MRST Formalism Applied to the Elastic Scattering at Oblique Incidence by a Fluid-Filled Cylindrical Borehole

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The three-channel unitary scattering $\mathbb{S}$ matrix is built up and analyzed in the case of the interaction between a fluid-filled cylindrical borehole in an elastic isotropic homogeneous medium and a plane harmonic wave at oblique incidence. The first incident [resp. outgoing] channel is associated to a longitudinal wave, while the two other incoming [resp. outgoing] channels are related to transverse waves each orthogonal to the others. The analysis of $\mathbb{S}$ in terms of Multichannel Resonant Scattering Theory (MRST) formalism reveals, as in the case of normal incidence, that $\mathbb{S}$ can be written exactly as the product of two unitary non diagonal scattering matrices respectively noted $\mathbb{S}^{(s)}$ and $\mathbb{S}^{(*)}$: $\mathbb{S}^{(s)}$ is the scattering by the empty "soft" cavity at oblique incidence containing non resonant contribution modes (geometrical and diffracted waves) and the Rayleigh mode, and $\mathbb{S}^{(*)}$ is the resonant scattering matrix containing the resonant scattering contributions of the fluid cylinder. The eigenvalues of $\mathbb{S}^{(*)}$ exhibit only one open eigenchannel carrying the whole resonant energy of the fluid cylinder , and two closed eigenchannels. The calculation of the associated eigenvector leads to the density matrix that governs the spreading of these resonances over the original scattering channels of $\mathbb{S}^{(*)}$: more precisely, its components involve mixing angles understandable in terms of partial half widths of the fluid borehole resonances. The de facto non commutativity between $\mathbb{S}$ factorization and eigenchannel separation is fully explained.

1 Introduction

In a recent paper [1], pioneering works of Solomon et al. [2] on multichannel scattering at normal incidence by a cylindrical fluid filled borehole were reinvestigated, in the light of the powerful tools of Multichannel Resonant Scattering Theory. Here is proposed to generalize the previous results to the case of the oblique incidence. Let consider a cylindrical hole (radius a) drilled along the *Oz* axis in an isotropic homogeneous solid (density ρ_s, longitudinal velocity c_L, transverse velocity c_T) and filled with a perfect fluid (density ρ_f, longitudinal velocity c_f). Three kinds of incident monochromatic plane wave, characterized by their potential (angular frequency ω), interact separately at oblique incidence α_L with the scatterer [3]:

$$\phi^{\text{inc}} = \mathrm{j}^{-1} \sum_{n=-\infty}^{\infty} \mathrm{j}^n \, \mathrm{J}_n(K_h r)\Omega_n(z,t) , \tag{1}$$

$$\left| A_{T_1}^{\text{inc}} \right\rangle = \mathrm{j} \sum_{n=-\infty}^{\infty} \mathrm{j}^n \, \mathrm{J}_n(Kr)\Omega_n(z,t)\begin{bmatrix}0 & 0 & 1\end{bmatrix}^t , \tag{2}$$

$$\left| A_{T_2}^{\text{inc}} \right\rangle = \frac{1}{K_T} \sum_{n=-\infty}^{\infty} \mathrm{j}^n \begin{bmatrix} \mathrm{j}K\,\mathrm{J}_{n+1}(Kr) \\ K\,\mathrm{J}_{n+1}(Kr) \\ K_z\,\mathrm{J}_n(Kr) \end{bmatrix} \Omega_n(z,t) . \tag{3}$$

They correspond respectively to an incident longitudinal L wave, an incident horizontal incident transverse T_1 wave, and an incident vertical transverse T_2 wave, written in cylindrical coordinates (r,θ,z) as sums of elementary cylindrical wave contributions. In the previous relations, $\Omega_n(z,t) = \exp\left[j\left(n\theta + K_z z - \omega t\right)\right]$, $K_L = \omega / c_L$ is the longitudinal wavevector norm, and $K_T = \omega / c_T$ is the transverse wavevector norm, while Snell's law provides $K_z = K_L \sin\alpha_L = K_T \sin\alpha_T$, and the projections of the previous wave vectors are $K_h = K_L \cos\alpha_L$ and $K = K_T \cos\alpha_T$. As the superscript P denotes the incoming channel, i.e. the polarization type of the incident wave (P=L, T_1, or T_2), the potentials scattered in the solid are defined as

