^+) to input of $\mathbf{Q}_2$ (p_2^+), and reciprocally.

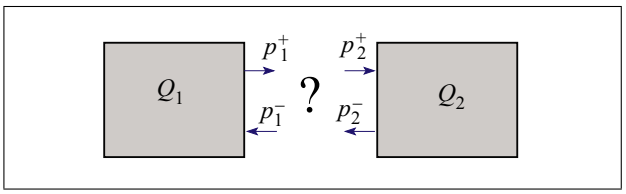


Figure 10. Connecting 2 two-port with variables (p^+, p^-).

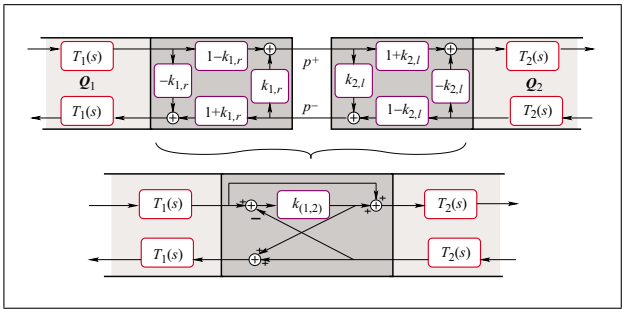


Figure 11. Junction of lossless cylinders (with $T_i(s) = \mathrm{e}^{-s\tau_i}$).

4.1. Junction of any pieces of pipe

Using the framework of Figure 9, it is possible to model, and then to simulate, the connection of two pieces of pipe, $\mathbf{Q}_1$ and $\mathbf{Q}_2$, with discontinuities of section, slope, curvature (and losses). The resulting two-port, $\mathcal{Q}_j^{1,2}$, of such a **j**unction is the connection of six cells, that is: $\mathcal{Q}_j^{1,2} \equiv \mathcal{Q}_{cl}^{r,1} \odot \mathcal{Q}_s^{r,1} \odot \mathcal{Q}_a^{r,1} \odot \mathcal{Q}_a^{l,2} \odot \mathcal{Q}_s^{l,2} \odot \mathcal{Q}_{cl}^{l,2}$.

With hypothesis on the order of discontinuity and on the shape of the pipe, this framework enables to recover some former waveguide models of the literature, using simplifications.

4.2. Junction of cylinders

Two lossless cylinders are considered. Two-ports $\mathbf{Q}_1$ and $\mathbf{Q}_2$ represent them, with section area $\mathrm{A}_{1,r}$ and $\mathrm{A}_{2,l}$.

For lossless cylinders, cells $\mathcal{Q}_s$ and $\mathcal{Q}_{cl}$ are identity systems and only cells $\mathcal{Q}_a$ are present. The resulting two-port of such a **j**unction is the connection $\mathcal{Q}_{j,a}^{1,2} \equiv \mathcal{Q}_a^{r,1} \odot \mathcal{Q}_a^{l,2}$. This junction is presented on top of Figure 11.

Merging two-ports at interface leads to an equivalent two-port which represents the junction of cylinders (cf. bottom of Figure 11). Using the algebraic laws of interconnections (cf. Appendix A1) and the expressions of $k_{1,r}$

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and $k_{2,l}$ (cf. section 3.2.3 §1), the coefficient of reflexion is given by

$$k_{(1,2)} = \frac{\mathrm{A}_{1,r} - \mathrm{A}_{2,l}}{\mathrm{A}_{1,r} + \mathrm{A}_{2,l}}. \tag{34}$$

The *Kelly-Lochbaum* framework for networks of lossless cylinders is recovered (see eg. [10]). This model corresponds to the 0-*order* of discontinuity.

Remark that variables p^+ and p^- have now disappeared after merging, and $k_{(1,2)}$ does not depend on Z_c.

4.3. Junction of cones

Consider two lossless cones represented by two-ports $\mathbf{Q}_1$ and $\mathbf{Q}_2$, with slopes $r'_{1,r}$ and $r'_{2,l}$, connected with continuity of section.

The continuity of section (or radius) implies $r_{1,r} = r_{2,l} = r_j$ and $k_{(1,2)} = 0$. And for lossless cones, cells $\mathcal{Q}_{cl}$ are identity systems. Consequently, only the effect of difference of slopes remains. The resulting two-port of such a **j**unction is the connection $\mathcal{Q}^{1,2}_{j,s} \equiv \mathcal{Q}^{r,1}_{s} \odot \mathcal{Q}^{l,2}_{s}$ (see Figure 12) where

$$R^s_{(1,2)}(s) = \frac{\alpha_{(1,2)}}{s - \alpha_{(1,2)}}, \tag{35}$$

$$\text{with} \qquad \alpha_{(1,2)} = \alpha_{1,r} + \alpha_{2,l} = +\frac{c_0}{2}\left(\frac{r'_{1,r} - r'_{2,l}}{r_j}\right). \tag{36}$$

The *Kelly-Lochbaum* framework is recovered for networks of lossless cones (see eg. [11]). It corresponds to the 1-*order* of discontinuity, with C^0-regularity of the shape of pipe.

