

To the Theory of Reflection of the Directed Sound Beams in Classical Schoch Experiments

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The new solution of the problem of the directed sound beams reflection at the interface between liquid half-space and solid half-space is analyzed in the report. This solution is received in a class of the generalized functions, which are the conjugate solutions of the appropriate not self-conjugate boundary problem. It is marked, that reflectance of spherical components of the total solution in the field of angles of complete internal reflection is less than one. The reflectance takes the minimal value when falling angle corresponds to generation of Rayleigh wave. The analytical expression for shift of the directed sound beams is received for reflection, which angular dependence precisely corresponds to observed dependence in classical Schoch experiments.

1 Introduction

In 1627 the Dutch scientist Snell and irrespective of him the French mathematicians Descartes formulated one of the first laws of the theory of wave processes occurring at the interface between two media. It is known as the law of reflection and refraction of light and sound beams. In essence the law postulates synphasy of distribution of falling, reflected and refracted waves along the border of the interface between two media. Self-evidence of the law appeared deceptive but the phenomenon of total internal reflection following from it is so fundamental that discussions around it have been going on since Newton's times. But until recently these discussions have been limited to optical phenomena only [1], [2].

A characteristic feature of the Snell - Descartes law is shown at total reflection of a falling sound wave from the interface between two media when power flow through the interface stops completely. The refracted wave becomes heterogeneous and corresponds to reactive inertial input impedance of the lower half-space and its amplitude decreases exponentially with depth. Incorrectness of that solution lies in that the heterogeneous wave transferring nonzero flow of power in the direction of propagation has no source of energy in the lower half-space and cannot receive necessary power through the mechanism of refraction from a source located in the upper half-space.

Understanding that incorrectness Newton offered the first and sole hypothesis about total internal reflection with the corresponding shift of a beam along the interface. The shift of the limited sound beams at their reflection in the field of angles of total internal reflection got the name of the Goos - Hanchen effect. Experimental observing of that shift in the classical Schoch experiments [3] seemed to have fully confirmed Newton's hypothesis but only in the part concerning geometrical distortions of spatial structure of a sound beam by horizontal shift. The question

about the source of energy transferred by a heterogeneous wave in the lower half-space remains unsolved as the refraction mechanism cannot be physically realized in homogeneous medium. Besides angular dependence of shift of the directed sound beams at reflection in the Schoch experiments appeared rather specific and has had neither physical nor mathematical substantiation until now.

The report presents the correct statement and solution of a problem on reflection of arbitrary sound beams applying the conjugated decisions of an initial non-conjugated problem.

2 Theory

Let's consider reflection of a spherical wave from the impedance interface between two media. The sound wave falls from the upper half-space which is characterized by density ρ_1 and sound velocity c_1 . The lower half-space has shift elasticity. Its density is ρ_2 , a longitudinal wave velocity is c_L and a shear wave velocity is c_T . Scalar potential of the sound field in the upper medium can be presented in the form of the following improper integral [4]:

$$\varphi_1(r, z) = \int_0^{\infty} \tilde{\varphi}_1(\xi, z) J_0(\xi r) \xi d\xi, \quad (1)$$

where $\tilde{\varphi}_1(\xi, z)$ in its turn is the solution of the problem for the cross operator with one impedance boundary condition

$$\frac{\partial^2 \tilde{\varphi}_1(\xi, z)}{\partial z^2} + (k_1^2 - \xi^2) \tilde{\varphi}_1(\xi, z) = -\delta(z - z_0); \quad z \geq 0, \quad (2)$$

$$z = 0; \quad p_1 + Z_{in} \frac{\partial \varphi_1}{\partial z} = 0, \quad (3)$$

$$Z_{in} = \frac{\rho_2 c_L}{\cos \theta} \cos^2(2\gamma) + \frac{\rho_2 c_t}{\cos \gamma} \sin^2(2\gamma),$$

where p_1 is sound pressure in the upper medium, θ is refraction angle for a longitudinal component and γ is refraction angle for a shear component, z_0 is vertical coordinate of the sound source, Z_{in} is input impedance of the lower half-space.