$$\phi^P = \mathrm{j}^{-1} \sum_{n=-\infty}^{\infty} \mathrm{j}^n b_n^{PL} H_n^{(1)}(K_h r)\Omega_n(z,t) , \tag{4}$$

$$\left|A^P\right\rangle = \frac{1}{K_T}\sum_{n=-\infty}^{\infty} \mathrm{j}^n \begin{bmatrix} \mathrm{j}Kb_n^{PT_2}\,\mathrm{H}_{n+1}^{(1)}(Kr) \\ Kb_n^{PT_2}\,\mathrm{H}_{n+1}^{(1)}(Kr) \\ K_z(b_n^{PT_1}+b_n^{PT_2})\,\mathrm{H}_n^{(1)}(Kr) \end{bmatrix} \Omega_n(z,t)\,, \quad (5)$$

where the unknown non-normalized scattering amplitudes b_n^{PQ} in the scattering channel Q (Q=L, T_1, or T_2) are calculated from the boundary equations at $r=a$:

$$\begin{cases} \left.\sigma_{rr}\right)_s + p = 0, \\ \left.u_r\right)_s + \left.u_r\right)_f = 0, \\ \left.\sigma_{r\theta}\right)_s = 0, \\ \left.\sigma_{rz}\right)_s = 0. \end{cases} \quad (6)$$

written for each type of incident polarization P. In this last relations, σ_{rr} is the normal stress, $\sigma_{r\theta}$ and σ_{rz} are tangential stresses,

$$p = \mathrm{j}^{-1}\sum_{n=-\infty}^{\infty} \mathrm{j}^n c_n^P\, \mathrm{J}_n(K_\perp r)\Omega_n(z,t) \quad (7)$$

is the fluid pressure ($K_\perp = K_f \cos\alpha_f$, $K_f = \omega / c_f$) and u_r the radial displacement. Subscripts s and f stand respectively for “solid” and “fluid”. The expressions of the scattering amplitudes are given in the Annex as functions of determinants obtained from the matricial form of Eqs. (6).

2 Scattering Matrix

2.1 Building up $\mathbb{S}$

For a given mode n, the 3×3 scattering $\mathbb{S}$ matrix

$$\mathbb{S} = \left[S^{PQ}\right]_{P,Q=L,T_1,T_2} \quad (7)$$

is built up from the scattering amplitudes b_n^{PQ}

$$\mathbb{S} = \mathbb{I} + 2\left[\chi^{PQ} b_n^{PQ}\right]. \quad (8)$$

Here, $\mathbb{I}$ is the identity matrix, and χ^{PQ} are normalization coefficients ensuring the unitarity of $\mathbb{S}$ ($\mathbb{S}\mathbb{S}^\dagger = \mathbb{S}^\dagger\mathbb{S} = \mathbb{I}$) as well as its symmetry ($\chi^{QP} b^{QP} = \chi^{PQ} b^{PQ}$). Values of these coefficients are

$$\begin{cases} \chi^{PP} = 1, \quad (P = L, T_1, T_2) \\ \chi^{LT_1} = X / X_T = (\chi^{T_1 L})^{-1}, \\ \chi^{LT_2} = X / X_z = (\chi^{T_2 L})^{-1}, \\ \chi^{T_1 T_2} = X_T / X_z = (\chi^{T_2 T_1})^{-1}, \end{cases} \quad (9)$$

given the normalized frequencies $X = Ka$, $X_T = K_T a$ and $X_z = K_z a$.

2.2 Elastic pole contributions in $\mathbb{S}$

From now on, a water-filled cavity inside aluminium medium will be considered ($\rho_s = 2790$ kg.m^{-3}, $c_L = 6380$ m.s^{-1}, $c_T = 3100$ m.s^{-1}, $\rho_f = 1000$ kg.m^{-3}, $c_f = 1470$ m.s^{-1}). moduli of the $\mathbb{S}$ components plotted versus normalized frequency $X_L = K_L a$ and longitudinal incidence angle α_L exhibit a background amplitude perturbed by sharp peaks (see Figure 1).

