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SH Wave Scattering at a Broken Tapered End of an Elastic Plate

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Summary
The paper focuses on the interaction of SH elastic waves in a plate with a tapered free end. The tapered free end is also considered after imposing a truncation to represent a broken tip. The plate is modeled as a semi-infinite elastic structure, which was assumed to have traction-free surfaces. The results of the analysis based on mode matching are presented for various combinations of normalized frequencies, angles of free end inclination, and sizes of broken tip part. The reported observations form important guidelines for the interpretation of experimental data when using horizontally polarized wave as a mean for nondestructive evaluation of elastic plates.

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1. Introduction

Nondestructive evaluation of plates is predominantly based on studying elastic wave scattering due to cracks, voids and other flaws included in these structural elements. Evaluating structural integrity conditions at the end of a plate requires studying the problem of elastic wave scattering right at the boundary. Literature in the area of "structural end diagnosis" is somehow limited.

The reflection of Lamb waves from the edge of a straight cut plate has been investigated in a number of papers. In an early study, Torvik [1] considered the reflection of the Lamb symmetric mode S. from a free end of a plate using a vibration mode expansion method. Gregory and Gladwell [2] studied the Lamb S_0 mode reflection from free and fixed ends of a semi-infinite elastic plate using a method of projection. Zhang *et al.* [3] utilized a photoelastic visualization technique to show reflected Lamb modes from the free end of a plate when the modes S_0 , A_0 and A_1 are incident on the free boundary. Cho and Rose [4] applied a boundary element method to study the Lamb wave mode reflection from a free end. Morvan *et al.* [5] conducted laser interferometer experiments on the reflection of Lamb waves at the free end of a plate over a range of frequencies and compared results with finite element outputs. Diligent and Lowe [6] focused on the reflection of the Lamb anti-symmetric mode A_0 from a free plate edge. Bian and Rose [7] applied a transverse resonance concept to investigate the reflections of the waveguide modes from normal interfaces and determined conditions for having no mode conversion at normal traction free or fixed end.

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The free ends of the structures considered by previously mentioned works are ideally straight, which is not the case in so many real situations. An interest was then raised in the reflection of a Lamb waves from inclined free edges of elastic plates. Al-Nassar and Sobh [8] used a mode matching technique to investigate the scattering of SH waves at inclined edges of plates having different angles of inclination. Wilkie- Chancellier *et al.* [9, 10, 11] performed finite element analyses and laser interferometer / vibrometer experiments on the reflection of the S_0 , A_0 and A_1 Lamb modes from an inclined edge of elastic plates having different values of the bevel angle. They proved that the edge mode was the resonance of different complex modes. Galán and Abascal [12] used a hybrid finite-element / boundary element technique to study Lamb mode conversion at inclined or perpendicular free edges of elastic plates for frequencies beyond the first cut-off frequency. It is also worth noting that elastic wave scattering was considered in plates with their thickness varying according to powerlaw shapes, thus creating wedges that ideally provide zero reflections [13, 14, 15].

It is the aim of this work to analytically consider elastic wave scattering at the tapered free end of an elastic plate. It is also aimed at considering the case of a broken edge. The mode matching procedure is done along a fictitious boundary inside the plate. In the left side of that boundary, the wave function expansions are represented by Bessel forms. While the right side of it, the wave function expansion is expressed by the plate solution. In this way, the traction free-surfaces are perfectly satisfied. By imposing the continuity of displacement and stresses between the Bessel solution and plate solution along the fictitious boundary, this is resulted in two matching equations. By integrating these two equations after multiplying them by appropriate weight functions, this leads to a system of linear al-

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gebraic equations for the reflection coefficient of the scattered field. Numerical accuracy of the solution is checked by verifying the energy balance. The scattering spectra of these two cases shall provide typical signatures of intact and broken plates. Range of inclination angles, sizes of broken tips, and frequencies have been considered and reported in this study. The outcome of the project should be of a value for ultrasonic nondestructive evaluation of elastic plates.

2. Problem formulation

The plate is assumed to be of uniform thickness, $2h$, and has traction-free surfaces. Two regions can be identified and these are plate region, Ω_1, and wedge region, Ω_2, as shown in Figure 1.

