Elasto-Viscoelastic Interaction at Sound Propagation in Polymeric Liquid Layer between Coaxial Shells

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Sound propagation in a waveguide consisting from two thin coaxial elastic shells with polymeric liquid in the gap is studied. It is assumed that external and internal shells are made from different isotropic elastic materials and have different widths. Dynamics of viscoelastic polymeric liquid in the gap is described by general linear hereditary model. The problem is solved in conjugated quasi-one-dimensional formulation. Dynamics of shells is treated using the Kirchhoff-Love approximation. Elastic deformations of shells in the sound wave are coupled with the liquid flow in the gap through appropriate dynamic and kinematic boundary conditions. The cross effect of liquid's rheology and compressibility in the momentum balance equation, which is usually small, is neglected. The dispersion equation for sound waves in the waveguide is obtained; it is valid in the low frequency range where the wave length is greater than the external shell radius. The limiting cases of a waveguide filled by ideal or pure viscous liquid are considered; in the former case the dispersion equation yields the water hammer speed in the system. Analysis of the dispersion equation has shown that rheological properties of the liquid highly influence the sound dispersion and attenuation in the low frequency region. The interplay of the liquid and shells parameters can change the wave speed in a wide range; the effect is characterized by plots of the sound speed vice relative shells width and polymer concentration in the solution.

1 Introduction

Elasto-viscous interactions were a major focus of interest in numerous studies of sound propagation in non-homogeneous systems. Such interactions have fundamental importance in variety of applications (nondestructive evaluation and measuring of properties, acoustics of waveguides and shells, submerged in liquid etc). Propagation of elastic waves in liquid-filled tube treated as a thin shell was studied theoretically in [1, 2]. Later linear analysis was extended with regard to material properties of the solid (viscoelasticity, prestressing, and anisotropy [3]) and fluid (viscoelastic liquids [4, 5]).

The target of the current paper is to describe the longitudinal low-frequency wave propagation in a gap between two thin-walled coaxial elastic shells, filled by viscoelastic liquid (solution of high-polymer in a low-molecular solvent). The analysis presented below is based on the results of the paper [6], where dispersion of sound waves in a similar waveguide, filled by purely viscous liquid, was studied.

2 Problem formulation

2.1 Equations of shells dynamics

The basic assumption of the model formulated below is that the diameter of the external shell is smaller than the sound wavelength. In this case the use of Kirchhoff - Love approximation leads to the following equations of the external ($j=1$) and internal ($j=2$) shell dynamics [6]:

$$i\nu_j k\hat{u}_r^{(j)}/\bar{R}_j + [k^2 - \lambda_j^{-1}\Omega^2(1-\nu_j^2)\delta_\rho^{(j)}]\hat{u}_x^{(j)} = 0 \quad (1)$$

$$Q_j\hat{u}_r^{(j)} - ik\nu_j\hat{u}_x^{(j)} - [(1-\nu_j^2)/(2\varepsilon_j\lambda_j)]\hat{p}_c^{(j)} = 0$$

$$Q_1 = 1-\Omega^2\lambda_1^{-1}(1-\nu_1^2), \quad Q_2 = 1/\varepsilon - \Omega^2\lambda_2^{-1}\varepsilon\delta_\rho^{(2)}(1-\nu_2^2)$$

$$\lambda_j = E_j/p_0, \quad t_0 = R_1(\rho_s^{(1)}/p_0)^{1/2}, \quad \tau = t/t_0$$

$$\varepsilon = R_2/R_1, \quad \bar{p}_c^{(j)} = \Delta p_j/p_0, \quad \delta_\rho^{(j)} = \rho_s^{(j)}/\rho_s^{(1)}$$