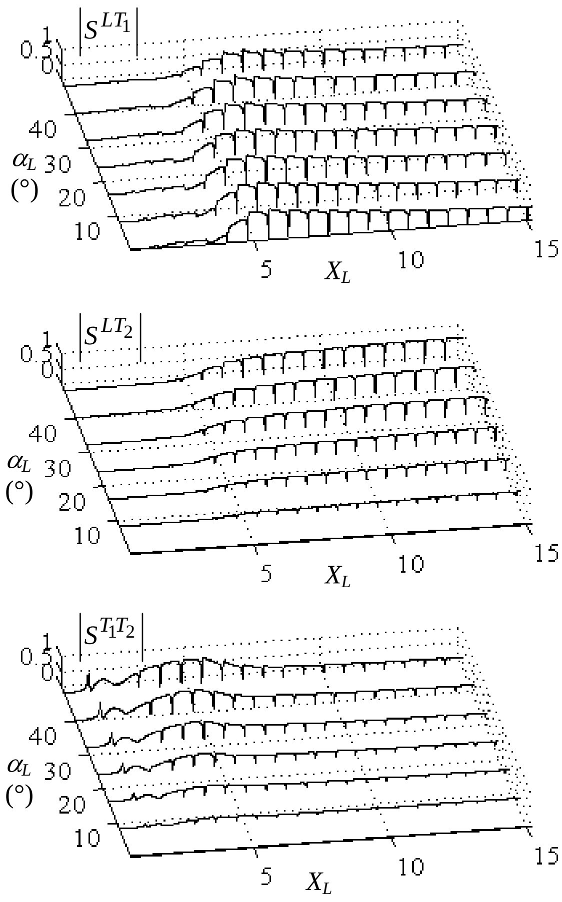

Figure 1 : Plots of $\left|S^{LT_1}\right|$, $\left|S^{LT_2}\right|$ and $\left|S^{T_1T_2}\right|$ in the (X_L,α_L) plane for mode n=5.

As in normal incidence [1-2], these peaks are due to the resonances of the fluid cylinder. Their frequencies $X_\ell^{(*)}$ are approximately given by the real part X'_p of the $\mathbb{S}$ poles $\underline{X}_p = X'_p + \mathrm{j}X''_p$. These poles are close to the roots $X_{0\ell}$ of the free mode equation $J'_n(X_\perp)=0$. In Figure 2, variations of X'_p and $X_{0\ell}$ (organized in trajectories denoted by F_ℓ , $\ell \in \mathbb{N}$) versus α_L are compared for mode $n=5$, exhibiting weak dependency on this parameter. In the following, the Multichannel Resonant Scattering Theory (MRST) formalism will be implemented to separate these resonant contributions from the background scattering in $\mathbb{S}$.

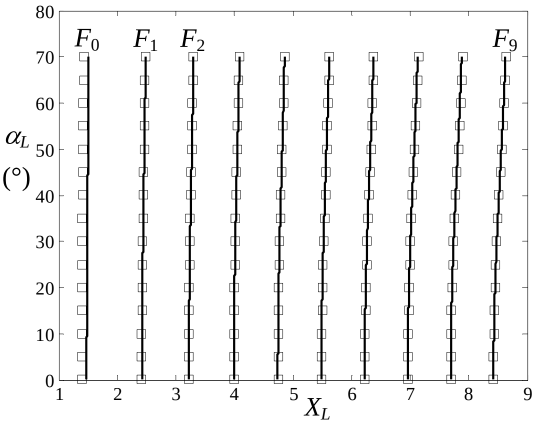

Figure 2: Free water cylinder modes (solid lines) compared to real parts X'_p of related poles (squares) versus incidence angle α_L (mode n=5).

