Reflection and transmission by randomly spaced cylindrical inclusions in a solid slab-like region: Twersky's theory

Sébastien Robert, Jean-Marc Conoir
Laboratoire d'Acoustique Ultrasonore et d'Electronique, LAUE, UMR CNRS 6068, Université du Havre, Place Robert Schuman, 76610 Le Havre, France, **{sebastien.robert , jean-marc.conoir}@univ-lehavre.fr**

V. Twersky developed in the sixties a theory describing the reflection and transmission process of scalar plane waves from slab-like regions of randomly distributed scatterers. The theory, which is based on Foldy's approximation and neglects "hole corrections", is an alternative approach to an exact calculation of multiple scattering that fails when the number of scatterers is important. Twerky's theory furnishes the properties of the coherent plane wave propagating inside the slab. From the point of view of the coherent wave, the slab may be described as an effective fluid medium; its reflection and transmission coefficients may be formally written as those of a fluid plate. In this work, Twersky's theory is extended to random distributions of cylindrical inclusions in elastic media. Before examining the reflection and transmission process of such multiple scattering media, our first objective is to theoretically validate the results predicted by the theory. With this aim in view, as an exact calculation of multiple scattering fails for purely random distributions, the idea consists in adapting Twersky's theory to distributions of scatterers periodic in one direction and random in the other one. Indeed, the reflection and transmission coefficients of such half-periodic media can be calculated by using a method originally developed for purely periodic media, i.e. phononic crystals. In our case, it consists in considering each half-periodic medium as a random distribution of linear periodic arrays of scatterers. This method furnishes exact reflection and transmission coefficients which can be successfully compared with those predicted by Twersky's theory.

1 Introduction

The problem of the multiple scattering of waves by a random distribution of scatterers is a subject receiving a great deal of attention since Foldy's work [1] in the forties. Depending on the concentration of scatterers, this problem has been treated following two main approaches. When the concentration is relatively low, the random multiple scattering medium can be characterized by the properties of the coherent wave, the "mean wave", propagating in the effective medium [2, 3, 4]. In such a case, for a random distribution of elastic shells in water bounded by two parallel planes (a "slab region"), a recent work [5] shows that the reflection and transmission coefficients of the effective medium may be formally written as those of a fluid plate. In the second approach, i.e. when the concentration of scatterers gets too high for a coherent wave to propagate in the effective medium, the propagation of the incoherent intensity is generally studied [6].

We consider here only the first approach. In this context, the model that has been chosen is Twersky's one [4] originally developed for the multiple scattering of scalar waves by slab-like regions of randomly distributed identical scatterers. The theory is extended here to the multiple scattering of elastic waves, longitudinal L and transverse T waves, by cylindrical inclusions in an elastic matrix. Before examining the reflection and transmission process of such multiple scattering media, our main objective is to validate the results given by Twersky's theory with an exact calculation of multiple scattering. This theory, based on Foldy's approximation, gives indeed approximate results, which have never been verified using another approach. With this validation in view, the choice of Twersky's theory is particularly interesting among all existing theories: it can be adapted to half-random media, i.e. to random distributions of linear periodic arrays of scatterers; and, contrary to purely random media, the coherent waves reflected and transmitted by a half-random medium can be obtained with an exact calculation of multiple scattering based on modal theories [7, 8]. Thus, the approximate reflection and transmission coefficients of the effective media obtained by adapting Twersky's theory to half-random media can be compared with those calculated in the exact way.

In the following, Section 2 presents Twersky's theory which is directly adapted to our problem, i.e. a half-random medium of cylindrical inclusions in an elastic matrix. Section 3 briefly describes the exact method of calculation mentioned above and, then, presents a comparison of the results obtained with both methods.

2 Twersky's theory

2.1 Presentation of the problem

The half-random medium is presented in Figure 1. It is composed of a random distribution of N linear, periodic and infinite arrays of cylindrical inclusions in an elastic medium. The thickness of the "slab" is denoted by *e*. The inclusions are identical, with radii *a*, and are infinitely long in the direction perpendicular to the *x*O*y* plane. As only plane waves having a wave front perpendicular to plane *x*O*y* are considered, the problem then is reduced to this plane. We suppose a time harmonic dependence $e^{-i\omega t}$ which will be omitted in the following and, for convenience, we shall consider only the case of a longitudinal incident plane wave. Let φ_{inc} be the displacement potential of this wave propagating in the direction given by unit vector $\hat{i}_L$; α_L is the incidence angle as described below.

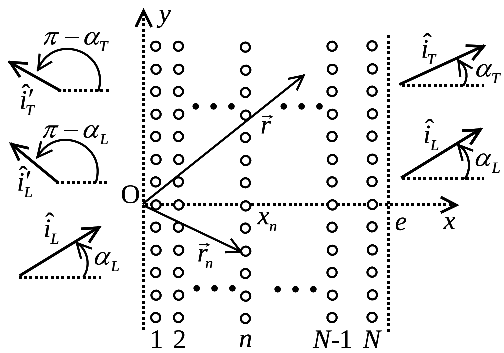

Figure 1: Slab-like region of thickness *e* containing a random distribution of *N* linear arrays. Vectors $\vec{r}$ and $\vec{r}_n$ respectively label an observation point and a point of an array *n*, the position of which is given by x_n.