$$\Omega = \omega t_0, \quad \bar{R}_j = R_j/R_1$$

Here R_j is the radius of the shell's middle surface; $2h_j$ is the shell width (it is supposed $\varepsilon_j = h_j/R_j << 1$); $\hat{u}_x^{(j)}, \hat{u}_r^{(j)}$ are complex amplitudes of the non-dimensional deformations of shells $\bar{u}_x^{(j)}, \bar{u}_r^{(j)}$ in the axial and radial directions, respectively; the symbols $\wedge$ and $^{-}$ denote complex amplitudes and dimensionless parameters, respectively; k is the non-dimensional wave number (with R_1 used for characteristic length); Ω - non-dimensional frequency; t – time; Δp_j is the contact pressure equal to the normal stress component in liquid at the pipe wall; p_0 - equilibrium pressure in the waveguide; $\rho_s^{(j)}$, E_j and ν_j are density, Young and Poisson modules of the shells material. The expressions for Q_j ($j=1,2$) in (1) were simplified with account for smallness of the bending stresses in the shell with respect to the membrane ones in the case of long waves [6]. The non-dimensional axial and radial deformations of shells and coordinates of the cylindrical coordinate system are defined as $\{\bar{u}_x, \bar{u}_r, \xi, \zeta\} = R_1^{-1}\{u_x, u_r, r, x\}$.

2.2 Boundary conditions

Kinematic boundary conditions at the liquid-shells interfaces ($\xi = 1-\varepsilon_1 \approx 1$ and $\xi = \varepsilon+\varepsilon_2 \approx \varepsilon$) mean:

$$\bar{v}_R^{(j)} = \dot{\bar{u}}_r^{(j)}, \quad \bar{v}_w^{(j)} = \dot{\bar{u}}_x^{(j)}$$

$$\{\bar{v}_R^{(j)},\ \bar{v}_w^{(j)}\} = (t_0/R_1)\{v_r,\ v_x\}_{r=R_j} \qquad (2)$$

$$\{\dot{\bar{u}}_r^{(j)},\ \dot{\bar{u}}_x^{(j)}\} = (t_0/R_1)\{\dot{u}_r^{(j)},\ \dot{u}_x^{(j)}\}, \quad j = 1,\ 2$$

where v_r, v_x are the radial and axial velocity components. In the low frequency range the pressure in the gap does not depend from the radial coordinate, which allows assuming $\bar{p}_c^{(1)} = -\bar{p}_c^{(2)}$. As a result, the dynamic boundary conditions take the form:

$$\bar{p}_c^{(1)} = \Delta\bar{p}_f - \bar{\tau}_R, \quad \bar{p}_c^{(2)} = -(\Delta\bar{p}_f - \bar{\tau}_R)$$

$$\Delta\bar{p}_f = \bar{p}_f - 1, \ \{\bar{p}_f,\ \bar{\tau}_R\} = p_0^{-1}\{p_f,\ \tau_R\} \qquad (3)$$

$$\tau_R = \frac{1}{\pi R_1^2(1-\varepsilon^2)}\int_{R_2}^{R_1} \tau_{rr}\cdot 2\pi r\,dr \approx \tau_{rr|R_1} \approx \tau_{rr|R_2}$$

Here p_f, τ_{rr} are pressure and normal deviatoric stress in the liquid and R_1, R_2 are used instead of internal radius of the external shell and external radius of the internal one, respectively.

2.3 Dynamics of viscoelastic liquid in the gap

It is supposed that polymeric liquid in the gap follows linear hereditary model [7]:

$$\tau_{ij} = 2\int_{-\infty}^{t} G(t-t_1)s_{ij}(t_1)dt_1 + 2\eta_s s_{ij}, \quad \Delta p_f = c_f^2\Delta\rho_f \qquad (4)$$

$$\Delta\rho_f = \rho_f - \rho_{f0}, \quad s_{ij} = e_{ij} - \frac{1}{3}(\nabla\cdot\vec{v})I$$

Here c_f is the sound speed in the liquid; ρ_{f0} - equilibrium density; $G(t - t_1)$ - relaxation function for the liquid; s_{ij} - deviator of the rate deformation tensor e_{ij}; η_s - solvent viscosity; $\vec{v}$ - vector of the liquid velocity. The liquid volume viscoelasticity is neglected in (4) because it has only minor effect on the sound propagation in the waveguide [7].