3 MRST

3.1 Background factorization

As a preliminary remark, it is useful to emphasize the relation

$$\mathbb{S}=(d_{11}d_{(r)}\mathbb{S}^{(r)}-d_{21}d_{(s)}\mathbb{S}^{(s)})/d \tag{10}$$

where $\mathbb{S}$ is expressed as the combination of two scattering matrices defined as follows :

$$\mathbb{S}^{(\mathrm{r})}=\mathbb{S}(\rho_f\to\infty) \tag{11}$$

is the unitary and symmetric scattering matrix of determinant $\det(\mathbb{S}^{(\mathrm{r})})=S^{(\mathrm{r})}=d^{(\mathrm{r})*}/d^{(\mathrm{r})}$ built up in a case of a rigid wall cavity, and

$$\mathbb{S}^{(\mathrm{s})}=\mathbb{S}(\rho_f=0) \tag{12}$$

is the unitary and symmetric scattering matrix of determinant $\det(\mathbb{S}^{(\mathrm{s})})=S^{(\mathrm{s})}=d^{(\mathrm{s})*}/d^{(\mathrm{s})}$ corresponding to the case of a “soft” wall cavity. The other coefficients occuring in relation (10) are defined in the Annex. In relation (10), the term d_{21} containing the free mode characteristic function of the fluid cylinder is weighted by $\mathbb{S}^{(\mathrm{s})}$: this point clearly indicates that this $\mathbb{S}^{(\mathrm{s})}$ can be used to remove the background from $\mathbb{S}^{(\mathrm{s})}$ in order to isolate the resonances. From this sign, let define the unitary scattering matrix

$$\mathbb{S}^{(*)}=\mathbb{S}^{(\mathrm{s})\dagger}\mathbb{S} \tag{13}$$

and the associated transition matrix

$$\mathbb{T}^{(*)}=\left[T^{(*)PQ}\right]=-\mathrm{j}(\mathbb{S}^{(*)}\text{-}\mathbb{I})/2\ . \tag{14}$$

The modulus of $\mathbb{T}^{(*)}$ components plotted in (X_L,α_L) plane (Figure 3) exhibit only resonance peaks located at the same frequencies than those noted on $\mathbb{S}$ components plot (Figure 1) suggesting at a first sight that $\mathbb{S}^{(\mathrm{s})}$ is the suitable background scattering matrix. In order to definitely confirm this assumption, the eigenvalues of $\mathbb{S}^{(*)}$ have to be evaluated.

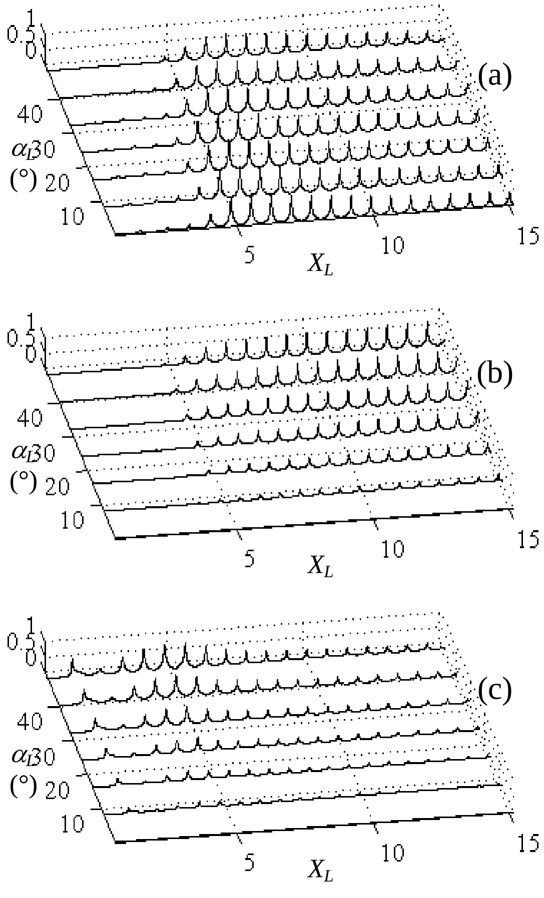

Figure 3: Plots of $\left|T^{(*)LT_1}\right|$ (a), $\left|T^{(*)LT_2}\right|$ (b), $\left|T^{(*)LT_2}\right|$ (c), in the (X_L,α_L) plane for mode n=5.