The multiple scattering by a single linear array was studied in a previous paper [7]. It was shown that such a "scatterer" is characterized by a cut-off frequency under which the reflection and transmission process of an incident plane wave is formally identical to that of a plane interface; it is characterized by reflection coefficients r^{LL}, r^{LT}, r^{TL}, r^{TT}, and transmission coefficients t^{LL}, t^{LT}, t^{TL}, t^{TT}, which may be calculated as part of a modal theory of multiple scattering [7, 8]. For a random distribution of *N* arrays submitted to a *L* plane wave as described in Figure 1, the half-random medium transmits under the cut-off frequency only a *L* plane wave and a *T* one, respectively in the directions given by unit vectors $\hat{i}_L$ and $\hat{i}_T$; associated transmission angles α_L and α_T obey Snell's law. As for the reflected waves, they propagate in the directions given by unit vectors $\hat{i}'_L$ and $\hat{i}'_T$; the associated reflection angles corresponding to $\pi - \alpha_L$ and $\pi - \alpha_T$.

From now on, each linear array *n* will be considered as only one "scatterer" *n* located by abscissa x_n.

2.2 Foldy-Twersky's equations

Inside the slab, at a point located by $\vec{r}$, the scalar displacement potential $\phi(\vec{r})$ and the only non-zero component $\psi(\vec{r})$ of the vector displacement potential result from a multiple scattering process, which may be writ-ten as follows

$$\phi(\vec{r}) = \varphi_{inc}(\vec{r}) + \sum_n \left[\hat{T}_n^{LL}\phi^e(\vec{r};\vec{r}_n) + \hat{T}_n^{TL}\psi^e(\vec{r};\vec{r}_n)\right], \quad (1)$$

$$\psi(\vec{r}) = \sum_n \left[\hat{T}_n^{TT}\psi^e(\vec{r};\vec{r}_n) + \hat{T}_n^{LT}\phi^e(\vec{r};\vec{r}_n)\right], \quad (2)$$

where $\varphi_{inc} = e^{i\vec{k}_L\cdot\vec{r}}$. (1) represents the *L* field as the sum of the incident *L* wave and the *L* fields scattered by all scatterers *n* ($1 \le n \le N$). (2) represents the sum of the *T* fields scattered by all scatterers *n*. Fields $\phi^e(\vec{r};\vec{r}_n)$ and $\psi^e(\vec{r};\vec{r}_n)$, often called effective fields, are respectively the *L* and *T* fields resulting from multiple scattering processes, which are incident on a scatterer *n*. So, (1) and (2) implicitly take into account multiple scattering processes. Each transition operator $\hat{T}_n^{l_1l_2}$ characterizes a $l_1 \to l_2$ scattering process by scatterer *n* (l_1 and l_2 stand for *L* or *T*). For the moment, it is not useful to define all transition operators since, in the following, we shall be interested in transition operators acting on mean fields, i.e. coherent fields, obtained by *configurational averages*.

Now, to find the equations satisfied by the coherent fields denoted by $\langle\phi(\vec{r})\rangle$ and $\langle\psi(\vec{r})\rangle$, let us introduce a spatial probability density $p(x_1, x_2\ldots, x_N)$ of finding a scatterer 1 at x_1, a scatterer 2 at x_2, and so forth. The coherent fields satisfy Equations (1) and (2) averaged over all possible positions of the *N* scatterers *n*; this expresses as Equations (3) and (4) below:

$$\langle\phi(\vec{r})\rangle = e^{i\vec{k}_L\cdot\vec{r}} + \sum_n \int_0^e \left[\hat{T}_n^{LL}\phi_n^e + \hat{T}_n^{TL}\psi_n^e\right] p(x_1,\ldots,x_N)\,dx_1\ldots dx_N,$$

$$\langle\psi(\vec{r})\rangle = \sum_n \int_0^e \left[\hat{T}_n^{TT}\psi_n^e + \hat{T}_n^{LT}\phi_n^e\right] p(x_1,\ldots,x_N)\,dx_1\ldots dx_N,$$

with $\phi_n^e \triangleq \phi_n^e(\vec{r};\vec{r}_n)$ and $\psi_n^e \triangleq \psi_n^e(\vec{r};\vec{r}_n)$. Under no approximation, both relations may be put in the forms

$$\langle\phi(\vec{r})\rangle = e^{i\vec{k}_L\cdot\vec{r}} + \int_0^e \left[\hat{T}_n^{LL}\langle\phi_n^e\rangle_n + \hat{T}_n^{TL}\langle\psi_n^e\rangle_n\right]\overline{n}(x_n)\,dx_n, \quad (5)$$

$$\psi(\vec{r}) = \int_0^e \left[\hat{T}_n^{TT}\langle\psi_n^e\rangle_n + \hat{T}_n^{LT}\langle\phi_n^e\rangle_n\right]\overline{n}(x_n)\,dx_n, \quad (6)$$

where $\overline{n}(x_n) = Np(x_n)$. In (5) and (6), both quantities $\langle\phi_n^e\rangle_n$ and $\langle\psi_n^e\rangle_n$ are the effective fields acting on a scatterer averaged over all possible configurations of all the over scatterers. To obtain equations satisfied by the coherent fields, we need here an approximation, which was introduced by Foldy [1]. When *N* is large, it consists in replacing fields $\langle\phi_n^e\rangle_n$ and $\langle\psi_n^e\rangle_n$ respectively by $\phi_n \triangleq \langle\phi(\vec{r}_n)\rangle$ and $\psi_n \triangleq \langle\psi(\vec{r}_n)\rangle$: this is *Foldy's approximation*. We then obtain integral equations of the coherent fields:

$$\langle\phi(\vec{r})\rangle = e^{i\vec{k}_L\cdot\vec{r}} + \int_0^e \left[\hat{T}_n^{LL}\langle\phi_n\rangle + \hat{T}_n^{TL}\langle\psi_n\rangle\right]\overline{n}(x_n)\,dx_n, \quad (7)$$

$$\left\langle \psi\left(\vec{r}\right)\right\rangle = \int_0^e \left[\hat{T}_n^{TT}\left\langle \psi_n\right\rangle + \hat{T}_n^{LT}\left\langle \phi_n\right\rangle\right]\overline{n}\left(x_n\right)dx_n \,. \qquad (8)$$