Remark: In [12, 14] the damping due to visco-thermal losses in propagation operators is taken into account, up to the same order of approximation ε (with $\Upsilon = 0$) of the acoustic model: $T(s) = \mathrm{e}^{-s\tau}\,\check{D}(s)$ where

$$\check{D}(s) := \mathrm{e}^{-\varepsilon\sqrt{s/c_0}L}. \tag{37}$$

4.4. Junction of constant-curvature pipe

Now a junction is considered with continuity of section ($\mathrm{A}_{1,r} = \mathrm{A}_{2,l}$) and continuity of slope ($r'_{1,r} = r'_{2,l}$), and two lossy flared pipes with different parameters are connected ($\Upsilon_1 \neq \Upsilon_2$ or $\varepsilon_1 \neq \varepsilon_2$).

C^1-regularity implies $k_{(1,2)} = 0$ and $R^s_{(1,2)} = 0$. The resulting two-port of such a **j**unction is the connection $\mathcal{Q}^{1,2}_{j,cl} \equiv \mathcal{Q}^{r,1}_{cl} \odot \mathcal{Q}^{l,2}_{cl}$. This junction is presented in Figure 13, where

$$R_{(1,2)}(s) = \frac{\Gamma_1(s) - \Gamma_2(s)}{\Gamma_1(s) + \Gamma_2(s)}, \tag{38}$$

$$\text{with} \qquad \Gamma_n(s) = \sqrt{\left(\frac{s}{c_0}\right)^2 + 2\varepsilon_n\left(\frac{s}{c_0}\right)^{\frac{3}{2}} + \Upsilon_n}, \quad \forall n \in \{1,2\}. \tag{39}$$

The framework of *Kelly-Lochbaum* for networks of flared pipes with constant curvature is recovered (see [17]).

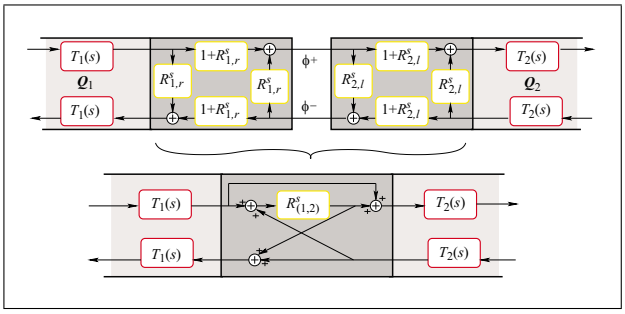


Figure 12. Junction of lossless cones (with $T_i(s) = \mathrm{e}^{-s\tau_i}$).

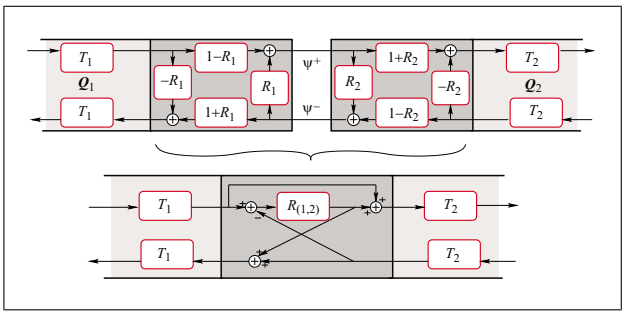


Figure 13. Junction of lossy flared pipes with constant curvature (where $T_i(s) = \mathrm{e}^{-\Gamma_i(s)L_i}$).

By comparing with former waveguide models, the improvement of this new model is that it allows junctions with continuity of section and slope, that is it can take into account the 2-*order* of discontinuity, with $r(\ell)$ C^1-regular.

The analytical model of a network of some pieces of pipe with constant curvature per piece and with continuity of section and slope at junctions is proved to be stable. This proof is done by analyzing stability and passivity for one two-port of piece of pipe. Then by induction reasoning, the passivity of the connexion of a two-port of piece of pipe with any passive system, leads to the passivity and the stability of the whole network. No details are given in the present paper, but a next article will be devoted to this proof.

5. Digital waveguides and simulation

5.1. Diffusive representation, approximation and digital derivation

Because of the square roots in $\Gamma(s)$, some involved transfer functions are irrational, and so they are not standard. These transfer functions have continuous lines of singularities in $\mathbb{C}$, which are named *cuts*. These cuts join some points (*branching points*) and the infinity.

If $\Upsilon = 0$, the function Γ has one branching point at $s = 0$. The cut $\mathbb{R}^-$ is chosen to preserve hermitian symmetry. Thereof, associated transfer functions have a continuous line of singularities on $\mathbb{R}^-$. Using the residue theorem, it has been shown (cf. eg. [25, 26, 27, 28]) that these functions are represented by a class of infinite-dimensional systems: *Diffusive Representations*. For any diffusive rep-

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resentation $H(s)$, analytic on $\mathbb{C} \setminus \mathbb{R}^-$:

$$H(s) = \int_0^\infty \frac{\mu_H(\xi)}{s+\xi} \mathrm{d}\xi, \tag{40}$$

$$\mu_H(\xi) = \frac{1}{2i\pi}\Big(H(-\xi + i0^-) - H(-\xi + i0^+)\Big). \tag{41}$$

For simulation in the time domain, in [29] for example, it has been proposed to approximate such diffusive representations by finite-dimensional approximations, given by $\widetilde{H}(s) = \sum_{j=1}^{j=J} \mu_j/(s+\xi_j)$, where J is the number of poles, $-\xi_j \in \mathbb{R}^-$ is the position of the jth pole and μ_j is its weight.