The basic assumptions of the liquid dynamics in the gap at acoustic excitation can be written in the form:

$$v_r << v_x, \ \delta = R_1/L << 1, \quad \frac{\partial^2 v_x}{\partial x^2} << \frac{1}{r}\frac{\partial v_x}{\partial r}$$

$$\frac{\partial^2 v_x}{\partial x^2} << \frac{\partial^2 v_x}{\partial r^2}, \quad \dot{u}_x^{(j)} << V, \quad \frac{\partial v_r}{\partial x} << \frac{\partial v_x}{\partial r}$$

Here V is the mean flow rate in the gap, defined from the relation:

$$V = \frac{2}{R_1^2(1-\varepsilon^2)}\int_{R_2}^{R_1} V_x r dr \ ,$$

where the relative axial velocity $V_x = v_x - \dot{u}_x^{(1)}$ is introduced instead of v_x. These assumptions allow writing the non-dimensional equations of liquid dynamics in the gap, averaged along the cross section, as follows [5]:

$$i\Omega\hat{V} = i\kappa k\hat{P} + \frac{2\kappa}{1-\varepsilon^2}(\hat{\tau}_1 - \varepsilon_2\hat{\tau}_2), \quad \hat{P} = \kappa^{-1}\bar{c}_f^2\hat{\rho},$$

$$i\Omega\hat{\rho} + \frac{2i\Omega}{1-\varepsilon^2}\left(\hat{u}_1^{(1)} - \varepsilon\,\hat{u}_1^{(2)}\right) - ik\hat{V} = 0 \qquad (5)$$

$$\kappa = \rho_s/\rho_{f0}$$

Here $\hat{P}$, $\hat{V}$, $\hat{\rho}$, $\hat{\tau}_1$, $\hat{\tau}_2$ are complex amplitudes of dimensionless perturbations of pressure, mean flow velocity, liquid density and shear components of deviatoric stresses in liquid at the walls of the gap. The transient frictions at liquid-solid interfaces are found from solution of the momentum balance equation for incompressible liquid flow in the gap (hydraulic approach [4]), which can be written in the form:

$$i\Omega\hat{V}_x = \hat{K} + \left((i\Omega)^{-1}\kappa G^* + \kappa\bar{\eta}_s\right)\left(\frac{d^2\hat{V}_x}{d\xi^2} + \frac{1}{\xi}\frac{d\hat{V}_x}{d\xi}\right)$$

$$\bar{K} = -\kappa\frac{\partial(\Delta\bar{p}_f)}{\partial\zeta} - \ddot{\bar{u}}_x, \quad \hat{K} = \bar{K}e^{-i\Omega\tau}$$

$$G^* = \int_0^\infty \frac{(\Omega\bar{\theta})\bar{F}(\bar{\theta})(1+i\Omega\bar{\theta})}{1+(\Omega\bar{\theta})^2}d\bar{\theta}, \quad \bar{\theta} = \theta/t_0$$

Here $\bar{F}(\bar{\theta})$ is the non-dimensional spectrum of relaxation times θ. The result has the form:

$$\hat{\tau}_1 - \varepsilon_2\hat{\tau}_2 = -4\bar{\eta}D\hat{V}, \quad \bar{\eta} = \bar{\eta}_s + G^*/(i\Omega) \qquad (6)$$

$$D = -\frac{1}{4}\cdot\frac{\mu T(\mu,\varepsilon)}{1-2\mu^{-1}(1-\varepsilon^2)^{-1}T(\mu,\varepsilon)},$$

$$T(\mu,\varepsilon) = [J_1(\mu) - \varepsilon J_1(\mu\varepsilon)]\cdot A + [Y_1(\mu) - \varepsilon Y_1(\mu\varepsilon)]\times$$

$$(1 - J_0(\mu\varepsilon)\cdot A)Y_0^{-1}(\mu\varepsilon), \quad \mu = i(i\Omega/(\kappa\bar{\eta}))^{1/2}$$

$$A = (1 - Y_0(\mu)/Y_0(\mu\varepsilon))(J_0(\mu) - J_0(\mu\varepsilon)Y_0(\mu)/Y_0(\mu\varepsilon))^{-1}$$

The physical meaning of the approximation used above is that we neglect both by small cross effect of the liquid viscosity and compressibility and by the influence of the radial deformations of shells on the losses at the liquid motion in the waveguide at the sound wave propagation.

3 Results and discussion

Now the boundary value problem is closed. The dispersion equation following from (1)-(3), (5), (6) can be written in the form:

$$az^3+bz^2+dz+m=0, \quad z=C^{-2}, \quad C=\Omega/k \qquad (6)$$

The coefficients *a*, *b*, *d*, *m* in (6) are defined by certain cumbersome formulas. In the limiting case $E_2 \to \infty$ (rigid internal shell) the term $m \to 0$ and equation (6) has only two non-zero roots, which follow the same dispersion equation as was obtained in [5]. The root $z=0$ in this case corresponds to infinite sound speed of the mode propagating in a rigid shell. Similar result takes place in the opposite case when the external shell is rigid and the internal one flexible.

