DOI: 10.61782/fa.2023.0046

THE FINITE SERIES METHOD IN ACOUSTIC SCATTERING WITH APPLICATION TO BESSEL BEAMS

Leonardo André Ambrosio [1*] **Gérard Gouesbet** [2]

[1] Department of Electrical and Computer Engineering, SEL/EESC, University of São Paulo, Brazil
[2] CORIA-UMR 6614 - Normandie Université. CNRS-Université et INSA de Rouen. Campus Universitaire du Madrillet, France

ABSTRACT

In acoustic scattering by spherical particles, the scalar velocity potential field can be written in terms of partial wave expansions using spherical wave functions. The expansion coefficients – the beam shape coefficients (BSCs) – embody the spatial information of the field and their exact determination is of utmost importance for reliable calculation of physical quantities of interest such as acoustic forces. In this work, the finite series (FS) method, a widely known method in the realm of optics and light scattering, is introduced to describe the BSCs of acoustic fields. The FS technique relies on Neumann expansion theorem and provide exact expressions for the BSCs of arbitrary-shaped beams. Examples of calculations with field reconstructions are presented for acoustic arbitrary-order Bessel beams. The results represent a first systematic approach to alternative methods for describing BSCs in acoustic scattering by spherical scatterers.

Keywords: *acoustic scattering, beam shape coefficients, finite series method, Gaussian beams, Bessel beams*

1. INTRODUCTION

In acoustic scattering by small spherical particles, a monochromatic, arbitrary-shaped incident scalar acoustic velocity potential $\psi_i(r,\theta,\phi)$ is usually expanded into a set of partial waves using spherical wave functions [1–3]:

$$\psi_i(r,\theta,\phi) = \psi_i^0 \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_n^{pw} g_n^m j_n(kr) P_n^{|m|}(\cos\theta) e^{im\phi}. \tag{1}$$

where a spherical coordinate system (r,θ,ϕ) is assumed whose origin $\mathcal{O}_P$ coincides with the center of the spherical scatterer. A time-harmonic factor $\exp(+i\omega t)$, with ω being the angular frequency, will be omitted throughout.

In Eqn. (1), $k = 2\pi/\lambda$ is the wave number (λ is the wavelength in the lossless propagating medium), n and m are integers ($0 \leq n < \infty$, $-n \leq m \leq +n$), ψ_i^0 is the complex field strength, $j_n(.)$ are spherical Bessel functions of the first kind and order n and $P_n^m(.)$ are associated Legendre polynomials with Hobson's convention [4]. The pre-factors $c_n^{pw} = (-i)^n(2n+1)$ ("pw" stands for "plane wave") are such that, for a plane wave and except for a phase factor, $g_n^m = 1$ for $m = 0$, $\forall n$, being 0 otherwise.

The expansion coefficients g_n^m are the beam shape coefficients (BSCs) and embody the spatial information of the wave relatively to a plane wave. They can be extracted explicitly from Eqn. (1) with the aid of orthogonal relations for $\exp(im\phi)$ and $P_n^m(\cos\theta)$ [5]. Such a procedure, called the *quadrature* technique, allows us to write the g_n^m's in terms of double integrals:

$$g_n^m = \frac{1}{4\pi c_n^{pw}} \frac{(n-|m|)!}{(n+|m|)!} \frac{2n+1}{j_n(kr)} \int_0^{2\pi} \int_0^{\pi} \frac{\psi_i(r,\theta,\phi)}{\psi_i^0} \times P_n^{|m|}(\cos\theta) e^{-im\phi} \sin\theta d\theta d\phi. \tag{2}$$

There are, in principle, at least two other techniques which can be introduced to extract analytical expressions

**Corresponding author*: *leo@sc.usp.br.*

for the BSCs of Eqn. (1), both of which have been largely explored for optical fields. The first is an approximate method which relies on the principle of localization of van de Hulst that associates rays to each partial wave, those rays being parallel to the propagation axis and located at specific transverse distances from it. It is known as the *localized approximation* [6, 7], with variants such as the integral localized approximation [8].

