Increased damping in irregular acoustic cavities

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One shows numerically, on various examples, that acoustic cavities with irregular geometries have specific damping properties. As expected, the increased damping, compared to a regular cavity, is related first to the larger surface to volume ratio. But more interestingly there exists an exaltation of the dissipation for those modes that are localized near the boundary and that raise due to the irregularity of the frontier.

1 Introduction

The study of the sprectrum of the wave equation in prefractal cavities shows eigenmodes with a small existence volume, that are localized near the frontier [1, 2, 3]. As the energy dissipation of a mode is, under certain assumptions, directly related to its existence volume, those modes therefore exhibit strong dissipation. However, the use of fractal structures as practical absorbing devices is delicate and one thus consider in the present study the properties of non fractal irregular cavities, as it appears that it is the irregularity rather than the fractality, that induces these effects of localization and increased damping.

2 Formulation

In this section one briefly gives the theoretical frame of our study and useful quantities, already introduced in earlier papers [2] to characterize the localization and energy dissipation in irregular cavities.

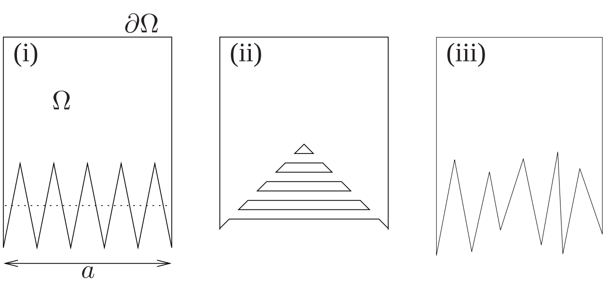

Figure 1: Examples of irregular cavities.

Consider a 2D cavity Ω with boundary $\partial\Omega$ as represented in Fig. 1. Assuming that this cavity is fulfilled with an adiabatic linear lossless medium, the acoustic pressure p, with time dependence $\exp(\mathrm{j}\omega t)$ omitted, satisfies the Helmholtz equation

$$\forall \mathbf{x} \in \Omega, \quad (\Delta + k^2)p = 0, \tag{1}$$

with boundary conditions

$$\forall \mathbf{x} \in \partial\Omega, \quad \mathbf{n} \cdot \boldsymbol{\nabla} p = -\mathrm{j}k\varepsilon p \tag{2}$$

at the walls, where $k = \omega/c_0$ is the wavenumber, c_0 is the sound velocity, and $\varepsilon(\omega) \in \mathbb{C}$ is the surface admittance of the walls.

2.1 Eigenvalue problem

One aims to calculate the eigenmodes of the cavity, in order to evaluate how the geometrical irregularity of the boundary modifies their amplitude distribution and damping. Assuming weak losses at the walls ($|\varepsilon| << 1$) the amplitude distribution of the "true" eigenmodes is well-approximated by the zero-loss cavity modes, *i.e.* the solutions $(\psi_n)_{n\in\mathbb{N}}$ of

$$\Delta\psi_n = -k_n^2\psi_n \tag{3}$$

that obey the boundary condition $\mathbf{n} \cdot \boldsymbol{\nabla}\psi_n = 0$ at the walls. One normalizes these modes by

$$\int_\Omega |\psi_n|^2 \, \mathrm{d}S = 1. \tag{4}$$

2.2 Characterization of the localization

The irregularity of the boundary of a fractal resonator is known to induce the appearance of localized modes, that is, modes whose amplitude distribution is confined in a restricted part of the resonator [1, 2, 3]. In order to characterize the localization of the eigenmodes ψ_n, one compute for each mode its *existence surface* S_n, defined by Thouless [4] as

$$S_n = \frac{1}{\int_\Omega |\psi_n|^4 \, \mathrm{d}S} \tag{5}$$

Thus, a mode will be considered as being *localized* if its existence surface S_n is significantly smaller than the surface area S of the cavity. In a rectangular cavity, and consequently in a square cavity, with "zero-loss" eigenmodes ψ_{mn}, $(m, n) \in \mathbb{N}^2$, the relative existence surface S_{mn}/S is 1 for $m = n = 0$, $2/3$ for m or $n = 0$, and $4/9$ for $m, n > 0$.