The dispersion equation (6) is simplified essentially in the low-frequency limit, when $\Omega \to 0$. In this case the inertia and longitudinal deformations of shells can be neglected. Taking into account also that contribution of the rheological term in normal stresses on the shell surface is small [4], one can write the dispersion equation in the form:

$$\frac{1}{C^2} = N\left(\frac{1}{\bar{c}_f^2}+\frac{1}{\bar{c}_1^2}+\frac{1}{\bar{c}_2^2}\right) \qquad (7)$$

$$\bar{c}_1^2 = \kappa(1-\varepsilon^2)\varepsilon_1\lambda_1, \quad \bar{c}_2^2 = \kappa(1-\varepsilon^2)\varepsilon^{-2}\varepsilon_2\lambda_2$$

$$N = 1+8\kappa\bar{\eta}D\cdot[i\Omega(1-\varepsilon^2)]^{-1}$$

This formula can be considered as generalization of the Korteweg relation [8] for the water hammer speed c_K in water filled thin-walled tube ($\varepsilon = 0$):

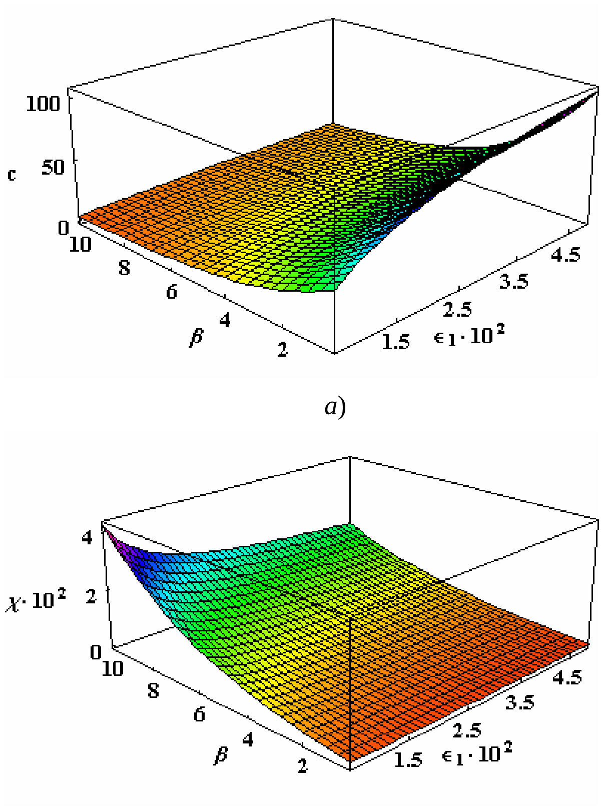
$$\frac{1}{\bar{c}_K^{\,2}} = \frac{1}{\bar{c}_f^2}+\frac{1}{\kappa\varepsilon_1\lambda_1} \qquad (8)$$

It follows from (7) that presence of the internal shell lowers the sound speed. For deformable internal shell the wave speed is smaller than for a rigid one.

In the general case the dispersion equation (6) was studied numerically. The speed and attenuation of the low frequency longitudinal mode were calculated for the aluminium made waveguide ($E_1 = E_2 = 7\cdot10^{10}$ N/m^2, $\nu_1 = \nu_2 = 0.34$, $\rho_s^{(1)} = \rho_s^{(2)} = 2.7\cdot10^3$ kg/m^3, $\varepsilon_1 = \varepsilon_2$) with $R_1 = 10^{-2}$ m, $p_0 = 10^5$ Pa, $\rho_{f0} = 850$ kg/m^3, $c_f = 1500$ m/s. Rheological properties of the polymeric liquid were evaluated according to discrete spectrum representation:

$$\bar{\eta}-\bar{\eta}_s = \frac{\bar{\eta}_p-\bar{\eta}_s}{z(\alpha_1)}\sum_{k=1}^{\infty}\frac{k^{\alpha_1}-i\omega\bar{\theta}_1}{k^{2\alpha_1}+(\omega\bar{\theta}_1)^2}$$