The poles are placed in $\mathbb{R}^-$ with a logarithmic scale, and the weights μ_j are obtained by a least-square optimization in the Fourier domain.

If $\Upsilon > 0$, Γ has two more branching points, which are complex conjugate. In this case, the diffusive representations are approximated with a finite sum of first and second order differential systems,

$$\widetilde{H}(s) = \sum_{j=1}^{j=J} \frac{\mu_j}{s+\xi_j} + \sum_{n=1}^{n=N} \left(\frac{w_n}{s+\gamma_n} + \frac{\overline{w}_n}{s+\overline{\gamma}_n} \right). \tag{42}$$

Transfer functions to approximate are: R_k, $R_{(k,k-1)}$ and D_k, for any k (see sections 3.2.3 (§4) and 4.4)[1]. Note that in models of [12] and [14], the transfer function $\breve{D}(s) = \mathrm{e}^{-\varepsilon_k \sqrt{s/c_0} L_k}$ has to be approximated with poles on $\mathbb{R}^-$.

Finally, a digital version of (42) is implemented using standard numerical approximations (see [30, 31, 9]).

5.2. Numerical comparisons of three simulations

In this section digital waveguide simulations of a *goal* pipe $\mathcal{P}_{goal}$ defined in section 5.2.1 are built approximating $\mathcal{P}_{goal}$ by straight (M1), conical (M2) or constant-curvature (M3) pieces of pipe. The sampling frequency F_s or length L are adapted so that no fractional delays are required. The number of pieces of pipe is chosen such that artefacts due to the discontinuities of the approximated shapes are rejected (see 5.2.2). The input impedances respectively obtained from (M1,M2,M3) are compared in the Fourier domain to the *goal* one (see 5.2.3).

Note that the three acoustic models take into account losses and the varying cross-section. By fixing an equivalent global complexity, the comparison is about the quality of approximations of the diffusive representations (cf. section 5.1).

[1] The order of approximation $(J+2N)$ depends on the desired quality: J mainly tunes the quality of approximation of effects due to visco-thermal losses, and N, that due to the curvature. For example, in the case of very large tubes, visco-thermal losses are negligible so that, the choice $J = 0$ can be done. Increasing J and N improves the approximation, and so the quality of the simulation, and decreasing J and N reduces the quality. In practice, choosing $J = 6$ and $N = 5$ leads to fully satisfactory results, in most cases (which corresponds to a 16^{th} order filter).

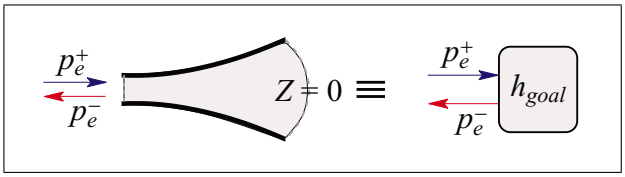


Figure 14. The impulse response h_{goal} is the reflexion of the virtual pipe.

5.2.1. *Goal* pipe, lengths and sampling frequencies

The *goal* pipe is defined by $r(\ell) = R_0 + \ell^\alpha$ on $\ell \in [0, L]$, with $R_0 = 2.5$ mm and $\alpha = 4$. This pipe is terminated at the right end by a zero impedance. Here, only digital simulation of acoustic tubes is studied, and the *goal* pipe is ideally open in order not to introduce a model of sound radiation.

The exact impulse response of the global reflexion function of the *goal* pipe is denoted $h_{goal}(t)$ (see Figure 14). Moreover, the input impedance $Z_{goal}(i\omega)$ is computed for each ω as follows: [step 1] for each (fixed) $s = i\omega$, numerically solve the linear Cauchy problem (1), (2) on $\ell \in [0, L]$ with the boundary condition $P(\ell = L, s) = 0$, $U(\ell = L, s) = 1$; [step 2] compute

$$Z_{goal}(s) = \frac{P(0,s)}{U(0,s)} = -\frac{\rho_0 s}{\mathrm{A}_0} \cdot \frac{P(\ell = 0, s)}{\partial_\ell P(\ell = 0, s)}, \tag{43}$$

with $s = \mathrm{i}\omega$. Results are presented in Figure 15 (curve C0).

For simulations based on (M3) which relies on the curvilinear length ℓ, the following choice is made to have no fractional delays: $F_s = 44100$ Hz and $L = Kc_0/F_s \approx 49.9$ cm with $K = 64$. From (3), the corresponding axial length is $L_z = \ell^{-1}(L) \approx 49.0$ cm so that the corresponding sampling rate is $F_z = Kc_0/L_z \approx 44924$ Hz for (M1).