The total displacement field, w^T, in the plate due to transversely shear wave is given by

$$w^T = w^{\text{inc}} + w^{\text{ref}}, \tag{1}$$

where the superscripts inc and ref in the above equation stand for incident and reflect field, respectively. Incident field, w^{inc}, consists of a single propagating mode propagating in the negative x-direction and the reflected field, w^{ref}, is composed of the reflected modes (propagating and non-propagating modes) in the plate region. The incident field is assumed to be time harmonic wave, $e^{j\omega t}$, and given by

$$w^{\text{inc}} = A_{\text{inc}} \cos\left(\frac{j\pi y}{2h}\right) e^{-j(k_x x - \omega t)}. \tag{2}$$

k and ω are wavenumber and frequency, respectively. The relationship between these two quantities is given by $n\pi = \sqrt{(2\omega h/c)^2 + (2kh)^2}$ whereas the reflected field (scattered) becomes

$$w^{\text{ref}} = \sum_{l=0}^{L} B_l \cos\left(\frac{l\pi y}{2h}\right) e^{j(k_l x + \omega t)}. \tag{3}$$

The displacement field in wedge region, Ω_2, is given by

$$w^{\text{tr}} = \sum_{n=0}^{N} \cos\left(\frac{n\pi\theta}{\alpha}\right)\left[C_n J\left(\frac{n\pi}{\alpha}, \frac{\omega r}{c_s}\right) + D_n Y\left(\frac{n\pi}{\alpha}, \frac{\omega r}{c_s}\right)\right] e^{j\omega t}, \tag{4}$$

where α is the wedge angle, r is the radial coordinate, c_s is the shear velocity, $(G/\rho)^{1/2}$, G and ρ stand for the shear modulus and density, respectively. Noting that, the free traction along the two boundaries Γ_1 and Γ_2 are perfectly satisfied.

Applying the free traction condition along the boundary, Γ_4, and the continuity of displacement and stress along the fictitious boundary, Γ_3, which are given by

$$\begin{aligned}
\tau\big|_{\text{wedge}} &= 0, \quad r = R_1, \quad 0 \le \theta \le \alpha,\\
w^T\big|_{\text{plate}} &= w^{\text{tr}}, \quad r = R_2, \quad 0 \le \theta \le \alpha,\\
\tau^T\big|_{\text{plate}} &= \tau^{\text{tr}}, \quad r = R_2, \quad 0 \le \theta \le \alpha,
\end{aligned}$$

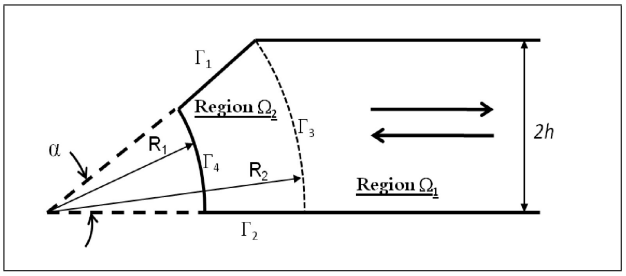

Figure 1. The plate divided into two sub-domains (Ω_1 and Ω_2) having free traction boundaries Γ_1, Γ_2 and Γ_4 in addition to the fictitious boundary Γ_3.

where w^{tr} is the displacement field in the wedge.

The stress field, τ^T, is given by

$$\tau^T\big|_{\text{plate}} = (\tau^{\text{inc}} + \tau^{\text{ref}})\big|_{\text{plate}}.$$

The stress, τ, takes the form

$$\tau = \tau_{xz} n_x + \tau_{yz} n_y,$$

where n_x and n_y are the direction cosine of the unit normal vector n along the boundary Γ_3. The above three boundary conditions lead to the following three equations.

$$\sum_{n=0}^{N} \cos\left(\frac{n\pi\theta}{\alpha}\right)\left[C_n J'\left(\frac{n\pi}{\alpha}, \frac{\omega r}{c_s}\right) + D_n Y'\left(\frac{n\pi}{\alpha}, \frac{\omega r}{c_s}\right)\right]\Big|_{r=R_1} = 0, \tag{5}$$

$$\left[A_{\text{inc}} \cos\left(\frac{j\pi}{2h} y\right) e^{-jk_x x} + \sum_{l=0}^{L} B_l \cos\left(\frac{l\pi}{2h} y\right) e^{jk_l x}\right]\Bigg|_{\substack{x=R_2\cos\theta \\ y=R_2\sin\theta}}$$

$$= \sum_{n=0}^{N} \cos\left(\frac{n\pi}{\alpha}\theta\right)\left[C_n J\left(\frac{n\pi}{\alpha}, \frac{\omega r}{c_s}\right) + D_n Y\left(\frac{n\pi}{\alpha}, \frac{\omega r}{c_s}\right)\right]\Big|_{r=R_2}, \tag{6}$$