3.2 $\mathbb{S}^{(*)}$ Eingenvalues

An elegant way to calculate the eigenvalues of $\mathbb{S}^{(*)}$ is to introduce expansion (10) in the unitarity relation $\mathbb{S}\mathbb{S}^{\dagger} = \mathbb{S}^{\dagger}\mathbb{S} = \mathbb{I}$. As $\mathbb{S}^{(\mathrm{r})}$ and $\mathbb{S}^{(\mathrm{s})}$ are also unitary , it follows that

$$q\mathbb{S}^{(\mathrm{s})\dagger}\mathbb{S}^{(\mathrm{r})} + q\mathbb{S}^{(\mathrm{r})\dagger}\mathbb{S}^{(\mathrm{s})} = \mathbb{I}\,, \qquad (15)$$

with $q = d^{(\mathrm{r})}d^{(\mathrm{s})*}/(d^{(\mathrm{r})}d^{(\mathrm{s})*} + d^{(\mathrm{r})*}d^{(\mathrm{s})})$. From this, the eigenmatrix $\mathbb{S}_{\mathrm{eig}}^{(\mathrm{s}\backslash\mathrm{r})}$ of $\mathbb{S}^{(\mathrm{s})\dagger}\mathbb{S}^{(\mathrm{r})}$ can be easily deduced by replacing this last term and its adjoint in (15) by the change of basis relation

$$\mathbb{S}^{(\mathrm{s})\dagger}\mathbb{S}^{(\mathrm{r})} = \mathbb{R}\mathbb{S}_{\mathrm{eig}}^{(\mathrm{s}\backslash\mathrm{r})}\mathbb{R}^{\dagger}\,, \qquad (16)$$

where $\mathbb{R}$ is a rotation matrix which components will be studied in the next section. We found

$$\mathbb{S}_{\mathrm{eig}}^{(s\backslash r)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S^{(s)*}S^{(r)} \end{bmatrix}. \qquad (17)$$

It results from (13) and (10) that

$$\mathbb{S}^{(*)} = \mathbb{R}\left(\frac{d_{11}d_{(r)}\mathbb{S}_{\mathrm{eig}}^{(s\backslash r)} - d_{21}d_{(s)}\mathbb{I}}{d}\right)\mathbb{R}^{\dagger}\,, \qquad (18)$$

providing obviously the relation between $\mathbb{S}^{(*)}$ and its eigenmatrix $\mathbb{S}_{\mathrm{eig}}^{(*)}$. Then, $\mathbb{S}_{\mathrm{eig}}^{(*)}$ has necessarily the same structure as $\mathbb{S}_{\mathrm{eig}}^{(s\backslash r)}$

$$\mathbb{S}_{\mathrm{eig}}^{(*)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & S^{(*)} \end{bmatrix} \qquad (19)$$

where the two first eigenvalues equal to 1 indicate two closed eigenchannels. The third eigenvalue

$$S^{(*)} = \det(\mathbb{S}^{(*)}) = \left(d^{(s)}/d^{(s)*}\right)d^{*}/d \qquad (20)$$

contains all the resonant interaction with the fluid cavity, as attested by the plot of the corresponding transition amplitude $T^{(*)} = -\mathrm{j}(S^{(*)} - 1)/2$, exhibiting resonant shapes with unitary amplitudes and centred close to each zeros $X_{0\ell}$ of $\mathrm{J}_n'(X_{\perp})$ (See Figure 4).

Indeed, as $T^{(*)}$ can be expressed as a function of determinants

$$T^{(*)} = \frac{d_{11}\,\mathrm{Im}(d_{(r)}^{*}d_{(s)})}{d_{11}d_{(r)}d_{(s)}^{*} - d_{21}\left|d_{(s)}\right|^{2}}\,, \qquad (21)$$

asymptotic expansions of $d_{(r)}$ and $d_{(s)}$ for $X_L >> n$ provide a quasi imaginary value of the ratio $d^{(r)}/d^{(s)}$. Then, the Taylor expansions of d_{11} and d_{21} at the vicinity of $X_{0\ell}$ allow to approximate $T^{(*)}$ as a typical Breit-Wigner function