5.2.2. Number of pieces of pipe for models (M1, M2, M3)

First, for (M1) with $\varepsilon = 0$, the continuous-time impulse response associated to pieces of pipe with length δz_1 writes

$$h_1(t) = \sum_{n=0}^{+\infty} a_n \delta(t - 2n\,\delta z_1/c_0), \tag{44}$$

where $\delta(.)$ denotes the Dirac distribution, and the a_n's can be deduced from reflexion coefficients k in (34). Hence, the choice $\delta z_1 = c_0/(2F_z)$ yields the digitalized impulse response $h(t_n) = a_n$ which avoids discontinuities due to the artificial intertwined "zeros" naturally involved by (M1). According to this well-known and standard result, the number of pieces of pipe is then chosen as $2K = 128$. Note that for $\varepsilon > 0$, (44) becomes smoother since the losses introduce dissipation through $\breve{D}$ in (37) which corresponds to the slowly decreasing impulse response (cf. [32]) $\tilde{d}(t) = (\beta/(2\sqrt{\pi t^3}))\,\mathrm{e}^{-\beta^2/(4t)}$ for $t > 0$ with $\beta = \varepsilon\,\delta z_1/\sqrt{c_0}$. But, this regularization is not sufficient to decrease the number $2K$ (see Figure A5a,b,c on page 489).

Second, for (M2) with $\varepsilon = 0$, the corresponding continuous-time impulse response h_2 involves decreasing

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exponentials (cf. [24, 33]) and digital versions involve first order filters so that the computational cost for one piece of pipe is twice greater for cones than for cylinders. To preserve a similar global computational cost and since the discontinuity of h_2 is smoothed by exponential responses, the number of pieces of pipe is chosen as $K = 64$ so that $\delta z_2 = 2\delta z_1$. In practice, this actually yields smooth results (see Figure A5d,e,f on page 489).

Third, (M3) leads to a naturally smooth impulse response h_3 and reducing the number of pieces of pipe does not introduce artefacts of discontinuity type but only influences the quality of approximation. In practice, taking only 4 pieces of pipe with constant curvature yields accurate results (see Figure A5g).

5.2.3. Digital waveguides, input impedances and comparisons

Each digital simulation based on (M1,M2,M3) is built with a Kelly-Lochbaum framework derived as in sections 4.2 to 4.4. In these networks, time domain versions of $T_i(s)$ (see (15), (16) and (37)) are implemented using a circular buffer for the delay part and using recursive digital filters straightforwardly deduced from optimisations of $D_i(s)$ and $\check{D}_i(s)$ following the method presented in section 5.1. Digital filters are also obtained for $R^s_{i,i+1}$, $R_{i,i+1}$ in the same way.

From the number of floatting point operations and the CPU point of view, parameters J, N involved in the approximations (42) of the diffusive representations are chosen in order to preserve the same algorithmic complexity. Then, the ξ_j's and γ_n are optimized to have the best results following the method proposed in [29].

In practice, the simulation network for (M1) involves $2K = 128$ delays and reflexion coefficients $k_{i,i+1}$, and $\check{D}_i$ are approximated using second order filters. The network for (M2) involves $K = 64$ circular buffers for delays, 64 first order filters for reflexions $R^s_{i,i+1}$ and 64 fourth order filters for approximating $\check{D}_i$. Finally, the network for (M3) involves 4 pieces of pipe: 8 circular buffers, 8 filters for approximating D_i, 5 filters for approximating $R_{i,i+1}$ with distinct orders but leading to the same global complexity as previous models (which are "globally" equivalent to 170^{th} order filters).

The input impedances corresponding to (M1,M2,M3) are respectively deduced from the discrete-time Fourier transforms R_1^{glob}, R_2^{glob}, R_3^{glob} of the digital impulse responses h_1, h_2 and h_3 of the global input reflexion (similar to Figure 14) using

$$\underline{Z}_{in}(f) = \frac{Z_c \pi R_0^2}{\rho_0 c_0} \cdot \frac{1 + R_{glob}(f)}{1 - R_{glob}(f)}, \qquad (45)$$

where f denotes the frequency.

These impedances and the *goal* impedance are plotted in Figure 15. To help comparisons, their spectral envelopes which links local maxima are also plotted. Frequency positions of maxima coincide for all models (with an error ±15 Hz), but not their amplitude. More precisely, for (M1) and

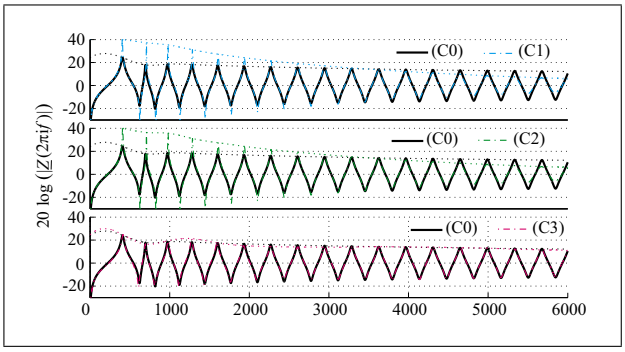


Figure 15. Comparisons between the *goal* impedance Z_{goal} and impedances obtained from (M1), (M2) and (M3). Curve (C0) is the impedance Z_{goal}. It is numerically computed using a standard Runge-Kutta method (cf. [34]) (Matlab©: ode23 with a relative error $\delta = 10^{-3}$). Curves (C1,C2,C3) are deduced from the simulation of 128 cylinders, of 64 cones and of 4 constant-curvature pipes, respectively. Dotted curves represent envelopes of maxima.