The second method, which is the subject of the present work, is the *finite series* (FS) method, first proposed by Gouesbet *et al.* for optical beams (particularly, Gaussian beams) in 1988 [9]. The FS invokes Neumann Expansion Theorem (NET) to provide expressions for the BSCs in terms of a finite series. In doing so, it avoids the integral over the polar coordinate θ, which is usually the most difficult to solve analytically. In addition, whenever the optical beam exactly satisfies Maxwell's equations, the BSCs so derived are exact and the partial wave reconstruction of the fields will exactly reproduce the intended, original electromagnetic wave.

For acoustic beams, both the LA and FS methods have been largely overlooked. To the best of our knowledge, no attempts have ever been made to develop and subsequently justify acoustic versions of the LA from Eqn. (1). Besides, the only attempt at introducing the FS approach for acoustic or ultrasonic beams was made by Zhang *et al.* in 2015 [10]. However, no systematic analysis was performed, and the authors limited themselves to Gaussian-like beams, that is, acoustic fields $\psi_i(r,\theta,\phi)$ which are neither solutions to the homogeneous scalar wave equation nor a solution to its more restricted paraxial version.

Therefore, in this work we formally present the FS method for acoustic fields. The formulas to be presented are valid for any arbitrary-shaped complex velocity potentials and not only to Gaussian-like beams. As an example, we extract the BSCs of arbitrary-order Bessel beams with relative ease, thus rendering the FS method a robust, new and promising technique for describing incoming acoustic/ultrasonic waves in acoustic scattering. The next section presents the mathematical details and computational calculations and simulations of BSCs and corresponding reconstructed fields. Then, our conclusions are presented.

2. THE FS METHOD FOR ACOUSTIC FIELDS AND EXAMPLES FOR BESSEL BEAMS

We start by first considering an equation of the form (see, e.g. Sec. 16.13 of Ref. [11], where we replaced cylindrical by spherical Bessel functions):

$$x^{\frac{1}{2}} g(x) = \sum_{n=0}^{\infty} c_n \sqrt{2x/\pi}\, j_n(x), \tag{3}$$

where x is a real quantity. According to the NET, if the function $g(x)$ in Eqn. (3) can be Maclaurin expanded:

$$g(x) = \sum_{n=0}^{\infty} b_n x^n, \tag{4}$$

then the possibly complex coefficients c_n are written in terms of the coefficients b_n by the following relation:

$$c_n = \frac{2n+1}{2} \sum_{j=0}^{\leq n/2} 2^{\frac{1}{2}+n-2j} \frac{\Gamma(\frac{1}{2}+n-j)}{j!} b_{n-2j}, \tag{5}$$

in which $\Gamma(n)$ is the Gamma function.

To see how one can take advantage of Eqn. (5) to find explicit expressions for the BSCs of Eqn. (1), let us set $x = kr$ and choose a particular value for θ, $\theta = \theta_0$. Multiplying both sides of Eqn. (1) by $\exp(-im'\phi)$ (m' is an integer) and integrating from 0 to 2π, one eventually finds:

$$\begin{aligned} x^{\frac{1}{2}} \int_0^{2\pi} \frac{\psi_i(x,\theta_0,\phi)}{\psi_i^0} e^{-im\phi} d\phi = \\ \sum_{n=0}^{\infty} \left[\pi\sqrt{2\pi} c_n^{pw} g_n^m P_n^{|m|}(\cos\theta_0) \right] \sqrt{2x/\pi}\, j_n(x). \end{aligned} \tag{6}$$