$$T^{(*)} \approx \frac{-\Gamma_{\ell}/2}{X_L - X_{0\ell} + j\Gamma_{\ell}/2} \qquad (22)$$

with resonance frequency and width approximately given by

$$\Gamma_{\ell}/2 = \frac{\rho_f}{\rho_s}\left(\frac{c_f}{c_L}\right)^2 \frac{1}{\cos^2\alpha_f}\left[\frac{X_T^2}{X_L}\mathrm{Im}\left(\frac{d^{(\mathrm{r})}}{d^{(\mathrm{s})}}\right)\right]_{X_{0\ell}} \,, \qquad (23.a)$$

$$X_{0\ell} = \frac{c_f}{c_L\cos\alpha_f}\left[n + 2\ell + \frac{1}{2}\right]\frac{\pi}{2}\,. \qquad (23.b)$$

These asymptotic values are compatibles with those found in normal incidence [1]. Relation (22) definitely demonstrates that $\mathbb{S}^{(*)}$ is a purely resonant scattering matrix and consequently that $\mathbb{S}^{(\mathrm{s})}$ is the adapted background to be removed from $\mathbb{S}$ in order to isolate the fluid resonances.

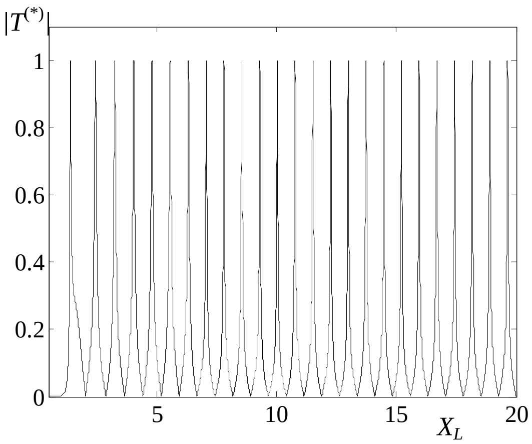

Figure 4: Plot of $\left|T^{(*)}\right|$ versus normalized frequency (n=5, α_L=50°).

3.3 Density Matrix

The change of basis between $\mathbb{S}^{(*)}$ ad $\mathbb{S}_{\mathrm{eig}}^{(*)}$ is ensured by the same rotation matrix

$$\mathbb{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ -r_{12}^* & r_{22} & r_{23} \\ -r_{13}^* & -r_{23}^* & r_{33} \end{bmatrix} \qquad (24)$$

that diagonalizes the product $\mathbb{S}^{(s)\dagger}\mathbb{S}^{(r)}$, as shown in relation (18)

$$\mathbb{S}^{(*)} = \mathbb{R}\mathbb{S}^{(*)}_{\mathrm{eig}}\mathbb{R}^{\dagger}.$$

By introducing relation (14) in this last change of basis relation, the spreading of the fluid cylinder resonances over the scattering channels is fully and conveniently characterized by the density matrix ρ such that

$$\mathbb{S}^{(*)} = \mathbb{I} + 2\mathrm{j}\rho T^{(*)}. \qquad (25)$$

This matrix defined as

$$\rho = \mathbb{R}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\mathbb{R}^{\dagger}, \qquad (26)$$

only depends on the normalized eigenvector associated to $S^{(*)}$, i.e. the third column $\begin{bmatrix} r_{13} & r_{23} & r_{33} \end{bmatrix}^{\mathrm{t}}$ of $\mathbb{R}$. An interesting way to understand the meaning of ρ, consists in emphasizing its role in relation (16) :

$$\mathbb{S}^{(s)\dagger}\mathbb{S}^{(r)} = \mathbb{I} + 2\mathrm{j}\rho T^{(s\backslash r)}_{\mathrm{eig}} \qquad (27)$$

with the transition amplitude associated to $T^{(s\backslash r)}_{\mathrm{eig}} = -\mathrm{j}(S^{(s)*}S^{(r)} - 1)/2$. It results that the coupling

$$\rho = \frac{\mathbb{S}^{(s)\dagger}}{S^{(s)*}}\frac{\mathbb{S}^{(r)} - \mathbb{S}^{(s)}}{S^{(r)} - S^{(s)}} \qquad (28)$$

of fluid resonances in the scattering channel of $\mathbb{S}^{(*)}$ depends on the “distance” between rigid and soft backgrounds. As a proof, the moduli of $\mathbb{T}^{(*)}$ components are compared in Figure 5 to the corresponding ρ components calculated by mean of relation (28).