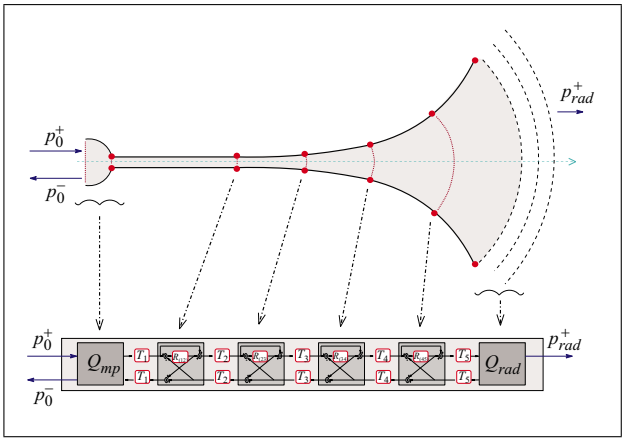


Figure 16. Making a virtual trombone.

(M2), amplitudes are significantly too high between 0 and 3000 Hz and too small in [4000, 6000] Hz. On the contrary, model (M3) better fits with the *goal* impedance with an accuracy smaller than 2 dB.

The reason is that many pieces of pipe are required (128 for (M1), 64 for (M2)), so to respect the given "global" algorithmic complexity only low order filters can be used to approximate the $\tilde{D}$'s which account for losses. On the contrary, with only 4 pieces of pipe, (M3) allows higher orders for approximations together with an equivalent global complexity and a best quality. Moreover, it is possible to adjust complexity and quality, by adapting the values of J and N for each approximation (cf. section 5.1).

5.3. Building a virtual instrument

This section deals with the resonator of a virtual trombone modeled by concatenating seven elements (see Figure 16, top): a mouth-piece, five pieces of pipe (1 cylinder and 4 flared pipes with constant curvature), and a radiation impedance.

The mouthpiece is modeled by an acoustic mass (cf. [35]), a resistance, and a compliance, and it is simulated

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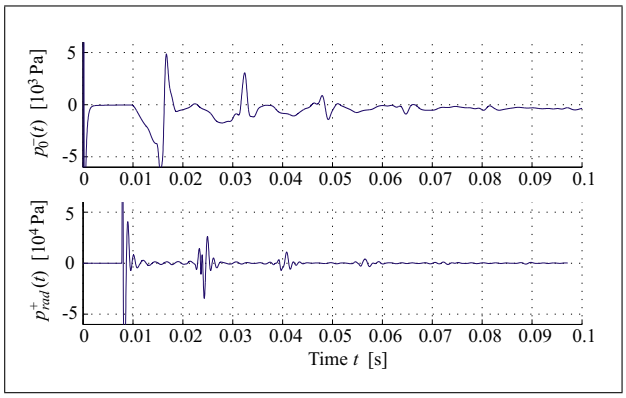


Figure 17. Input (top) and output (bottom) impulse responses (digital time simulation).

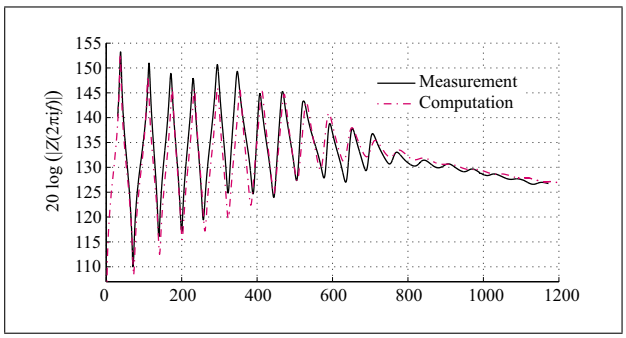


Figure 18. Measured and computed impedances.

by a second order system. Because of the flaring at the end of the horn of brass instruments, the radiation model of [36] seems well adapted for this application; here this model is used. The cylinder models the slide of the trombone and the four flared pieces of pipe model the horn. Parameters (lengths and curvatures) of these pieces of pipe have been empirically tuned such that the approximated shape fits the radius measured on a real trombone (see Appendix A2 for details). The measurements correspond to a Courtois trombone.

Note that, with $r_{min} \approx 7$ mm and $r_{max} \approx 11$ cm (the extreme radii), the *Webster-Lokshin* model is valid at least with $0.13 \ll f < 915$ Hz (for validation of losses and longitunal propagation). For trombones, this limitation is satisfactory considering that the coupling of the resonator and the excitator (the lips, cf. eg. [37]), mainly involves lower modes.

The approximated shape is C^1-regular, and so the *Kelly-Lochbaum* framework is recovered for the whole pipe of the trombone, leading to Figure 16 (bottom). Figure 18 compares the impedance measured on the real trombone to that computed from the digital time simulation of the impulse response.

Figure 18 shows that the modelisation is globally correct, but with some significant errors in the spectral envelop around 300 and 600 Hz. This difference comes from the rough empirical approximation of the original profil. Nevertheless, some recent works will allow better automatic approximations of the profile (cf. [38]).

6. Conclusions and perspectives

In this paper, the *Webster-Lokshin* model has been used to build digital waveguides taking into account both visco-thermal losses at the wall, and curvature of the shape of pipe.

A "generalized" framework has been derived in which delays and effects due to the shape are separated (section, slope and curvature). Considering particular cases of regularity at junctions of pieces of pipe, three kinds of Kelly-Lochbaum networks are recovered. One of them makes it possible to consider junctions with continuity of section and slope, and discontinuity of curvature, which means the C^1-regularity of the shape.