In view of that, if

$$g(x) = \frac{1}{\pi\sqrt{2\pi}} \int_0^{2\pi} \frac{\psi_i(x,\theta_0,\phi)}{\psi_i^0} e^{-im\phi} d\phi, \tag{7}$$

comparison between Eqn. (3)-Eqn. (5) and Eqn. (6) can be shown to lead to

$$\begin{aligned} g_n^m = & \frac{2n+1}{2 c_n^{pw} P_n^{|m|}(\cos\theta_0)} \\ & \times \sum_{j=0}^{\leq n/2} 2^{\frac{1}{2}+n-2j} \frac{\Gamma(\frac{1}{2}+n-j)}{j!} b_{n-2j}. \end{aligned} \tag{8}$$

Therefore, if one can find a function $g(x)$ from $\psi_i(r,\theta,\phi)$ in accordance with Eqn. (7), which involves a single integral over ϕ and that can be Maclaurin expanded, the BSCs associated with this particular acoustic potential field can be readily calculated from Eqn. (8).

In Eqn. (8), the choice of θ_0 can lead to singularities if $P_n^{|m|}(\cos\theta_0) = 0$. A usual choice is $\theta_0 = \pi/2$ [10, 12], for which $P_n^{|m|}(0) \neq 0$ if $n-m$ even, and 0 if $n-m$ is odd. In this case, Eqn. (8) provides only a subset of the full set of BSCs, viz., those for which $n-m$ is even. The other subset ($n-m$ odd) can be found by working with the derivative of $\psi_i(r,\theta,\phi)$ with respect to $\cos\theta$ and using the fact that, for $\theta_0 = \pi/2$, $dP_n^m(\cos\theta)/d\cos\theta \neq 0$ when $n-m$ is odd, being 0 when $n-m$ is even.

So, for $\theta_0 = \pi/2$ (or $\cos\theta_0 = 0$), Eqn. (8) is valid for $n-m$ even. Differentiating Eqn. (1) with respect to $\cos\theta$ and proceeding as before, one then finds the subset of BSCs for $n-m$ odd. Without going into all the details, it can be shown that:

$$g(x) = \frac{1}{\pi\sqrt{2\pi}} \int_0^{2\pi} \frac{1}{\psi_i^0} \frac{d\psi_i(x,\theta,\phi)}{d\cos\theta}\Big|_{\theta=\pi/2} e^{-im\phi} d\phi, \tag{9}$$

$$g_n^m = \frac{2n+1}{2c_n^{pw}\left[dP_n^{|m|}(\cos\theta)/d\cos\theta\right]\Big|_{\theta=\pi/2}} \times \sum_{j=0}^{\leq n/2} 2^{\frac{1}{2}+n-2j} \frac{\Gamma(\frac{1}{2}+n-j)}{j!} b_{n-2j}, \tag{10}$$

with the Maclaurin coefficients b_n extracted from Eq. (9).

As an example, let us consider an ideal v-th order arbitrary order Bessel beam (BB) propagating along $+z$:

$$\psi_i(r,\theta,\phi) = \psi_i^0 J_v(k_\rho r\sin\theta) e^{iv\phi} e^{-ik_z r\cos\theta}, \tag{11}$$

with $k_\rho = k\sin\alpha$ ($k_z = k\cos\alpha$) being the radial (longitudinal) wave numbers, α being the axicon angle. For $(n-m)$ even, inserting Eqn. (11) into Eqn. (7) leads to:

$$g(x) = \sqrt{2/\pi} J_v(x\sin\alpha)\delta_{m,v}, \tag{12}$$

where $\delta_{i,j}$ is the Kronecker delta. The Maclaurin series associated with Eqn. (12) follows from the definition of Bessel functions [5]:

$$J_v(z) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!\Gamma(j+v+1)} \left(\frac{z}{2}\right)^{2j+v}, \tag{13}$$

so that, after setting $n = 2j+v$ in Eqn. (13) and substituting it in Eqn. (12), it can be shown that

$$b_n = \sqrt{\frac{2}{\pi}} \frac{\epsilon(n; v-1)(-1)^{\frac{n-v}{2}}}{\left(\frac{n-v}{2}\right)!\Gamma\left(\frac{n+v+2}{2}\right)} \left(\frac{\sin\alpha}{2}\right)^n \delta_{m,v}, \tag{14}$$

where $\epsilon(n;u) = 0$ for $n \leq u$, 1 otherwise. From Eqn. (14) that, and as already verified using quadratures [13], the only non-zero BSCs in Eqn. (8) are those with $m = v$.