Let's write the solution of problem (2) - (3) for the reflected wave:

$$\varphi_{refl}(r, z) = -\frac{i}{2} \int_{-\infty}^{\infty} V(\xi) e^{-ik_{31}(z+z_0)} H_0^{(2)}(\xi r) \frac{\xi d\xi}{k_{31}}, \quad (4)$$

where V is a reflection coefficient for the liquid - solid body interface.

$$V = \frac{k_{31}[k_{3t} \cos^2(2\gamma) + k_{3L} \sin^2(2\gamma)] - \rho_{12} k_{3L} k_{3t}}{k_{31}[k_{3t} \cos^2(2\gamma) + k_{3L} \sin^2(2\gamma)] + \rho_{12} k_{3L} k_{3t}}, \quad (5)$$

$$k_{31} = k_1 \cos \theta_1, \quad k_{3t} = k_1 \sqrt{c_{1L}^2 - \sin^2 \theta_1},$$

$$k_{3L} = k_1 \sqrt{c_{1t}^2 - \sin^2 \theta_1}, \quad \xi = k_1 \sin \theta_1, \quad \theta_1 - \text{falling angle,}$$

$$\cos(2\gamma) = 1 - \frac{2 \sin^2 \theta_1}{c_{1t}^2}, \quad \sin(2\gamma) = \frac{2 \sin \theta_1 \sqrt{c_{1t}^2 - \sin^2 \theta_1}}{c_{1t}^2},$$

$$c_{1L} = \frac{c_1}{c_L}, \quad c_{1t} = \frac{c_1}{c_t}.$$

Considering the asymptotic expression for the Hankel function we shall present integral (4) as follows:

$$\varphi_{refl}(r, z) = (i2\pi r)^{-1/2} \int_{-\infty}^{\infty} V(\xi) e^{-i\Phi_1(\xi)} \frac{\sqrt{\xi} d\xi}{k_{31}}, \quad (6)$$

$$\Phi_1(\xi) = \xi r + k_{31}(z + z_0),$$

To construct a solution for the field of reflected waves as a cut eliminating indeterminacy of radical k_{31} we shall choose the necessary condition for existence of integral (6):

$$\text{Im} \Phi_1(\xi) \leq 0, \quad (7)$$

and the cuts eliminating indeterminacy of radicals k_{31} and k_{3t} are defined in the radiation condition

$$\text{Re} k_{3L} \geq 0, \quad \text{Re} k_{3t} \geq 0. \quad (8)$$

The radiation conditions in the form (8) are a mathematical expression of the principle of causality according to which power flow must be directed from a source of energy to loading which role plays the lower half-space, i.e. from cause to effect, not vice versa. Indeed the substantial part of input impedance of the lower half-space is positively the certain value according to (8) whereas the reactive part of input impedance can have any sign in full conformity with the principle of equal realizability of reactive impedances of two types.

There are specific problems connected with construction of the solution corresponding to conditions (8) resulting from the fact that the sound field of the lower half-space in the field of total internal reflection angles is described by inhomogeneous waves. The amplitude of inhomogeneous waves either decreases exponentially for the regular components, or increases exponentially for the generalized components at removal from the interface. Probably the first time that solutions of this type were considered theoretically was at the analysis of roots of the characteristic equation describing the Sholte waves of a regular or generalized type at the interface between the liquid and solid half-spaces [5].

Use of components with the exponentially increasing amplitude in the total solution has no physical restrictions if generalized functions (according to L.G. Sobolev) with the finitary range of definition are used for their description. However this question was solved extremely gracefully in practice in the work of French acousticians [6], in which such waves were found experimentally at the water - plexiglas interface and the water - polyvinyl chloride interface.