Moreover, this model does not impose a number of pieces of pipe, contrarily to models of connections of straight and conical pieces of pipe. In section 5.2, whereas 128 cylinders (M1) or 64 cones (M2) are required to model the *goal* pipe, 4 constant curvature pieces of pipe are enough to get good results (M3). Using this improvement, digital real-time simulations has been implemented in C/C++ language.

In the particular case of C^1-regularity (with $\Upsilon(\ell) > 0$), the stability and passivity of transfer functions involved in the global Kelly-Lochbaum network can be proved analytically (in the Laplace domain). This specific aspect of the work will be presented in a future paper.

However, for negative curvatures, some transfer functions involved in the Kelly-Lochbaum network are unstable. This phenomenon is quite complicated, but is now well understood (see eg. [7]). Very similar phenomena appear at junctions of cones, when the difference of slopes (on both size of the junction) is negative (cf. [24, 14]). In [39], the simple case of a single convergent cone has been studied, and the corresponding Kelly-Lochbaum network has been proved to be internally stable if it is represented by its minimal realization, in the sense of the state-space representations of systems.

Hence, an approach to cope with the problem of negative curvature could be based on an extension of solutions which have been proposed for networks of cones.

In this paper, the example of a virtual trombone is presented to illustrate some results of simulations. The application of the presented model to the simulation of the vocal tract will be considered after solving the problem of negative curvatures.

Appendix

A1. Connecting 2 two-ports

Connecting 2 two-ports consists in branching output of one two-port to input of the other, and reciprocally as top of Figure A1 shows.

Merging these two-ports into a unique equivalent two-port is interesting for some reasons:

- First, in some cases, the merging allows to simplify the framework.
- It can be used to prove the equivalence between some different forms of a two-port system.

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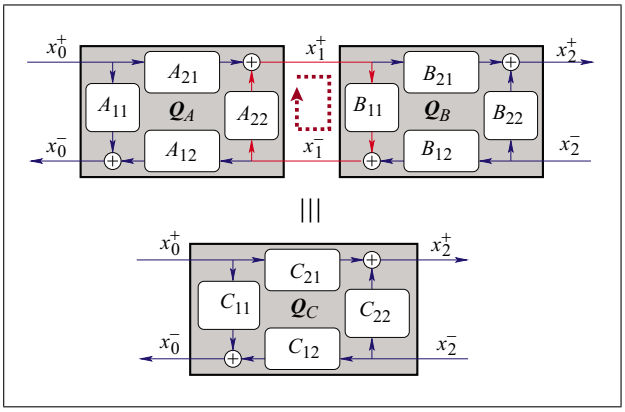


Figure A1. Connecting two-ports and removing instantaneous loops.

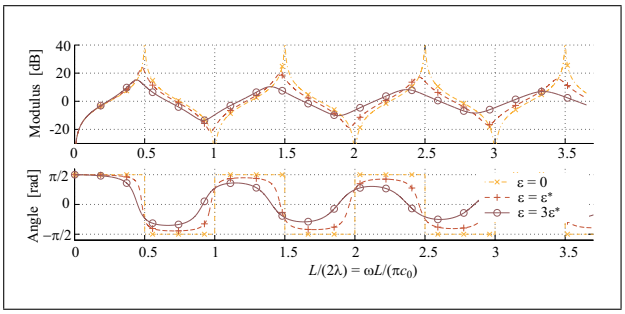


Figure A2. Influence of losses on the impedance of an open cylinder, H_{11}^{-1}. Parameters: $r_0 = r_L = 5\,\text{mm}$, $L = 1\,\text{m}$, $\Upsilon = 0\,\text{m}^{-2}$.

- As Figure A1 shows, there is an instantaneous loop at interface which cannot be simulated numerically.

In Figure A1, $\mathbf{Q}_C$ is defined as the equivalent two-port of the connection of two-ports $\mathbf{Q}_A$ and $\mathbf{Q}_B$. In [22], the following notation is defined: $\mathbf{Q}_C \equiv \mathbf{Q}_A \odot \mathbf{Q}_B$, where $\equiv$ is the equivalent relation and $\odot$ is the connection operator.

Algebraic expressions of elements of $\mathbf{Q}_C$ are

$$C_{11} = A_{11} + \frac{A_{12}B_{11}A_{21}}{1 - A_{22}B_{11}}, \tag{A1}$$

$$C_{12} = \frac{B_{12}A_{12}}{1 - A_{22}B_{11}}, \tag{A2}$$

$$C_{21} = \frac{A_{21}B_{21}}{1 - A_{22}B_{11}}, \tag{A3}$$

$$C_{22} = B_{22} + \frac{B_{12}A_{22}B_{21}}{1 - A_{22}B_{11}}, \tag{A4}$$

for which $A_{22}(s)B_{11}(s) \neq 1, \forall s/\Re e(s) > 0$ is required to guarantee the stability of the feedback loop.

A2. Table of parameters of section 5.3

In Tables A1 and A2, the subscript c means the *cup* of the mouth-piece of the trombone and b means its *backbore*. r_c and V_c are the radius and the volume of the cup, L_b and r_b are the length and the equivalent radius of the backbore. C_a is the acoustic compliance of the cup, R_a and M_a are the acoustic resistance and the acoustic mass associated to the backbore (see [35]).