In nuclear and particle physics [4], ρ components are often expressed using mixing angles. Indeed, as the components of the third column of the rotation matrix $\mathbb{R}$ can be written

$$\begin{cases} r_{13} = -\sin\beta\cos\varepsilon e^{-j(2\eta-\gamma-\tau)} \\ r_{23} = -\sin\beta\sin\varepsilon e^{-j(2\eta-\gamma+\tau)} \\ r_{33} = \cos\beta e^{-j2\eta} \end{cases} \qquad (29)$$

the coupling angles β, ε, and the phase lags η, γ and τ allow to define all components of $\mathbb{R}$. Expressing now the modulus of each of these components in terms of partial widths

$$\Gamma_{LL} = \Gamma\sin^2\beta\cos^2\varepsilon, \qquad (30)$$

$$\Gamma_{T_1T_1} = \Gamma\sin^2\beta\sin^2\varepsilon, \qquad (31)$$

$$\Gamma_{T_2T_2} = \cos^2\beta, \qquad (32)$$

the resonant energy conservation is then merely provided by the trace of ρ

$$\mathrm{Tr}(\rho) = \frac{\Gamma_{LL}}{\Gamma} + \frac{\Gamma_{T_1T_1}}{\Gamma} + \frac{\Gamma_{T_2T_2}}{\Gamma} = 1. \qquad (33)$$

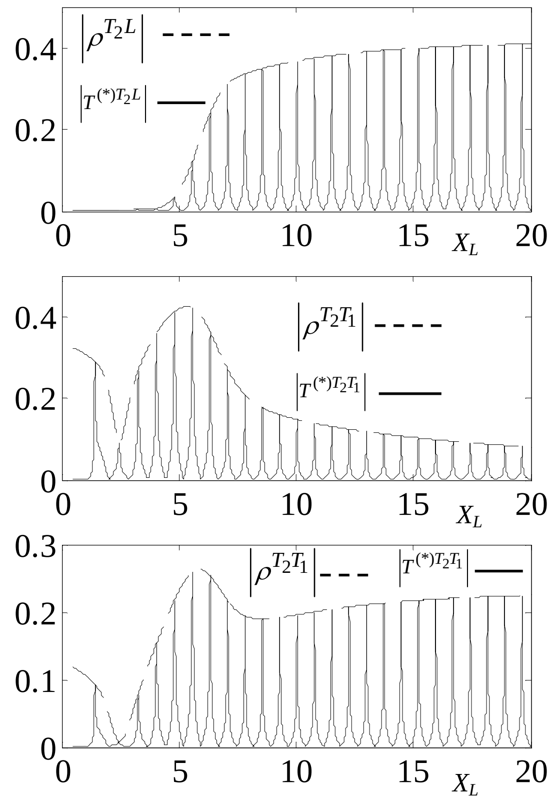

Figure 5: Plots of $|T^{(*)T_2Q}|$ (Q=L,T_1,T_2) and their envelop $|\rho^{T_2Q}|$ versus normalized frequency X^L (n=5, α_L=50°).

4 Summary

The results presented in this paper have allowed us to extend from normal to oblique incidence previous studies performed on a fluid-filled cavityin a solid.

Using the factorization $\mathbb{S}^{(*)} = \mathbb{S}^{(s)\dagger}\mathbb{S}$ for each n mode, the resonances of the fluid cylinder are isolated in $\mathbb{S}^{(*)}$, while other non resonant behaviour is confined in $\mathbb{S}^{(s)}$. The calculus of $\mathbb{S}^{(*)}$ eigenvalues and eigenvectors allow to predict the amount of resonant energy scattered in each outgoing channel and connect these quantities with the scattering by soft and rigid empty cavities.