In the classical solution of the boundary problem on reflection [4] the cuts and the top sheets of the Riemann surface are defined by the following conditions:

$$\text{Im} k_{3L} \leq 0, \quad \text{Im} k_{3t} \leq 0.$$

However this solution satisfies neither the principle of causality nor the principle of equal realizability of two reactive input impedances of the inertial and elastic character. These two impedances are realized by inhomogeneous waves of the lower half-space with the exponentially decreasing amplitude and the exponentially increasing amplitude. A reflection coefficient received from the solution which does not contain generalized components is a reflection coefficient of a plane wave $V_{pl} = V$. In contrast to it the reflection coefficient of a spherical wave V_{sph} received with allowing for conditions (7) and (8) will be defined by a linear combination of reflection coefficients for the conjugated solutions, which in the aggregate provide for the field continuity in terms of normal velocities and stress at the interface:

$$V_{sph} = \begin{cases} V_{pl} = V_{pl}(k_{3L}, k_{3t}); & \theta_1 \leq \theta_{cr,1} = \arcsin c_{1L} \\ \frac{3}{4} V_{pl} + \frac{1}{4} V_{ns}^*; & \theta_{cr,1} \leq \theta_1 \leq \theta_{cr,2} = \arcsin c_{1t} \\ \frac{5}{8} V_{pl} + \frac{1}{8} (V_L + V_t + V_{Lt}); & \theta_{cr,2} \leq \theta_1 \leq \frac{\pi}{2} \end{cases} \quad (9)$$

where $V_L = V_L(k_{3L}^*, k_{3t})$ is a coefficient of reflection from the half-space where a longitudinal wave is generalized and a shear wave is regular,

$V_t = V_t(k_{3t}^*, k_{3L})$ is a coefficient of reflection from the half-space where a shear wave is generalized and a longitudinal wave is regular, $V_{Lt} = V_{Lt}(k_{3L}^*, k_{3t}^*)$ is a coefficient of reflection from the half-space where both longitudinal and shear waves are generalized.

The spherical component in the field of reflected waves is described in the following canonical expression

$$\varphi_{refl}(r, z) \approx \frac{V_{sph}}{R} e^{-ik_t R}, \quad R^2 = r^2 + (z + z_0)^2,$$

$$V_{sph}(\theta_{10}) = |V_{sph}(\theta_{10})| e^{i\psi_{sph}(\theta_{10})}, \quad \theta_{10} = \arctg \frac{r}{z + z_0}.$$

The solution of a more complicated problem on reflection of the directed sound beams from the impedance interface in fact comes to definition of horizontal shift of a beam at reflection and explanation of its specific angular dependence.

$$\Delta_{sph}(\theta_{10}) = \left. \frac{\partial \psi_{sph}(\xi)}{\partial \xi} \right|_{\xi = k_t \sin \theta_{10}},$$

where θ_{10} is falling angle of the directed beam.

3 Results

Angular dependences of reflection coefficients V_{pl} and V_{sph} were calculated for the water – solid half-space interface with various characteristics of the solid half-space. In Fig. 1a, b reflection coefficients are presented for a case when the solid half-space corresponds to the sea sediment layer with the following parameters: $\rho_{12} = 1/1.6$, $c_{1L} = 1.5/1.75$, $c_{1t} = 15$, ($c_t = 100$ m/c) and $c_{1t} = 3$, ($c_t = 500$ m/c).