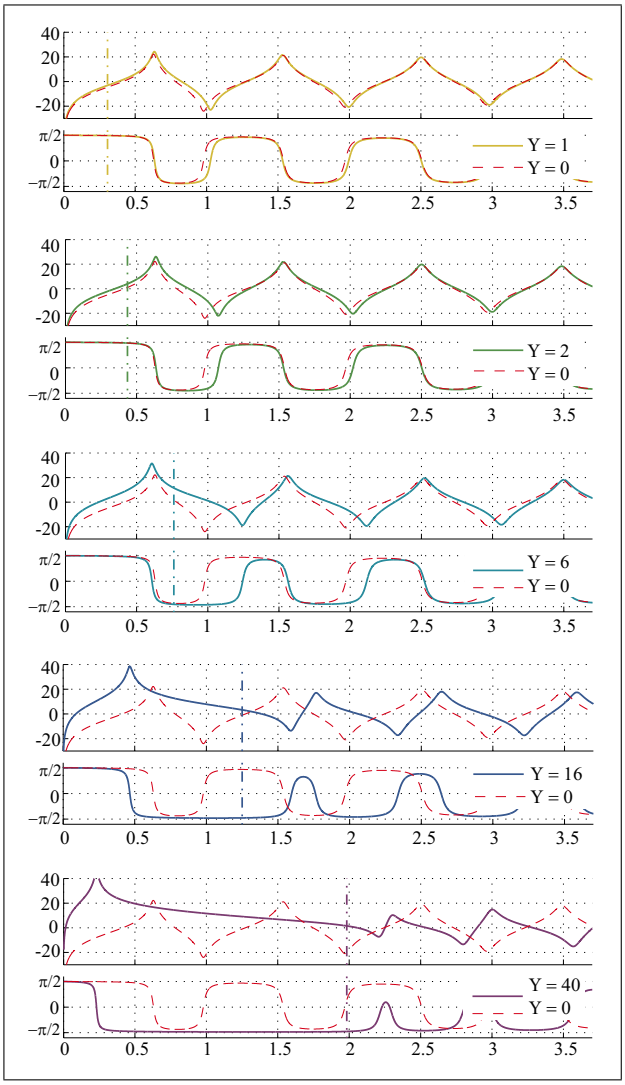


Figure A3. Influence of the curvature on the impedance of an open pipe, H_{11}^{-1}. Parameters: $r_0 = 5\,\text{mm}$, $r_L = 10\,\text{mm}$, $L = 1\,\text{m}$. Abscissa: kL/π.

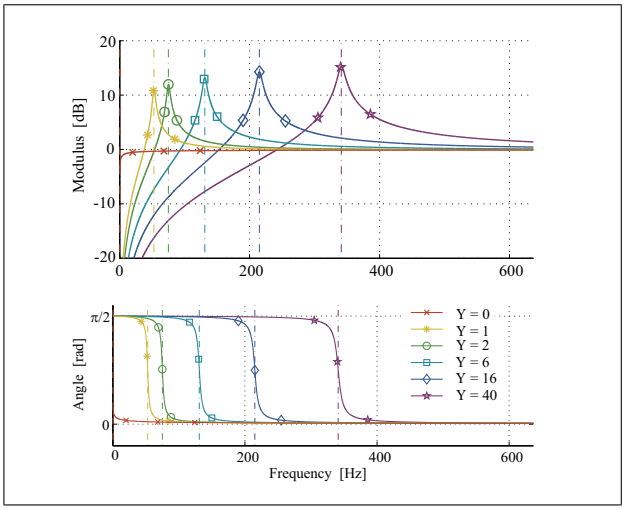


Figure A4. Influence of the curvature on the impedance of a semi-infinite pipe, $G_{11}^l(s = 2i\pi f)$. Parameters: $r_l = 5\,\text{mm}$, $\varepsilon = 0.0525\,\text{m}^{-1/2}$, $\zeta_l = 0\,\text{m}^{-1}$.