As a final remark, the order of the previous processing performed on $\mathbb{S}$, that is

(i) remove a background contribution from $\mathbb{S}$,

(ii) calculate eigenmatrix and density matrix associated to $\mathbb{S}^{(*)}$,

cannot be exchanged. Indeed, calculating first eigenmatrix $\mathbb{S}_{\text{eig}}$ of $\mathbb{S}$, provide a change of basis relation

$$\mathbb{S} = \mathbb{R}_1 \mathbb{S}_{\text{eig}} \mathbb{R}_1^{\dagger}, \tag{34}$$

where all the eigenchannels are open and a part of the resonant energy of the fluid contributes to the components of the rotation matrix $\mathbb{R}_1$. From this starting point, it is difficult, and even impossible, to isolate fluid resonances.

5 Annex

For each incident polarization P, the boundary equations (6) written at $r = a$ give rise to the linear system $\mathbb{D}\left|B^P\right\rangle = \left|N^P\right\rangle$ where the vector

$$\left|B^P\right\rangle = \begin{bmatrix} c_n^P & b_n^{PL} & b_n^{PT_1} & b_n^{PT_2} \end{bmatrix}^t$$

contains the unknown scattered amplitudes, and $\left|N^P\right\rangle$ the incident contributions. The components of the 4×4 matrix $\mathbb{D} = \left[d_{ij}\right]_{i,j=1\to 4}$ are

$$d_{11} = \frac{\rho_f}{\rho_s} X_T^2 \, \mathrm{J}_n(X_\perp),$$

$$d_{12} = ((X_T^2 - 2X_z^2 - 2n^2)\mathrm{H}_n^{(1)}(X_h) + 2X_h \mathrm{H}_n^{(1)'}(X_h)),$$

$$d_{13} = 2n(X\,\mathrm{H}_n^{(1)'}(X) - \mathrm{H}_n^{(1)}(X)),$$

$$d_{14} = -2(X\,\mathrm{H}_n^{(1)'}(X) + (X^2 - n^2)\mathrm{H}_n^{(1)}(X)),$$

$$d_{21} = -X_\perp \mathrm{J}'_n(X_\perp), \; d_{22} = -X_h \mathrm{H}_n^{(1)'}(X_h),$$

$$d_{23} = n\,\mathrm{H}_n^{(1)}(X), \; d_{24} = X\,\mathrm{H}_n^{(1)'}(X),$$

$$d_{31} = 0, \; d_{32} = 2n(X_h \mathrm{H}_n^{(1)'}(X_h) - \mathrm{H}_n^{(1)}(X_h)),$$

$$d_{33} = ((X^2 - 2n^2)\mathrm{H}_n^{(1)}(X) + 2X\,\mathrm{H}_n^{(1)'}(X)),$$

$$d_{34} = -2n(X\,\mathrm{H}_n^{(1)'}(X) - \mathrm{H}_n^{(1)}(X)),$$

$$d_{41} = 0, \; d_{42} = 2X_h X_z \mathrm{H}_n^{(1)'}(X_h),$$

$$d_{43} = -X_z n \mathrm{H}_n^{(1)}(X), \; d_{44} = X(X^2 - X_z^2)\mathrm{H}_n^{(1)'}(X),$$

where $X_\perp = K_\perp a$ and $X_h = K_h a$. Denoting by $d = d_{11}d^{(r)} - d_{11}d^{(s)}$ the determinant of $\mathbb{D}$, and by $d_{(r)}$ and $d_{(s)}$ the determinants related to the rigid and soft cavity scattering respectively, the scattered amplitudes are easily obtained by writing

$$b_n^{PQ} = d^{PQ} / d$$

where d^{PQ} is the modified determinant of $\mathbb{D}$ in which the components of the column associated to the outgoing Q channel have been replaced by the opposite and conjugate column components associated to the outgoing P channel, for instance,

$$d^{T_1T_2} = \begin{vmatrix} d_{11} & d_{12} & d_{13} & -d_{13}^* \\ d_{21} & d_{22} & d_{23} & -d_{23}^* \\ 0 & d_{32} & d_{33} & -d_{33}^* \\ 0 & d_{42} & d_{43} & -d_{43}^* \end{vmatrix}.$$

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