These equations will be called in the following *Foldy-Twersky's equations*.

2.3 Properties of the coherent waves

We now suppose that the N scatterers, i.e. the N linear arrays, are uniformly distributed in the slab of thickness e. Foldy-Twersky's equations then become

$$\left\langle \phi\left(\vec{r}\right)\right\rangle = \mathrm{e}^{\mathrm{i}\vec{k}_L\cdot\vec{r}} + \overline{n}\int_0^e \left[\hat{T}_n^{LL}\left\langle \phi_n\right\rangle + \hat{T}_n^{TL}\left\langle \psi_n\right\rangle\right]dx_n \,, \quad (9)$$

$$\left\langle \psi\left(\vec{r}\right)\right\rangle = \overline{n}\int_0^e \left[\hat{T}_n^{TT}\left\langle \psi_n\right\rangle + \hat{T}_n^{LT}\left\langle \phi_n\right\rangle\right]dx_n \,, \quad (10)$$

with $\overline{n} = N/e$. With Figure 1, it is clear that a coherent field at $\vec{r} = x\hat{x} + y\hat{y}$ (or at $\vec{r}_n = x_n\hat{x} + y_n\hat{y}$) differs from those at $\vec{r} = x\hat{x}$ (or at $\vec{r}_n = x_n\hat{x}$) only by the phase factor introduced by the L wave incident on the slab, i.e.

$$\begin{aligned}\left\langle \phi\left(x\right)\right\rangle &= \mathrm{e}^{\mathrm{i}k_L x\cos\alpha_L} \\ &+ \overline{n}\,\mathrm{e}^{\mathrm{i}k_L\left(y_n - y\right)\sin\alpha_L}\int_0^e \left[\hat{T}_n^{LL}\left\langle \phi_n\right\rangle + \hat{T}_n^{TL}\left\langle \psi_n\right\rangle\right]dx_n \,,\end{aligned} \quad (11)$$

$$\left\langle \psi\left(x\right)\right\rangle = \overline{n}\,\mathrm{e}^{\mathrm{i}k_L\left(y_n - y\right)\sin\alpha_L}\int_0^e \left[\hat{T}_n^{TT}\left\langle \psi_n\right\rangle + \hat{T}_n^{LT}\left\langle \phi_n\right\rangle\right]dx_n \,, (12)$$

where $\phi_n \triangleq \phi\left(x_n\right)$ and $\psi_n \triangleq \psi\left(x_n\right)$. Until now, except for mode conversions, there is practically no difference with Twersky's work [4]. Actually, the main difference lies in the definition of the action of the transition operators on coherent fields since we do not consider the same scatterers. In our case, as the fields scattered by each scatterer are composed of transmitted and reflected plane waves satisfying Snell's Law, the action of a transition operator $\hat{T}_n^{TL}$ on a coherent field $\left\langle \psi\left(x_n\right)\right\rangle$, for instance, must be written as

$$\hat{T}_n^{TL}\left\langle \psi_n\right\rangle = \tilde{T}^{TL}\left(\hat{i}_T,\hat{i}_L\right)\left[\left\langle \psi_n\right\rangle\right]\mathrm{e}^{\mathrm{i}k_L\left[\left(y-y_n\right)\sin\alpha_L + \left(x-x_n\right)\cos\alpha_L\right]}$$

for a transmission $T{\to}L$, and for a reflection $T{\to}L$

$$\hat{T}_n^{TL}\left\langle \psi_n\right\rangle = \tilde{T}^{TL}\left(\hat{i}_T,\hat{i}'_L\right)\left[\left\langle \psi_n\right\rangle\right]\mathrm{e}^{\mathrm{i}k_L\left[\left(y-y_n\right)\sin\alpha_L - \left(x-x_n\right)\cos\alpha_L\right]},$$

remembering that all unit vectors, i.e. $\hat{i}_L$, $\hat{i}_T$, $\hat{i}'_L$ and $\hat{i}'_T$, are described in Figure 1. Developing the actions of all transition operators on all coherent fields in (11), this integral equation may be decomposed as

$$\left\langle \phi\left(x\right)\right\rangle = \phi_+\left(0,x\right) + \phi_-\left(x,e\right), \qquad (13)$$

with

$$\begin{aligned}\phi_+\left(0,x\right) &= \mathrm{e}^{\mathrm{i}\gamma_L x}\left[1 + \overline{n}\int_0^x \tilde{T}^{LL}\left(\hat{i}_L,\hat{i}_L\right)\left[\left\langle \phi_n\right\rangle\right]\mathrm{e}^{-\mathrm{i}\gamma_L x_n}dx_n\right] \\ &+ \mathrm{e}^{\mathrm{i}\gamma_L x}\,\overline{n}\int_0^x \tilde{T}^{TL}\left(\hat{i}_T,\hat{i}_L\right)\left[\left\langle \psi_n\right\rangle\right]\mathrm{e}^{-\mathrm{i}\gamma_L x_n}dx_n \,,\end{aligned} \quad (14)$$