where $\bar{\eta}_p$ is Newtonian viscosity of the liquid, $z(\alpha_1)$ - the Riemann zeta function of the spectral distribution parameter α_1 ($\theta_k = \theta_1/k^{\alpha_1}$), θ_1 – the maximum relaxation time in the spectrum. The Newtonian viscosity of the solution was computed from the Martin relation [7] $\eta_p/\eta_s = 1+\beta exp(k_M\beta)$ with $k_M = 0.4$ and $\eta_s = 0.1 Pa\cdot s$. The relaxation times θ_k were found according to the Rouse theory ($\alpha_1 = 2$) from equation $\bar{\theta}_1 = 0.608\bar{\eta}_s A\exp(k_M\beta)$ with $A = 500$ (the non-dimensional parameter $A = [\eta]Mp_0/R_G$, where $[\eta]$ - characteristic viscosity of the solution, M - molecular mass of the polymer, R_G - universal gas constant, β - reduced polymer concentration).

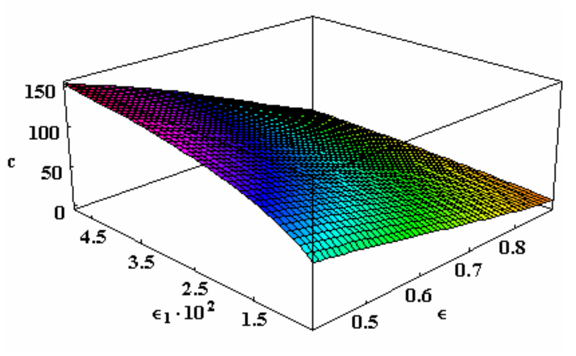


a)

b)

Figure 1: Dimensionless sound speed and attenuation versus reduced polymer concentration and relative shell width

Results of numerical simulations within rheological characterization of the liquid, described above, are presented on the Figure 1 in the form of 3-D plots of the non-dimensional wave speed $c = \Omega / \mathrm{Re}\{k\}$ and attenuation χ = - $\mathrm{Im}\{k\}$ versus reduced polymer concentration β and relative width of the shell ε_1 for ε = 0.7. The plots on the Figure 2 illustrate the dependency of the sound speed and attenuation from the relative gap and shell widths, for polymeric solution with $\eta_s = 0.5 \cdot 10^{-3} Pa \cdot s$, α_1 = 2, $\eta_p = 0.5 Pa \cdot s$, $\theta_1 = 0.01 s$. For all plots $\Omega = 0.1$.



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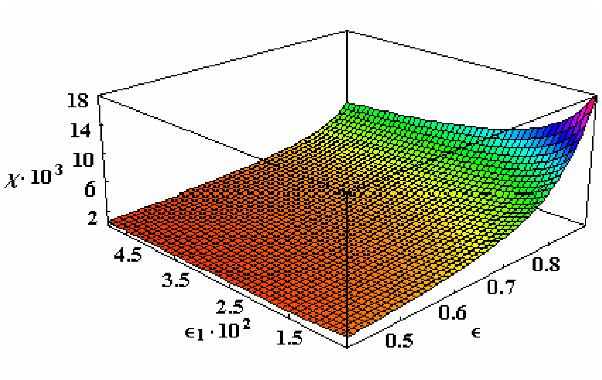

b)

Figure 2: Dimensionless sound speed and attenuation versus relative gap and shell widths

As it follows from the Figure 1, the increase in polymer concentration raises attenuation and leads to slowing of the wave speed. Numerical data show that for the same values of the system parameters the sound speed in a pure viscous liquid with viscosity η_p is essentially less than that one in a similar viscoelastic liquid, which is explained by the frequency dependent dynamic viscosity of polymeric solution.

The shells width highly influences the wave propagation – the more flexible are the gap walls (smaller ε_1 values) the less is the sound speed and larger its attenuation. As a result, the use of acoustic method for characterization of the liquid rheology will be more indicative in the case of a waveguide with thicker walls. Note that with growth of ε_1 the sound speed on the Figure 1, *a*, for $\beta = 0$ approaches the value, following from (7).

The plots on the Figure 2 show that the smaller is the gap between the shells, the less is the sound speed – the result is explained by the losses increase with narrowing of the waveguide (Figure 2, *b*). It follows from the data that attenuation of the wave is highly sensitive not only to the liquid viscosity and shells flexibility, but also to the relative gap width. In this respect the gap and the shells widths have similar effect on the wave propagation – their increase is resulted in attenuation reduction and raises the wave speed.

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