2.3 Characterization of the dissipation

The energy dissipation in a cavity can be characterized by its quality factor Q. If ω is the pulsation of excitation, Q is defined as ω times the ratio of the time averaged stored energy E to the time averaged energy dissipated per unit time W:

$$Q = \omega E/W. \tag{6}$$

If $p_n = P_n S^{1/2} \psi_n$ is the pressure field corresponding to the mode ψ_n, the associated dissipated energy W_n, that is the total outflow through the cavity walls, is

$$W_n = \int_{\partial\Omega} \frac{|p_n|^2}{2\rho_0 c_0} \mathrm{Re}(\epsilon) \, \mathrm{d}L, \tag{7}$$

giving thus

$$W_n = \frac{P_n^2 S}{2\rho_0 c_0} \mathrm{Re}(\epsilon) \int_{\partial\Omega} |\psi_n|^2 \, \mathrm{d}L, \tag{8}$$

while the stored energy is written

$$E_n = \int_\Omega \frac{1}{4\rho_0 c_0^2} \left(\frac{|\vec{\nabla} p_n|^2}{k^2} + |p_n|^2 \right) \mathrm{d}S. \tag{9}$$

Assuming that the eigenmodes are approximated by the "zero-loss" eigenmodes, the stored energy E_n at the resonance ($k = k_n$) can be written

$$E_n = \frac{P_n^2 S}{2\rho_0 c_0^2} \tag{10}$$

It follows that the quality factor Q_n for the mode n calculated at its resonance is

$$Q_n = \frac{k_n \Lambda_n}{\mathrm{Re}(\varepsilon)}, \text{where } \frac{1}{\Lambda_n} = \int_{\partial\Omega} |\psi_n|^2 \, \mathrm{d}L \tag{11}$$

As shows Eq. (11), the higher the amplitude on the boundary, the lower the value of the length Λ_n and the lower the quality factor. Therefore, throughout this paper, the energy dissipation will be measured by $1/\Lambda_n$, since it appears to be a convenient way to relate the amplitude distribution of the modes to the resulting losses.

Consider now the fundamental mode $\psi_0 = 1/\sqrt{S}$; its amplitude distribution is uniform in the cavity and the associated dissipation parameter $1/\Lambda_0$ is thus L/S, where L is the perimeter length of the cavity. Therefore, for a given surface area S, the energy dissipation of this mode is proportional to the perimeter length. Although for some of the non trivial modes the energy dissipation also increases in a manner that is roughly proportional to the perimeter length, it will be shown that, in general, an increase of the irregularity do not modify the dissipation only by increasing the length of the dissipative wall. Therefore, to clearly point out the effect of the localization on the energy dissipation, losses will be measured by the normalized factor $S/L\Lambda_n$. With this normalization the dissipation factor of the trivial mode ψ_0 is 1 for any cavity.

In a square cavity with "zero-loss" eigenmodes ψ_{mn}, $S/L\Lambda_{mn}$ is equal to 1 for $m = n = 0$, $3/2$ for m or $n = 0$, and 2 for $m, n > 0$.

3 Results

Consider a 2D rigid square cavity of side a, one side of which is made of N triangular wedges of height $a/2$, as shown in Fig. 1(i). The surface area of the cavity, $S = a^2$, is independent of N and the perimeter length is a linear function of $\sqrt{1 + N^2}$.

3.1 Localization and attenuation

The relative existence surface S_n/S of the first eigenmodes of three cavities of type (i), with 5, 10 and 20 wedges respectively, are plotted as function of the nondimensional frequency ka/π in Fig. 2(a).

One clearly see on this first plot that the irregular shape of the boundary, whatever the number of wedges, causes a noticeable decrease of the mean existence surface of the modes, compared to a square cavity: apart from the fundamental mode, whose relative existence surface is independent of the geometry, almost all of the modes have a relative existence surface smaller than 4/9, the minimum value in a square cavity.

Among all of these modes, whose number in the chosen frequency interval is naturally increasing with the perimeter to surface ratio, two families can be distinguish: a first family, composed of the less localized modes (typically, $S_n/S > 0.2$) of which the number is globally constant when increasing the number of wedges, and a second, composed of an increasing number of very localized modes ($S_n/S < 0.2$). These modes are con-