$$\left\{A_{\text{inc}}\left[-jk_i \cos\left(\frac{j\pi}{2h} y\right) n_x - \frac{j\pi}{2h} \sin\left(\frac{j\pi}{2h} y\right) n_y\right] e^{-jk_l x} + \sum_{l=0}^{L}\left[jk_l \cos\left(\frac{l\pi}{2h} y\right) n_x - \frac{l\pi}{2h} \sin\left(\frac{l\pi}{2h} y\right) n_y\right] e^{jk_l x}\right\}\Bigg|_{\substack{x=R_2\cos\theta \\ y=R_2\sin\theta}}$$

$$= \sum_{n=0}^{N} \frac{\omega}{c_s} \cos\left(\frac{n\pi}{\alpha}\theta\right)\left[C_n J'\left(\frac{n\pi}{\alpha}, \frac{\omega r}{c_s}\right) + D_n Y'\left(\frac{n\pi}{\alpha}, \frac{\omega r}{c_s}\right)\right]\Big|_{r=R_2}. \tag{7}$$

J' and Y' mean d(..)/dr. By considering N ($L = N$) reflected modes and choosing weight functions of the form

$$f_w = \cos\left(\frac{m\pi}{\alpha}\theta\right), \quad m = 0, 1, 2, \ldots, N. \tag{8}$$

These three continuity equations can be integrated through the angle of the wedge after multiplying them with a set of weight functions. By this way, it leads to a set of algebraic equations ($3N$ equations) in which the coefficients, B_l, C_n and D_n can be determined after setting intensity of the incident wave, A_{inc}, to be one.

3. Results and discussion

Focus of this work is directed towards the wave scattering occurring when a traveling monochromatic wave of certain frequency arrives at the free inclined broken end of an elastic plate. A semi-infinite plate of a homogeneous, isotropic, linearly elastic material is considered. The elastic plate is made of stainless steel and has a density and a shear modulus of 7.8 g/cm^3 and $0.81 \cdot 10^{11}$ Pa, respectively.

A time-harmonic SH wave propagates in the plate and is incident upon the inclined end of the plate. Since the end of the plate is considered to be stress free, the wave scattering problem reduces to determining the formed reflected wave field. Only the traveling modes have the capability of carrying the incident wave energy back into the plate. These traveling modes are supplemented by a series of end modes, which are associated with purely imaginary wave numbers, in order to satisfy a traction-free end conditions. The number of modes, N, in the wave expansion function has been chosen to be 15 modes as to include all the propagating and non-propagating ones. The accuracy of the results is determined by studying the error norm in the energy balance.

In order to understand wave scattering, a single SH mode is incident on the free end of the plate and reflected SH modes are examined. The results are presented in terms of the reflected energy share of each reflected mode. Figure 2 shows the energy shares of scattered (reflected) modes due to the incidence of the zeroth symmetric mode at the free tapered end of an elastic plate. Modes are allowed to propagate at the normalized frequency ($\Omega = 2\omega h/\pi c$) of 1.5. The end of the plate is allowed to take various angles of inclination (from 15° to 90°). It is noted that both the zeroth symmetric mode and the first anti-symmetric mode are reflected. Mode conversion is observed to occur in plates with ends of various inclination angles. The most substantial mode conversion occurs in plates with ends having 60° to 70° angles of inclination. In a plate with an end inclined at 47°, only the symmetric mode is reflected, which indicated the formation of an elastic standing wave pattern. Figure 2 also shows energy shares of reflected modes from plate tapered broken tips with an r_1/r_2 ratio of 0.5. It is observed that the most substantial mode conversion occurs in plates with broken ends having 78° to 84° angles of inclination. In a plate with an end inclined at 20° and 53°, only the symmetric mode is reflected, indicating the formation of standing wave pattern. As a check for the accuracy of the analysis, energy conservation is preserved as the reflected energies sum to the incident energy.

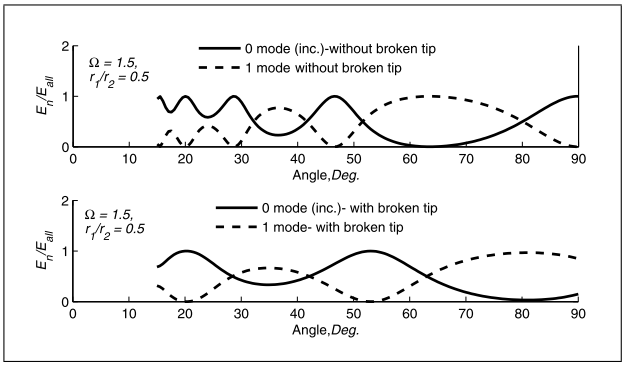

Figure 2. Energy shares of scattered (reflected) modes due to the incidence of the zeroth mode at inclined edges (Top); and broken inclined edges (bottom) of a plate for various angles of inclination.