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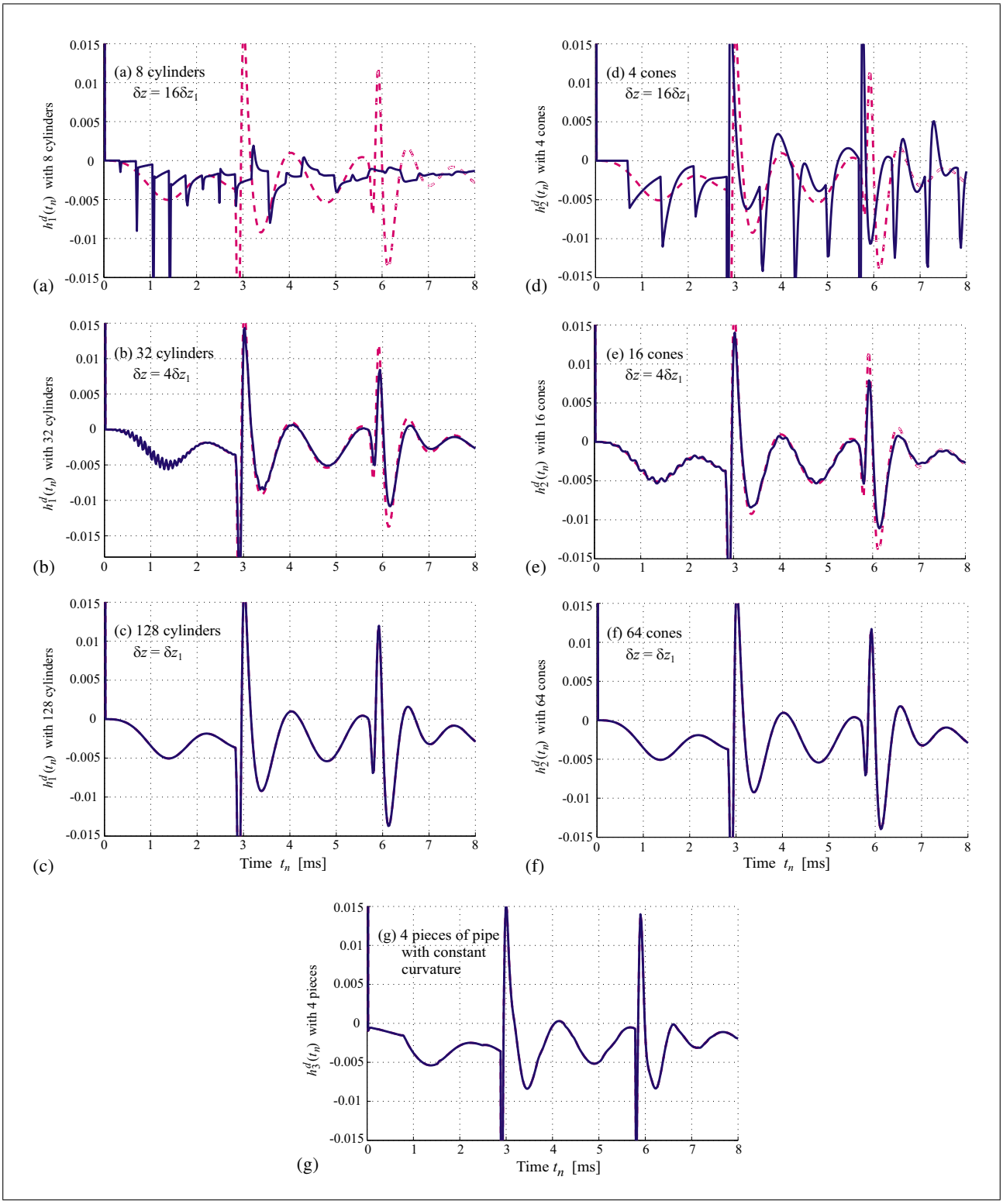


Figure A5. Digital impulse responses of simulations in the discrete time domain (cf. section 5.2.2, page 485): (a) Connection of 8 cylinders, (b) connection of 32 cylinders, (c) connection of 128 cylinders, (d) connection of 4 cones, (e) connection of 16 cones, (f) connection of 64 cones, (g) connection of 4 constant-curvature pipes. The dashed curves are the correctly sampled impulse responses for cylinders and cones.

Note that parameters C_a, R_a, M_a, r'_l, r'_r and ε are calculated from the others. For constant curvature, $\varepsilon(\ell)$ varies with $r(\ell)$, ε is chosen as the mean of $\varepsilon(\ell)$ along the piece of pipe.

A3. Effects of curvature and losses on Webster-Lokshin transfer functions

The input impedance of a cylinder which is ideally open at right (that is $P_L = 0$) is drawn in Figure A2 for vari-

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Table A1. Mouthpiece parameters.

r_c (m) $14.5 \cdot 10^{-3}$	V_c (m^3) $10.8 \cdot 10^{-6}$	L_b (m) $52.5 \cdot 10^{-3}$	r_b (m) $4.5 \cdot 10^{-3}$
C_a ($m^3\ P_a^{-1}$) $7.59 \cdot 10^{-11}$	M_a ($kg\ m^{-4}$) 990	R_a ($N\ s\ m^{-5}$) $5.87 \cdot 10^{-3}$	

Table A2. Pieces of pipe parameters.

Number	1	2	3	4	5
r_l (mm)	6.9	6.9	10.4	22	43.7
r_r (mm)	6.9	10.4	22	43.7	110
r'_l	0	0	0.013	0.055	0.5
r'_r	0	0.013	0.055	0.5	0.99
L (m)	1.615	0.544	0.352	0.103	0.085
Y (m^{-2})	0	3	9	120	60
ϵ ($m^{-1/2}$)	0.0507	0.0422	0.0248	0.0116	0.0033

ous loss coefficients: $\epsilon = 0$ (no losses), $\epsilon = \epsilon^*$ computed from the standard physical values, and $\epsilon = 3\epsilon^*$ (artificially overestimated losses).

Figure A3 compares the effects of the curvature value on the input impedance $Z_{in} = 1/H_{11}$ of an ideally open pipe (see also Figure 1). Vertical dashed lines represent the cutoff frequency ω_c for each curvature.

Figure A4 compares the effects of the curvature value on the input impedance $Z_{in} = 1/H_{11}^l$ of a semi-infinite pipe (see also Figure 7). Vertical dashed lines represent the cutoff frequency for each curvature.

Acknowledgement

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