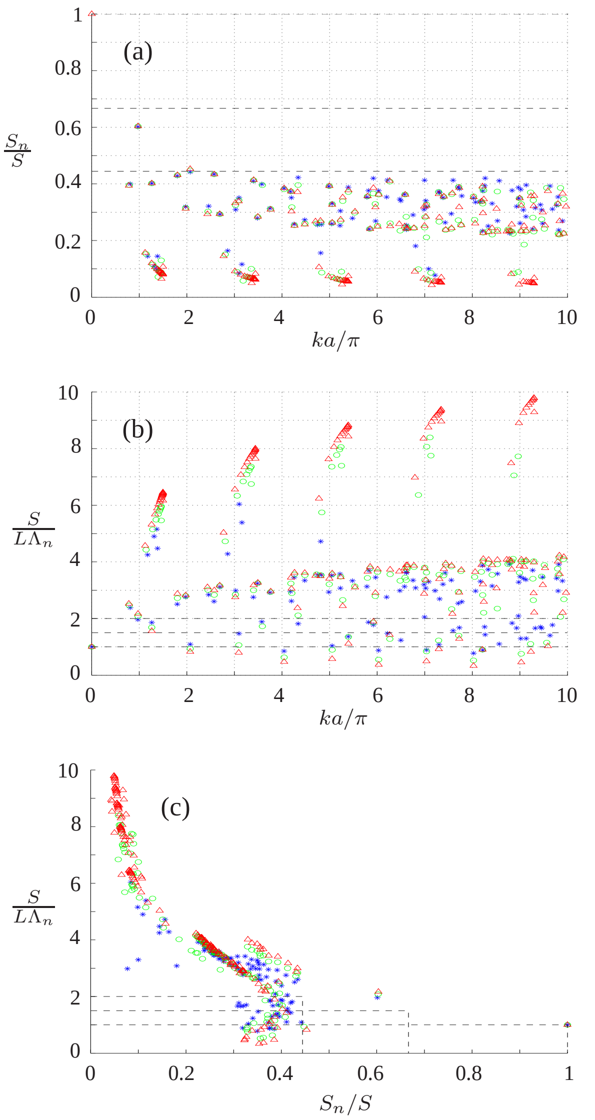

Figure 2: a) Relative existence surface and (b) energy dissipation, as measured by $S/L\Lambda_n$, for the first eigenmodes of the cavity of type (i) with ($*$) 5, ($\circ$) 10 and ($\triangle$) 20 wedges respectively. (c) Energy dissipation vs the relative existence surface. The dashed lines indicates the possible values for a square cavity.

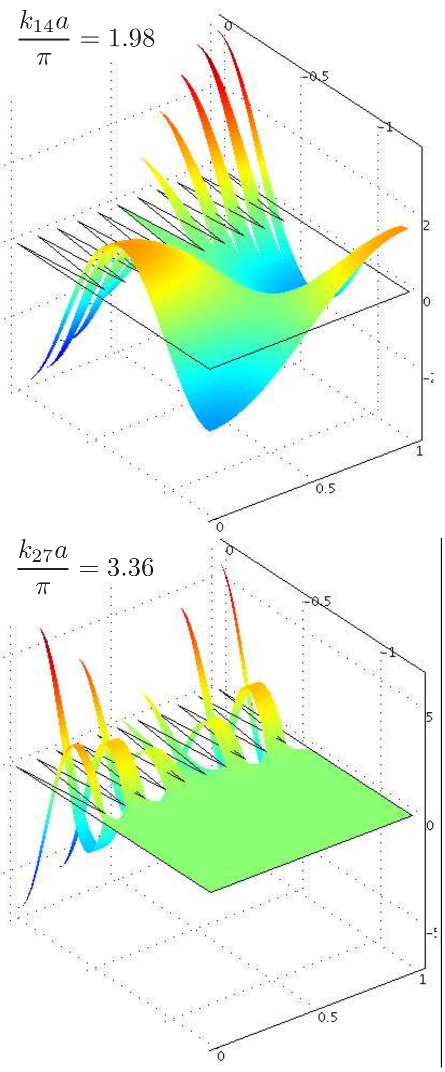

Figure 3: Pressure distribution for two eigenmodes in the cavity of type (i) with 10 wedges: (a) non localized mode ($n = 14$, the eigenmodes are sorted in increasing order of their eigenfrequency) and (b) localized mode ($n = 27$).

fined near the irregular wall, in the "sub-cavities" formed by the wedges (Fig. 3), and their eigenfrequencies, because the sub-cavities are identical, are grouped near frequencies $ka/\pi \sim 1, 3, 5, 7, \ldots$ If one considers the sub-cavities as one-dimensional resonators with length $a/2$, these frequencies correspond to $\lambda/4, 3\lambda/4, \ldots$ resonances.

The energy dissipation of the modes, as measured by $S/L\Lambda_n$, is plotted in Fig. 2(b): for almost all of the modes it is larger than in a square cavity. Besides, this plot, as the previous one, allows us to distinguish the already mentionned two families of modes. The first family, previously characterized as being composed of weakly localized modes, appears now to corresponds to the less damped modes (typically, $S/L\Lambda_n < 4$). The mean energy attenuation of these modes increases in a manner that is roughly proportionnal to the perimeter length. The second family of modes was characterized as being composed of strongly localized modes; it now appears to correspond to the more damped modes. More generally, as shows Fig. 2(c), plotting the energy dissipation as function of the relative existence surface reveals a global trend that is an increase of the damping with irregularity and localization.