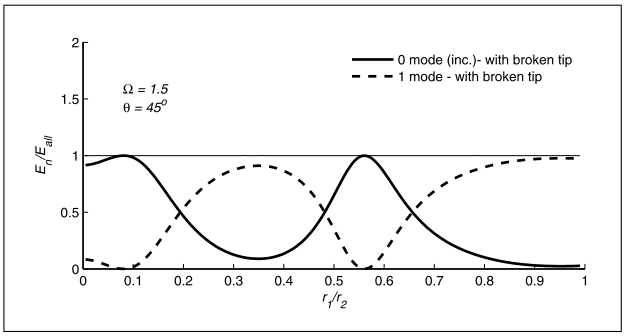

Figure 3. Energy shares of scattered (reflected) modes due to the incidence of the zeroth mode on 45° inclined broken edge of a plate with different sizes of broken parts.

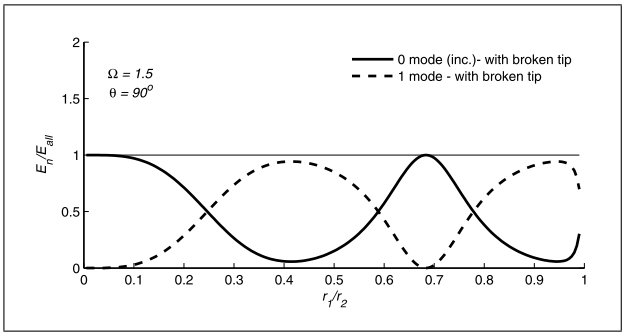

Figure 4. Energy shares of scattered (reflected) modes due to the incidence of the zeroth mode on 90° straight edge of a plate with different sizes of broken parts.

To investigate the effect of the size of the broken tip of elastic plates on the scattered (reflected) field, Figures 3 shows energy shares of scattered (reflected) modes due to the incidence of the zeroth mode on 45° inclined broken edge of a plate with different sizes of broken parts. The broken part of the inclined free edge is allowed to take a variety of sizes expressed in terms of the r_1/r_2 ratio and ranges from 0.01 to 0.99. Single mode reflection occurs when $r_1/r_2 = 0.08$ and 0.56; and substantial mode conversion occurs when r_1/r_2 is in the range 0.3–0.4 or above 0.92. Figure 4. shows energy shares of scattered (reflected) modes due to the incidence of the zeroth mode on a verti-

cally straight edge of a plate having a broken corner with different sizes of broken parts. Single mode reflection occurs when r_1/r_2 is less than 0.07 or at 0.68; and substantial mode conversion occurs when r_1/r_2 is in the range 0.39-0.43 or at 0.96. These findings suggest a potential quantitative rather than qualitative characterization of the inclines edges of the plate and the evaluation of the size of broken free edges.

The existence of broken tip at the end of tapered plate has significantly altered the scattering characteristics as presented in terms of energy shares of scattered modes. The size of the broken part was also found to influence mode scattering. Therefore, the presented study provided wave scattering information at three levels of investigation; i) when the end of a plate was tapered rather than straight, ii) if the tapered end of a plate was broken or intact, and iii) how significant the effect of the broken part size was.

4. Conclusion

Elastic wave scattering at the free tapered end of an elastic plate due to an incident monochromatic wave was investigated. The tapered end may have a truncation to represent a broken tip. The governing equations, based on a semi-analytical model, were solved to satisfy traction-free end conditions to determine all propagating waves and end modes for the plate. The amplitudes of the traveling waves and end modes were the basis for representing the wave reflection phenomenon at the free end. Detailed numerical illustrations were given for scattering occurring during the interactions of incident SH modes with inclined free edge having various inclination angles and different sizes of broken tips at selected frequencies. A significant difference was noted in the nature of mode conversion and the formation of standing wave patterns. Hence, under an experimental setup, one should not only predict if the tip of the plate is broken or not, but also one should provide an estimation of the size of the broken tip from reading the scattering spectrum. A future improvement of this study includes the consideration of broken tip with irregular geometry resembling realistic fracture lines. The study also needs to be extended for the case of Lamb wave scattering due to its frequent use in the field of nondestructive evaluation.

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