$$\begin{aligned}\phi_-\left(x,e\right) &= \mathrm{e}^{-\mathrm{i}\gamma_L x}\,\overline{n}\int_x^e \tilde{T}^{LL}\left(\hat{i}_L,\hat{i}'_L\right)\left[\left\langle \phi_n\right\rangle\right]\mathrm{e}^{\mathrm{i}\gamma_L x_n}dx_n \\ &+ \mathrm{e}^{-\mathrm{i}\gamma_L x}\,\overline{n}\int_x^e \tilde{T}^{TL}\left(\hat{i}_T,\hat{i}'_L\right)\left[\left\langle \psi_n\right\rangle\right]\mathrm{e}^{\mathrm{i}\gamma_L x_n}dx_n \,,\end{aligned} \quad (15)$$

where $\gamma_L = k_L\cos\alpha_L$. In a similar way, Equation (12) may be written as

$$\left\langle \psi\left(x\right)\right\rangle = \psi_+\left(0,x\right) + \psi_-\left(x,e\right), \qquad (16)$$

with

$$\begin{aligned}\psi_+\left(0,x\right) &= \mathrm{e}^{\mathrm{i}\gamma_T x}\,\overline{n}\int_0^x \tilde{T}^{TT}\left(\hat{i}_T,\hat{i}_T\right)\left[\left\langle \psi_n\right\rangle\right]\mathrm{e}^{-\mathrm{i}\gamma_T x_n}dx_n \\ &+ \mathrm{e}^{\mathrm{i}\gamma_T x}\,\overline{n}\int_0^x \tilde{T}^{LT}\left(\hat{i}_L,\hat{i}_T\right)\left[\left\langle \phi_n\right\rangle\right]\mathrm{e}^{-\mathrm{i}\gamma_T x_n}dx_n \,,\end{aligned} \quad (17)$$

$$\begin{aligned}\psi_-\left(x,e\right) &= \mathrm{e}^{-\mathrm{i}\gamma_T x}\,\overline{n}\int_x^e \tilde{T}^{TT}\left(\hat{i}_T,\hat{i}'_T\right)\left[\left\langle \psi_n\right\rangle\right]\mathrm{e}^{\mathrm{i}\gamma_T x_n}dx_n \\ &+ \mathrm{e}^{-\mathrm{i}\gamma_T x}\,\overline{n}\int_x^e \tilde{T}^{LT}\left(\hat{i}_L,\hat{i}'_T\right)\left[\left\langle \phi_n\right\rangle\right]\mathrm{e}^{\mathrm{i}\gamma_T x_n}dx_n \,,\end{aligned} \quad (18)$$

where $\gamma_T = k_T\cos\alpha_T$. (13) and (14) show that each coherent field is the sum of two coherent fields: one propagates, with direction $\hat{i}_L$ or $\hat{i}_T$, i.e. in the direction of increasing x, and the other propagates in direction $\hat{i}'_L$ or $\hat{i}'_T$, i.e. in the direction of decreasing x. Now, to find the equations satisfied by the coherent fields $\phi_\pm$ and $\psi_\pm$, let us decompose $\left\langle \phi\left(x_n\right)\right\rangle$ and $\left\langle \psi\left(x_n\right)\right\rangle$ in (14), (15) and in (17), (18). This gives the following equations, respectively numbered (19), (20), (21) and (22):

$$\begin{aligned}\phi_+ &= \mathrm{e}^{\mathrm{i}\gamma_L x} \\ &+ \mathrm{e}^{\mathrm{i}\gamma_L x}\overline{n}\int_0^x\left\{\tilde{T}^{LL}\left(\hat{i}_L,\hat{i}_L\right)\left[\phi_+\right] + \tilde{T}^{LL}\left(\hat{i}_L,\hat{i}_L\right)\left[\phi_-\right]\right\}\mathrm{e}^{-\mathrm{i}\gamma_L x_n}dx_n \\ &+ \mathrm{e}^{\mathrm{i}\gamma_L x}\overline{n}\int_0^x\left\{\tilde{T}^{TL}\left(\hat{i}_T,\hat{i}_L\right)\left[\psi_+\right] + \tilde{T}^{TL}\left(\hat{i}_T,\hat{i}_L\right)\left[\psi_-\right]\right\}\mathrm{e}^{-\mathrm{i}\gamma_L x_n}dx_n,\end{aligned}$$

$$\begin{aligned}\phi_+ &= \mathrm{e}^{-\mathrm{i}\gamma_L x}\overline{n}\int_x^e\left\{\tilde{T}^{LL}\left(\hat{i}_L,\hat{i}'_L\right)\left[\phi_+\right] + \tilde{T}^{LL}\left(\hat{i}_L,\hat{i}'_L\right)\left[\phi_-\right]\right\}\mathrm{e}^{\mathrm{i}\gamma_L x_n}dx_n \\ &+ \mathrm{e}^{-\mathrm{i}\gamma_L x}\overline{n}\int_0^x\left\{\tilde{T}^{TL}\left(\hat{i}_T,\hat{i}'_L\right)\left[\psi_+\right] + \tilde{T}^{TL}\left(\hat{i}_T,\hat{i}'_L\right)\left[\psi_-\right]\right\}\mathrm{e}^{\mathrm{i}\gamma_L x_n}dx_n,\end{aligned}$$