Any kind of geometrical irregularity will lead to the same conclusions, whatever the shape and dimensions of the boundary (a cavity with rough walls also causes localization). For example, one gives in Fig. 4(a) the energy dissipation of the first eigenmodes in a cavity of type (ii) (cf. Fig. 1) as function of their relative existence surface. The cavity is composed of a single triangular wedge of height $a/2$, bored with 9 parallel holes. The surface area is a^2 as for the cavity of type (i) and its perimeter length differ by less than 4% from the one of the cavity of type (i) with 10 wedges. Again one can see the increase of the damping with the localization of the eigenmodes. Besides, the comparison with the results for the cavity of type (i) with 10 wedges shows that, although these two cavities have the same surface area and perimeter length, and therefore similar mode densities, the cavity with one bored wedge exhibits more localized modes, with however smaller damping parameters.

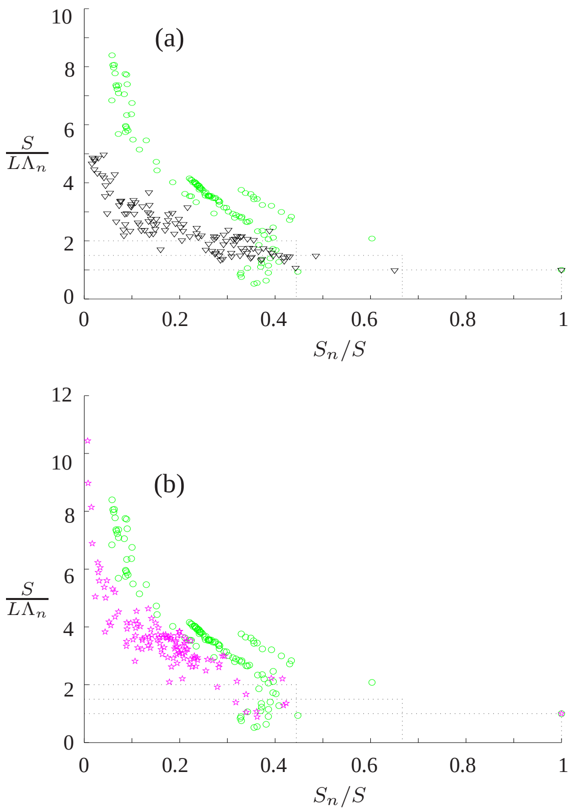

Figure 4: Energy dissipation vs the relative existence surface : (○) cavity of type (i) (cf. Fig. 1) with 10 wedges, (▿) cavity of type (ii) with 9 holes, (⋆) cavity of type (iii) with 10 wedges.

3.2 Asymmetric structure

Due to our choice for the wall shape in previous examples, notably the cavity of type (i) composed of N identical wedges, the geometrical irregularity of the cavity is, in fact, quite “regular”. It is obvious that this regularity of the wall shape, as the rotation invariance of the prefractal cavity in the earlier studies, limits the confinement of the eigenmodes amplitude distribution. Thus it can be expected that breaking this "regularity" will increase the localization of the modes. To this end one modifies the geometry of the cavity of type (i) by randomly varying, up to 20% of their initial value, the base and and height of each wedge, as soon as the abscissa or each vertex. While doing this disymmetrization one ensures that the surface area and perimeter length of the cavity differ by less than 0.01% from their initial value. As one guess the same results can be observed on the evolution of the damping, localization and localized mode density with the perimeter length, as soon as the relation between localization and energy dissipation (Fig. 4(b)). Compared to the symmetrical case, the wedges, and consequently the sub-cavities they form near the boundary, are no longer identical. Therefore the localized mode no longer appears in groups near frequencies $ka/\pi = 2n+1$, $n \in \mathbb{N}$ as was shown on Figs. 2(a) and (b), but are more spread in the spectrum, more localized, and with higher energy dissipation.

4 Summary

The influence of the geometrical irregularity of an acoustic cavity on the localization and energy dissipation of the eigenmodes has been shown numerically. Also has been shown a very general trend that is the increase of the energy dissipation with the localization. These numerical simulations have been made assuming very weak losses at the walls, which is a strong hypothesis. One now aim to generalize these results to any kind of absorbing boundary conditions, and our results at the moment show that the irregularity, whatever the boundary condition, always induces localization.

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