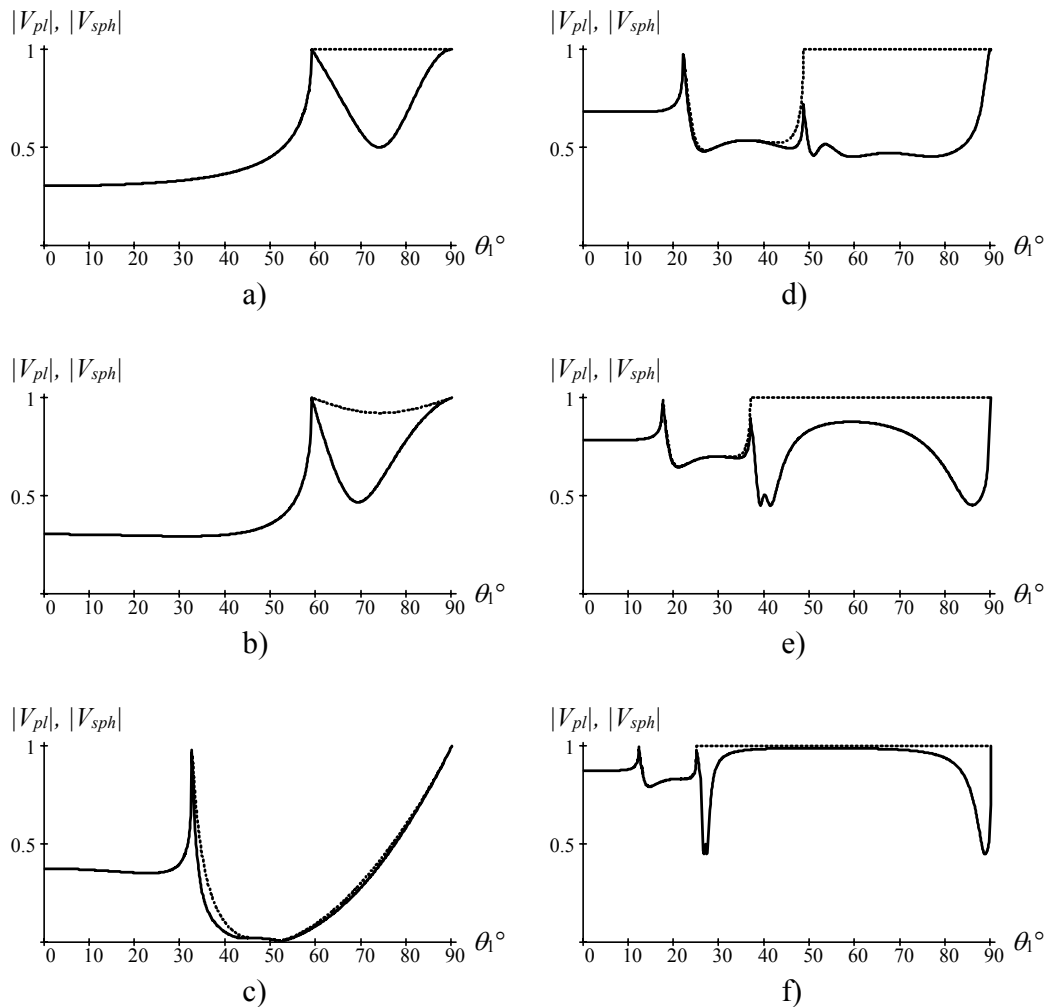


Fig. 1. Angular dependence of reflection coefficients (continuous line) and (dotted) for liquid half-space – solid half-space interface: a), b) water – sea sediment layer; c) water – plexiglas; d), e) water – basalt; f) xylol – aluminium.

In this case shift elasticity of the sea sediment layer practically has no influence on angular dependence of reflection coefficients and the sediment layer itself with low shift elasticity can be perfectly considered equivalent liquid.

Fig. 1c explains angular dependence of reflection coefficients at the water – plexiglas interface ($\rho_{12} = 1/1.18$, $c_{1L} = 1.48/2.75$, $c_{1r} = 1.48/1.39$) which turns out very transparent for the angles which are larger than the first critical angle. In all the three considered cases speed of a shear wave satisfied conditions $c_i \leq c_1 \leq c_L$. In such media in particular surface waves of the generalized type were found experimentally in the work [6].

Fig. 1d - f explain angular dependence of reflection coefficients in the case of the half-space rigid enough for which conditions $c_1 \leq c_i \leq c_L$ are satisfied. The basic difference in behaviour of reflection coefficients V_{sph} , V_{pl} can be noted in the field of angles exceeding the second critical angle, and Fig. 1d, e correspond to sea bottom rock bed such as basalt with the following parameters: $\rho_{12} = 1/2$, $c_{1L} = 1.5/4$, $c_{1r} = 2c_{1L}$ and $\rho_{12} = 1/2.5$, $c_{1L} = 1.5/5$, $c_{1r} = 2c_{1L}$ accordingly.

Fig. 1f corresponds to the xylol - aluminium interface ($\rho_{12} = 0.86/2.7$, $c_{1L} = 1.35/6.4$, $c_{1r} = 2c_{1L}$) in the classical Schoch experiments [3] on experimental supervision of shift of the directed sound beams at total internal reflection.