$$\begin{aligned}\psi_+ &= \mathrm{e}^{\mathrm{i}\gamma_T x}\overline{n}\int_x^e\left\{\tilde{T}^{TT}\left(\hat{i}_T,\hat{i}_T\right)\left[\psi_+\right] + \tilde{T}^{TT}\left(\hat{i}_T,\hat{i}_T\right)\left[\psi_-\right]\right\}\mathrm{e}^{-\mathrm{i}\gamma_T x_n}dx_n \\ &+ \mathrm{e}^{\mathrm{i}\gamma_T x}\overline{n}\int_0^x\left\{\tilde{T}^{LT}\left(\hat{i}_L,\hat{i}_T\right)\left[\phi_+\right] + \tilde{T}^{LT}\left(\hat{i}_L,\hat{i}_T\right)\left[\phi_-\right]\right\}\mathrm{e}^{-\mathrm{i}\gamma_T x_n}dx_n,\end{aligned}$$

$$\begin{aligned}\psi_- &= \mathrm{e}^{-\mathrm{i}\gamma_T x}\overline{n}\int_x^e\left\{\tilde{T}^{TT}\left(\hat{i}_T,\hat{i}'_T\right)\left[\psi_+\right] + \tilde{T}^{TT}\left(\hat{i}_T,\hat{i}'_T\right)\left[\psi_-\right]\right\}\mathrm{e}^{-\mathrm{i}\gamma_T x_n}dx_n \\ &+ \mathrm{e}^{-\mathrm{i}\gamma_T x}\overline{n}\int_0^x\left\{\tilde{T}^{LT}\left(\hat{i}_L,\hat{i}'_T\right)\left[\phi_+\right] + \tilde{T}^{LT}\left(\hat{i}_L,\hat{i}'_T\right)\left[\phi_-\right]\right\}\mathrm{e}^{-\mathrm{i}\gamma_T x_n}dx_n.\end{aligned}$$

In these new integral equations, all transition operators now act on coherent fields $\phi_\pm$ and $\psi_\pm$. Let us define their action on these coherent fields, for instance in (19). A physical interpretation leads us to write

$$\tilde{T}^{LL}\left(\hat{i}_L,\hat{i}_L\right)\left[\phi_+\left(0,x_n\right)\right] = \tilde{T}_+^{LL}\phi_+\left(0,x_n\right), \qquad (23)$$

$$\tilde{T}^{LL}\left(\hat{i}_L,\hat{i}_L\right)\left[\phi_-\left(x_n,e\right)\right] = \tilde{R}_-^{LL}\phi_-\left(x_n,e\right), \qquad (24)$$

$$\tilde{T}^{LL}\left(\hat{i}_L,\hat{i}_T\right)\left[\psi_+\left(0,x_n\right)\right] = \tilde{T}_+^{TL}\psi_+\left(0,x_n\right), \qquad (25)$$

$$\tilde{T}^{TL}\left(\hat{i}_T,\hat{i}_L\right)\left[\psi_-\left(x_n,e\right)\right] = \tilde{R}_-^{TL}\psi_-\left(x_n,e\right). \qquad (26)$$

As the left hand side of Equation (19) is a L coherent wave $\phi_+\left(0,x\right)$ propagating in the direction of increasing x, and $\phi_+(0,x_n)$ is a L coherent wave incident on scatterer n propagating in the same direction, (23) necessarily represents a $L\to L$ forward scattering by scatterer n. Scatterer n is a linear array, it follows that $\tilde{T}_+^{LL}=t^{LL}-1$, remembering that t^{LL} is the $L\to L$ transmission coefficient of each linear array. Quantity $\phi_-(x_n,e)$ represents a L coherent wave propagating in the direction of decreasing x. Consequently, (24) represents a $L\to L$ backscattering by scatterer n and it follows that $\tilde{R}_-^{LL}=r^{LL}$, where r^{LL} is the $L\to L$ reflection coefficient of each linear array. (25) and (26) represent similar scattering processes but for mode conversions $T\to L$. Then, we can define $\tilde{T}_+^{TL}=t^{TL}$ and $\tilde{R}_-^{TL}=-r^{TL}$. As in the last relation, the minus sign is introduced each time a T (L) incident wave propagating in the direction of decreasing x is converted into a L (T) wave after scattering by scatterer n. The supercripts '+' and '–' at operator transitions index the direction of propagation of the coherent waves incident on a scatterer n.