A similar procedure can be applied when $(n-m)$ odd. Without going into the details, one eventually finds that, in this case, the Maclaurin coefficients, Eqn. (14) reads as:

$$b_n = -i\sqrt{\frac{2}{\pi}} \frac{\epsilon(n;v)(-1)^{\frac{n-v-1}{2}}\cos\alpha}{\left(\frac{n-v-1}{2}\right)!\Gamma\left(\frac{n+v+1}{2}\right)} \left(\frac{\sin\alpha}{2}\right)^{n-1} \delta_{m,v} \tag{15}$$

which should be used in the context of Eq. (10).

As an example, let $f = \omega/2\pi = 2.5$ MHz and a BB of order $v = 2$ in water. From the above formulas, we compared Eqn. (1) with Eqn. (11) using the FS method. Results for the field intensity $|\psi(r,\theta,\phi)|^2$ at $z = 0$ are shown in Fig. 1 for $\alpha = 35°$. To truncate the sum in Eqn. (1), we adopted Wiscombe's criterion [14], but lower percentage error can be achieved if we set a higher maximum n, i.e., if we increase the number of partial waves that composes $\psi_i(r,\theta,\phi)$ in Eqn. (1). Tests have been conducted for several positions in space, and it has been verified that the equations here presented agree, in the limit $n \to \infty$ in Eqn. (1), with the values obtained from quadratures [13].

3. CONCLUSIONS

A new method has been proposed for the evaluation of beam shape coefficients in acoustic scattering based on Neumann expansion theorem called the finite series. For velocity potentials which exactly satisfy the scalar homogeneous Helmholtz equation, the coefficients so derived are exact, thus providing an alternative technique beyond quadratures. The finite series can be used with advantage whenever double integrals in the quadrature approach cannot be solved analytically, which might be the case of Laguerre-Gauss beams, finite-energy Bessel and helicoidal beams, and structured beams in general, thus paving the way for advanced investigations and force calculations in acoustic scattering by spherical particles.

4. FUNDING

This work was supported by CNPq (406949/2021-2, 309201/2021-7) and FAPESP (2021/06121-0), Brazil.