It follows from the Schoch experiment that shift of the directed sound beams at reflection appears only at the falling angle corresponding to generation of the Rayleigh wave ($\theta_1 = 27^\circ$), and does not take place at smaller angles including the second critical angle or at large angles. It is perfectly obvious that the shift of the directed sound beam occurs exactly because the falling angle $\theta_1 = 27^\circ$ in the summary field presented in all conjugated solutions has no total internal reflection.

For this angle reflection coefficient V_{sph} has a global minimum ensuring power transparency of the interface and transport of the sound beam energy along the interface in the lower half-space by the value of spatial shift. Let's note thus, that complexity of the reflection coefficient is a necessary, but not a sufficient condition for realization of spatial shift of the directed sound beams at reflection.

The necessary and sufficient conditions for such a shift appear to be satisfied only if there exists a window of power transparency of the interface within a range of angles of total internal reflection that is provided by the interference mechanism if the summary field comprises all possible components including generalized ones. All earlier theories which are briefly reviewed in the work [7] explain quite well the value of the Goos -

Hanchen shift that is proven experimentally in the cases when the shift itself is observed. However none of these theories explains the evident selectivity of the shift from falling angle and impossibility of its realization in the whole range of angles of total internal reflection as it follows from the Schoch experiment.

For the required correspondence we shall enter the effective value of the shift for a voluntary beam of sound waves $\Delta_E(\xi)$ set by the angular spectrum $\Phi(\xi)$ as a formula of power averaging

$$\Delta_E(\xi) = \frac{\int_{k_{sp}}^{k_i} |\Phi(\xi)|^2 D_E(\xi) \Delta_{sph}(\xi) d\xi}{\int_{k_{sp}}^{k_i} |\Phi(\xi)|^2 d\xi}, \quad (10)$$

where $\Delta_{sph}(\xi) = \frac{\partial \psi_{sph}}{\partial \xi}$ is the value of the shift in the

classical geometry-optical definition expressed through a derivative from a phase of reflection coefficient $V_{sph}(\xi)$,

$$D_E(\xi) = \left(1 - |V_{sph}(\xi)|^2\right), \quad V_{sph}(\xi) = |V_{sph}(\xi)| e^{i\psi_{sph}(\xi)},$$

$$\xi = k_1 \sin \theta_1, \quad \theta_1 \in (\theta_{1,cr}, \frac{\pi}{2}),$$

$k_{cr} = k_2$ for the interface between two liquid media; $k_{cr} = k_1$ for the interface between the liquid half-space and the solid half-space, $V_{sph}(\xi)$ is an effective reflection coefficient of a spherical wave determined for all types of waves including regular and generalized waves; $D_E(\xi)$ is a transparency coefficient in terms of the interface energy.

For the well directed sound beam falling at the interface under angle θ_{10} in the range of overcritical falling angles, formula (10) is simplified and will be transformed into the following form

$$\Delta_E(\theta_{10}) = \Delta_{sph}(\theta_{10}) D_E(\theta_{10}). \quad (11)$$

Formula (11) together with Fig. 1f explains completely all the features of the Schoch experimental results, which are cited practically in all monographs devoted to the Goos - Hanchen effect and its experimental supervision in acoustics.

Numerical estimations of the value of the normalized shift of the directed sound beam Δ_E / λ_1 by formulas (9), (11) are shown in Fig. 2 (curve 3). For comparison the same Figure shows the value of the normalized shift calculated by the classical formula from work [4], (curve 1), and the value of the normalized shift (curve 2) is calculated through a phase of coefficient $V_{sph}(\xi)$:

$$\Delta_{sph} = \frac{\partial \psi_{sph}}{\partial \xi}, \quad V_{sph}(\theta_1) = |V_{sph}(\theta_1)| e^{i\psi_{sph}(\theta_1)},$$

where a reflection coefficient $V_{sph}(\xi)$ is determined by formula (9).