Finally, defining the action of all transition operators in (19), (20), (21) and (22), we obtain the following system (27) of four coupled integral equations:

$$\phi_+=\mathrm{e}^{\mathrm{i}\gamma_L x}+\mathrm{e}^{\mathrm{i}\gamma_L x}\int_0^x\left[\bar{T}_+^{LL}\phi_++\bar{R}_-^{LL}\phi_-+\bar{T}_+^{TL}\psi_++\bar{R}_-^{TL}\psi_-\right]\mathrm{e}^{-\mathrm{i}\gamma_L x_n}dx_n,$$

$$\phi_-=\mathrm{e}^{-\mathrm{i}\gamma_L x}\int_x^e\left[\bar{R}_+^{LL}\phi_++\bar{T}_-^{LL}\phi_-+\bar{R}_-^{TL}\psi_++\bar{T}_-^{TL}\psi_-\right]\mathrm{e}^{\mathrm{i}\gamma_L x_n}dx_n,$$

$$\psi_+=\mathrm{e}^{\mathrm{i}\gamma_T x}\int_0^x\left[\bar{T}_+^{TT}\psi_++\bar{R}_-^{TT}\psi_-+\bar{T}_+^{LT}\phi_++\bar{R}_-^{LT}\phi_-\right]\mathrm{e}^{-\mathrm{i}\gamma_T x_n}dx_n,$$

$$\psi_-=\mathrm{e}^{-\mathrm{i}\gamma_T x}\int_x^e\left[\bar{R}_+^{TT}\psi_++\bar{T}_-^{TT}\psi_-+\bar{R}_-^{LT}\phi_++\bar{T}_-^{LT}\phi_-\right]\mathrm{e}^{\mathrm{i}\gamma_T x_n}dx_n,$$

with, for instance, $\bar{R}_+^{TT}=\bar{n}\tilde{R}_+^{TT}$. Following Twersky's theory [4], we shall solve the above system supposing the solutions may be written as

$$\phi_\pm=A_\pm^L\mathrm{e}^{\mathrm{iK}x}+B_\pm^L\mathrm{e}^{\text{-iK}x}+C_\pm^L\mathrm{e}^{\mathrm{iK}'x}+D_\pm^L\mathrm{e}^{\text{-iK}'x},\quad(28)$$

$$\psi_\pm=A_\pm^T\mathrm{e}^{\mathrm{iK}x}+B_\pm^T\mathrm{e}^{\text{-iK}x}+C_\pm^T\mathrm{e}^{\mathrm{iK}'x}+D_\pm^T\mathrm{e}^{\text{-iK}'x},\quad(29)$$

where K and K' are complex effective wave numbers. Here, the difference with Twersky's work lies in the fact that there are two wave numbers to be determined, K and K', which are associated to L and T coupled coherent waves propagating in the slab.

First of all, to calculate K and K', as well as all amplitudes of the coherent waves, the procedure consists in substituting (28) and (29) into (27) and, secondly, in expanding their right hand side. Next, the characteristic equation satisfied by K and K' is obtained by applying *Lorentz-Lorenz law*, which consists in equating to zero the factors of all propagative terms with effective wave number K or K'. Applying Lorentz-Lorenz law to the four equations of system (27) leads to four similar systems of four equations, which can be summarized by the following system (30):

$$\begin{cases}(x-\gamma_L)X_+^L+\mathrm{i}\left[\bar{T}_+^{LL}X_+^L+\bar{R}_-^{LL}X_-^L+\bar{T}_+^{TL}X_+^T+\bar{R}_-^{TL}X_-^T\right]=0\\(x+\gamma_L)X_-^L+\mathrm{i}\left[\bar{R}_+^{LL}X_+^L+\bar{T}_-^{LL}X_-^L+\bar{R}_+^{TL}X_+^T+\bar{T}_-^{TL}X_-^T\right]=0\\(x-\gamma_T)X_+^T+\mathrm{i}\left[\bar{T}_+^{TT}X_+^T+\bar{R}_-^{TT}X_-^L+\bar{T}_+^{LT}X_+^L+\bar{R}_-^{LT}X_-^L\right]=0\\(x+\gamma_T)X_-^T+\mathrm{i}\left[\bar{R}_+^{TT}X_+^T+\bar{T}_-^{TT}X_-^T+\bar{R}_+^{LT}X_+^L+\bar{T}_-^{LT}X_-^L\right]=0\end{cases}$$

In system (30), symbol X stands for A when x = K, B when x = -K, C when x = K' and D when x = -K'. The four systems of four equations then lead to four homogeneous systems, with the amplitudes X as unknowns. Finally, equating to zero their determinant gives four equations of dispersion equivalent. For x = K, for instance, the equation of dispersion may be written as

$$\begin{vmatrix}S_+^{LL}-\mathrm{i}K & \bar{R}_-^{LL} & \bar{T}_+^{TL} & \bar{R}_-^{TL}\\ \bar{R}_+^{LL} & S_-^{LL}+\mathrm{i}K & \bar{R}_+^{TL} & \bar{T}_-^{TL}\\ \bar{T}_+^{LT} & \bar{R}_-^{LT} & S_+^{TT}-\mathrm{i}K & \bar{R}_-^{TT}\\ \bar{R}_+^{LT} & \bar{T}_-^{LT} & \bar{R}_+^{TT} & S_-^{TT}+\mathrm{i}K\end{vmatrix}=0\ ,(31)$$

where $S_\pm^{LL}=\mathrm{i}\gamma_L+T_\pm^{LL}$ and $S_\pm^{TT}=\mathrm{i}\gamma_T+T_\pm^{TT}$. It is interesting to precise that this equation may be analytically solved. It gives all solutions: i.e. K = K_L and K = K_T (as well as K= -K_L and K= -K_T), these wave numbers are associated to the coherent waves $\phi_\pm$ and $\psi_\pm$.