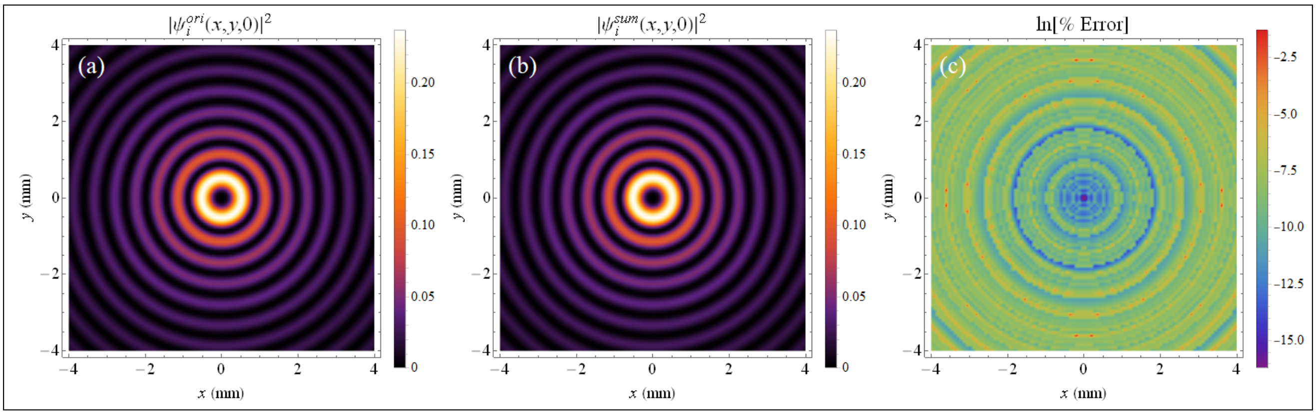

Figure 1. (a) Original $[|\psi_i^{ori}(r,\theta,\phi)|^2)$, Eqn. (11)] and (b) reconstructed $[|\psi_i^{sum}(r,\theta,\phi)|^2$, Eqn. (1)] field intensities at the xy plane ($z = 0$) for a 2nd order BB with $\alpha = 35°$, $f = 2.5$ MHz, in water (speed of sound of 1540 m/s). (c) Percent error (in logarithmic scale) using Wiscombe's criterion to truncate Eqn. (1).

5. REFERENCES

[1] J. J. Faran, "Sound scattering by solid cylinders and spheres," *The Journal of the Acoustical Society of America*, vol. 23, no. 4, pp. 405–418, 1951.

[2] G. T. Silva, "An expression for the radiation force exerted by an acoustic beam with arbitrary wavefront (L)," *The Journal of the Acoustical Society of America*, vol. 130, no. 6, pp. 3541–3544, 2011.

[3] D. Baresch, J.-L. Thomas, and R. Marchiano, "Three-dimensional acoustic radiation force on an arbitrarily located elastic sphere," *The Journal of the Acoustical Society of America*, vol. 133, no. 1, pp. 25–36, 2013.

[4] L. Robin, *Fonctions Sphériques de Legendre et Fonctions Sphéroïdales, vol. 1, 2 3*. Paris, France: Gauthier-Villars, 1957.

[5] G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists*. Burlington, MA, USA: Harcourt/Academic Press, 2001.

[6] G. Gréhan, B. Maheu, and G. Gouesbet, "Scattering of laser beams by Mie scatter centers: numerical results using a localized approximation," *Appl. Opt.*, vol. 25, pp. 3539–3548, Oct 1986.

[7] G. Gouesbet and G. Gréhan, *Generalized Lorenz-Mie Theories*. Springer International Publishing, 2 ed., 2017.

[8] K. F. Ren, G. Gouesbet, and G. Gréhan, "Integral localized approximation in generalized Lorenz–Mie theory," *Appl. Opt.*, vol. 37, pp. 4218–4225, Jul 1998.

[9] G. Gouesbet, G. Gréhan, and B. Maheu, "Expressions to compute the coefficients g_n^m in the generalized Lorenz-Mie theory using finite series," *Journal of Optics*, vol. 19, pp. 35–48, jan 1988.

[10] X. Zhang, Z. Song, D. Chen, G. Zhang, and H. Cao, "Finite series expansion of a Gaussian beam for the acoustic radiation force calculation of cylindrical particles in water," *The Journal of the Acoustical Society of America*, vol. 137, no. 4, pp. 1826–1833, 2015.

[11] G. N. Watson, *A Treatise on the Theory of Bessel Functions*. Cambridge, UK: Cambridge University Press, 1944.

[12] G. Gouesbet, B. Maheu, and G. Gréhan, "Light scattering from a sphere arbitrarily located in a Gaussian beam, using a Bromwich formulation," *J. Opt. Soc. Am. A*, vol. 5, pp. 1427–1443, Sep 1988.

[13] Z. Gong, P. L. Marston, and W. Li, "T-matrix evaluation of three-dimensional acoustic radiation forces on nonspherical objects in Bessel beams with arbitrary order and location," *Phys. Rev. E*, vol. 99, p. 063004, Jun 2019.

[14] W. J. Wiscombe, "Improved Mie scattering algorithms," *Appl. Opt.*, vol. 19, pp. 1505–1509, May 1980.