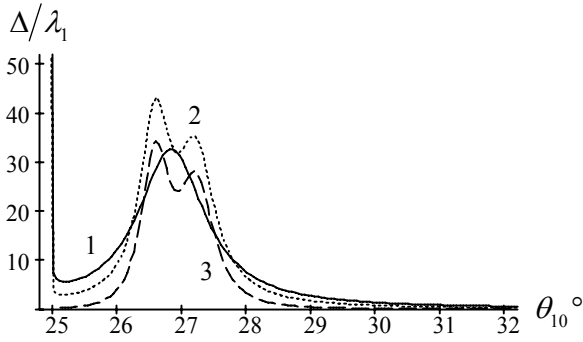


Fig. 2. Angular dependence of the Goos - Hanchen shift for the xylol – aluminium interface:

$$1 - \Delta_{pl} = \frac{\partial \psi_{pl}}{\partial \xi}; \quad 2 - \Delta_{sph} = \frac{\partial \psi_{sph}}{\partial \xi}; \quad 3 - \Delta_E = \frac{\partial \psi_{sph}}{\partial \xi} D_E.$$

For the interface of water – sea sediment layer as well as for the interface between two liquid media with total internal reflection in view of behaviour of the effective reflection coefficient $V_{sph}(\xi)$ formula (11) is automatically followed by a conclusion about absence of any shift of the directed beam for the critical falling angle $\theta_{10} = \theta_{1,cr}$ and for the grazing fall $\theta_{10} \rightarrow \frac{\pi}{2}$. In this case angular dependence of value $\Delta_E(\theta_{10})$ is illustrated in fig. 3 (curve 3).

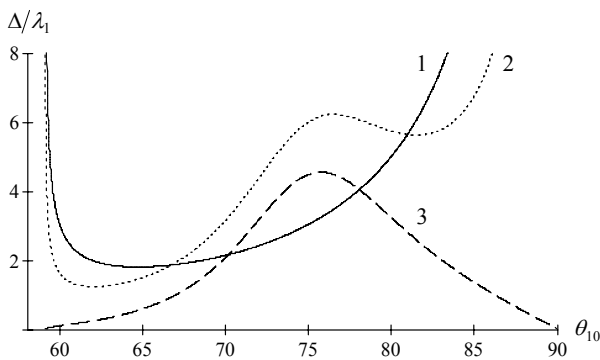


Fig. 3. Angular dependence of the Goos - Hanchen shift for the two liquid media interface:

$$1 - \Delta_{pl} = \frac{\partial \psi_{pl}}{\partial \xi}; \quad 2 - \Delta_{sph} = \frac{\partial \psi_{sph}}{\partial \xi}; \quad 3 - \Delta_E = \Delta_{sph} D_E.$$

The maximal value of the shift corresponds to the angle of minimum of a reflection coefficient (maximum of power transparency of the interface). The shift itself is quite small ($\Delta_E \approx 4.5\lambda_1$) in comparison with the shift of a beam at the xylol – aluminium interface observed in the Schoch experiment ($\Delta_E \approx 33.4\lambda_1$). Curves (1) and (2) in the figure correspond to the classical shift calculated through a phase of reflection coefficients $V_{pl}(\xi)$ and $V_{sph}(\xi)$ accordingly.

This conclusion is evidently at variance with the estimations of spatial shift produced on the basis of classical representations about complete reflection as well as with the results of the specified theoretical estimations given, for example, in work [7]. However these estimations have no experimental confirmation. Therefore, giving preference to the Schoch experimental results, it is necessary to note splendid conformity to these results of the description (10), (11) received using conjugated solutions of the initial not self-conjugated problem and generalized functions corresponding to them.

4 Summary

Let's sum up the comparison of the theoretical and experimental researches of the phenomenon of total internal reflection with shift, having noted the following features.

- The reflection coefficient of a spherical wave determined for the total field using conjugated solutions of the initial not self-conjugated problem differs from the reflection coefficient of a plane wave in the classical solution in the field of angles of total internal reflection ($V_{sph} \neq V_{pl}$).
- Reflection coefficient V_{sph} in the field of angles of total internal reflection is less than one because of interference interaction of waves of the conjugated solutions with an estimation $|V_{sph}|_{min} = 0.5$.
- Power transparency of the interface in the field of angles of total internal reflection is a necessary condition for existence of the beam shift at reflection.
- Power transparency of the water - sea sediment layer interface, maximal in the range of grazing angles 12–15°, provides for abnormally large illumination of the bottom half-space [8] at small grazing angles.

References

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