Let suppose that the equation of dispersion (31) has been solved. Now, to calculate the sixteen amplitudes of the coherent waves, we need to build an inhomogeneous system of sixteen equations. Among them, twelve are given by Lorentz-Lorenz law, removing one of the four equations of each system defined by (30). Four other equations are given by the *extinction theorem*, which consists in equating to zero the factors of all propagative terms with wave number k_L or k_T: this condition then 'extinguishes' the incident wave inside the slab. Extinction theorem furnishes only one inhomogeneous equation, which is

$$A_+^L+B_+^L+C_+^L+D_+^L=1\,,\quad(32)$$

and three homogeneous equations, i.e.

$$A_+^T+B_+^T+C_+^T+D_+^T=0\,,\quad(33)$$

$$A_-^L\mathrm{e}^{\mathrm{i}K_L}+B_-^L\mathrm{e}^{\text{-i}K_L}+C_-^L\mathrm{e}^{\mathrm{i}K_T}+D_-^L\mathrm{e}^{\text{-i}K_T}=0\,,\quad(34)$$

$$A_-^T\mathrm{e}^{\mathrm{i}K_L}+B_-^T\mathrm{e}^{\text{-i}K_L}+C_-^T\mathrm{e}^{\mathrm{i}K_T}+D_-^T\mathrm{e}^{\text{-i}K_T}=0\,.\quad(35)$$

The system obtained may be easily solved using Cramer's method. When all amplitudes are known, reflection coefficients R_{eff}^{LL} and R_{eff}^{LT} of the effective medium are respectively defined by ϕ_- and ψ_- at x = 0, and transmission coefficients T_{eff}^{LL} and T_{eff}^{LT} by $\phi_+\mathrm{e}^{-\mathrm{i}k_L x\cos\alpha_L}$ and $\psi_+\mathrm{e}^{-\mathrm{i}k_T x\cos\alpha_T}$ at $x = e$. It is important to notice that for a T wave incident on the slab, the formalism practically stays unchanged: the right hand sides of Eqs. (32) and (32) respectively become equal to 1 and 0.

In the next section, we shall restrict our study to the case where $\alpha_L = 0°$ since we have not noted relevant information for $\alpha_L \neq 0°$. In addition, as no mode conversion occurs for $\alpha_L = 0°$ (no T coherent wave then propagates in the slab), the theory is considerably simplified, and allows us to rewrite the transmission and reflection coefficients of the effective medium as those of a solid slab in a solid medium, i.e.

$$R_{eff}^{LL} = \frac{4\tau^{LL} e^{i(K_L - k_L)e}}{\left(\tau^{LL}+1\right)^2 + \left(1-\tau^{LL}\right)^2 e^{2iK_L e}}, \qquad (36)$$

$$T_{eff}^{LL} = \frac{\left({\tau^{LL}}^2+1\right) + \left(1-{\tau^{LL}}^2\right) e^{2iK_L e}}{\left(\tau^{LL}+1\right)^2 + \left(1-\tau^{LL}\right)^2 e^{2iK_L e}}, \qquad (37)$$

where

$$\tau^{LL} = \frac{\overline{R}_+^{LL} - S_+^{LL} + iK_L}{\overline{R}_+^{LL} + S_+^{LL} - iK_L} \qquad (38)$$

can be physically interpreted as a ratio of acoustic impedances between both media. The second remark is that, making e tend to infinity, we can obtain local reflection and transmission coefficients R_{ij}^{LL} and T_{ij}^{LL} (with $i, j = 1, 2$) at each local boundary between the slab (medium 2) and the outside medium (medium 1):

$$R_{12}^{LL} = \frac{1-\tau^{LL}}{1+\tau^{LL}}, \quad T_{12}^{LL} = \frac{2}{1+\tau^{LL}}, \qquad (39)$$

$$R_{21}^{LL} = \frac{\tau^{LL}-1}{\tau^{LL}+1}, \quad T_{21}^{LL} = \frac{2\tau^{LL}}{\tau^{LL}+1}. \qquad (40)$$

These results are particularly interesting since we shall be able to interpret our results in the next section developing (36) or (37) into Debye series.

3 Results and discussion

Before examining the results coming from Twersky's theory, let us validate them using another approach, which does not rest on an approximation. It consists in considering a position x_n of a scatterer n as a random variable, with $0 \le x_n \le e$, satisfying an uniform probability density (i.e. $p(x_n) = 1/e$), and in carrying out a great number M of random drawings. For each configuration m of the disorder, reflection and transmission coefficients R_m^{LL} and T_m^{LL} can be easily calculated adopting a method commonly encountered in the study of multilayers [8]. It is an iterative method based on Debye's series expansions, which describe the multiple reflections between two consecutive "interfaces" or linear arrays. Once coefficients R_m^{LL} and T_m^{LL} are calculated for all drawings m, they can be averaged:

$$R_{eff}^{LL} = \frac{1}{M}\sum_{m=1}^{M} R_m^{LL}, \quad T_{eff}^{LL} = \frac{1}{M}\sum_{m=1}^{M} T_m^{LL}. \qquad (41a\text{-}b)$$

Our computations have been performed for a half-random medium composed of $N = 100$ linear arrays of empty cylindrical inclusions (of radius $a = 1$ mm) in an aluminum matrix with the relevant parameters: density $\rho = 2700$ kg/m^3; longitudinal velocity $c_L = 6380$ m/s; transverse velocity $c_T = 2700$ m/s. The period of each periodic array is $d = 3$ mm. The thickness of the slab has been chosen sufficiently large, $e = 1$ m, to obtain a relatively low density of arrays: $\overline{n} = 100$ arrays/m; a low density minimizes indeed the probability of finding two arrays at a same abscissa, which is physically unacceptable. Figure 2-a below presents the moduli of the reflection coefficients of the effective medium calculated with both methods: Twersky's theory (heavy solid lines) and the "exact" method based on random drawings (light solid lines). The transmission coefficients are presented in Figure 2-b. All coefficients are plotted versus $k_L e$. The exact ones have been obtained carrying out $M = 10^4$ drawings.

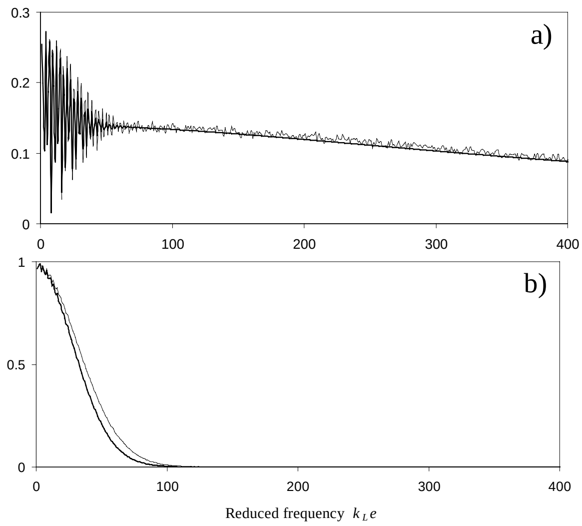

Figure 2: Comparison of both reflection coefficients a) and both transmission coefficients b) of the effective medium obtained with both methods of calculation.

Figure 2-a shows that both reflection coefficients are superimposed in a remarkable way on the whole frequency range. In the low frequency range, we notice damped oscillations, which are perfectly reproduced by both coefficients. A slight difference can be seen in Figure 2-b between both transmission coefficients but even so the agreement stays good. Thus, in conclusion, we can state that Twersky's theory furnishes exact results, even though it rests on an approximation, which is Foldy's approximation. In addition, we must precise that other calculations performed for $\alpha_L \neq 0°$ led to similar results. This validation of Twersky's theory represents the central result of our work since, to our knowledge, is the first time that an effective medium theory has been numerically verified.

Twersky's theory being validated, let us interpret its results presented in Figure 2. A first observation is that there is little reflection compared to transmission. The reflection coefficient is maximum in the low frequency range where it presents fast damped oscillations. In a complementary way, the transmission coefficient rapidly decreases with frequency: for $k_L e \gtrsim 100$, there is no coherent wave transmitted by the slab anymore. Except for low frequencies, these observations clearly show that the coherent wave is too damped to propagate over long distances in the slab. Its high damping results from a strong incoherent scattering, which was noted plotting the reflected and transmitted coherent intensities. They are defined by replacing R_m^{LL} and T_m^{LL} in (41a-b) respectively by $\left|R_m^{LL}\right|^2$ and $\left|T_m^{LL}\right|^2$. To reinforce this analysis, and to explain the damped oscillations of the reflection coefficient, the latter may be written in an alternative way, using Debye series, which allow the description of the slab as a Fabry-Perot interferometer. The series is expressed using the local reflection and transmission coefficients defined by (39) and (40), i.e.

$$R_{eff}^{LL} = R_{12}^{LL} + T_{12}^{LL} T_{21}^{LL} R_{21}^{LL} e^{2iK_L e} \sum_{p=0}^{+\infty} {R_{21}^{LL}}^p e^{2piK_L e} . \quad (42)$$

In Figure 3, the overall reflection coefficient (plotted in solid lines) of the slab is compared to both first terms of Debye series (plotted in heavy solid lines): the first one corresponds to a wave reflected at the first interface of the slab, i.e. to a specular reflection; and the second one corresponds to a wave which propagates back and forth between both interfaces, i.e. a distance 2*e*.

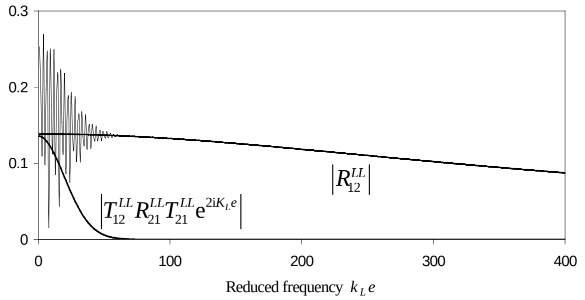

Figure 3: Comparison of the reflection coefficient of the effective medium with both first terms of (42).

For $k_L e \gtrsim 100$, we notice in Figure 3 that the reflection by the slab becomes identical to the specular one: the coherent wave does not propagate inside the slab because of its too high attenuation. Actually, the coherent wave significantly propagates in the slab in a narrow low frequency range, in which the wave propagating back and forth between both interfaces can interfere with the specular one. This interferential phenomenon explains the damped oscillations observed.

4 Conclusion

In this paper, we have presented a generalization of Twersky's theory to describe the reflection and transmission process by a slab-like region of randomly cylindrical inclusions in a solid medium. In addition, we have theoretically validated the theory adapting it to a medium periodic in one dimension and random in the other one, for which an exact calculation of multiple scattering may be also performed. Next, a brief analysis of the reflection and transmission by the effective medium has been presented. The main result is that the coherent wave is globally too damped to propagate in the medium. This is due to a too high concentration of scatterers, which introduces a main incoherent scattering compared to the coherent one.

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