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PROGRESS IN MODELING THE STATISTICAL DISTRIBUTIONS OF RANDOMLY SCATTERED SOUND

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ABSTRACT

Statistical distributions for the amplitude, intensity, and phase of randomly scattered sound are important in a variety of problems involving noise characterization, detection, communication, beamforming, and remote sensing. This paper discusses some recent progress in modeling the distributions of scattered sound and the situations to which they apply. In particular, several extensions to the gamma distribution are described: the *compound gamma* for signals that have been scattered with randomly varying strength, the *variance gamma* for the complex products (covariances and cross spectra) between pairs of sensors, and the *compound variance gamma* for signals at pairs of sensors with varying scattering strength. Furthermore, a new joint amplitude-phase distribution, termed the *phase-modulated Rice*, is described, which is appropriate for signals that are scattered by inhomogeneities spanning a broad range of spatial scales as occurs in atmospheric turbulence. This distribution employs the basic (unmodulated) Rice distribution for scattering by relatively small (Fresnel-zone scale) turbulent eddies, while the phase is modulated with a von Mises distribution to represent the impact of relatively strong large-scale turbulence.

Keywords: *scattering, turbulence, phase modulation, variance gamma distribution, Rice distribution*

1. INTRODUCTION

Randomization of the amplitude, intensity, and phase of acoustic signals by the environment is important in a variety of problems involving noise characterization, detection, communication, beamforming, and remote sensing. The randomization results from scattering and multipath effects caused by turbulence in the atmosphere and ocean, ground and ocean bottom roughness, buildings in an urban environment, trees in a forest, and ocean surface waves. An example of these effects in an urban-like scenario is provided in Figure 1, which shows a finite-diffrence time-domain (FDTD) calculation involving interactions of sound waves with buildings. Similarly, scattering of sound by atmospheric turbulence is illustrated in Figure 2, which shows a parabolic equation (PE) calculation involving upwind propagation over the ground. (Details on the FDTD method used to calculate Figure 1 can be found in Chapter 12 of Ostashev and Wilson [1]; details on the PE method used for Figure 2 are in Chapter 11.)

Many statistical models have been used to describe randomly scattered signals, including the gamma and Nakagami distributions for fully saturated (zero-mean) signals [2–5], the Rice distribution for weakly scattererd (non-zero mean) signals [3,6], and the K-distribution [7,8] for signals subject to random variability in the scattering strength.

This paper reviews some recent progress in modeling of distributions for scattered sound and other applications involving randomized signals. In Sec. 2, distributions for signals at a single sensor are discussed, first in the limit of full saturation (complete randomization), for which the basic gamma distribution is applies, and then incomplete saturation, for which the noncentral form of the gamma distribution is needed. Then, in Sec. 3, distributions for the complex products (covariances and cross spectra) between *pairs* of sensors are discussed. In particular, the *variance gamma* distribution is introduced for modeling of the complex product in full saturation conditions.

Lastly, in Sec. 4, the concept of modulation is dis-

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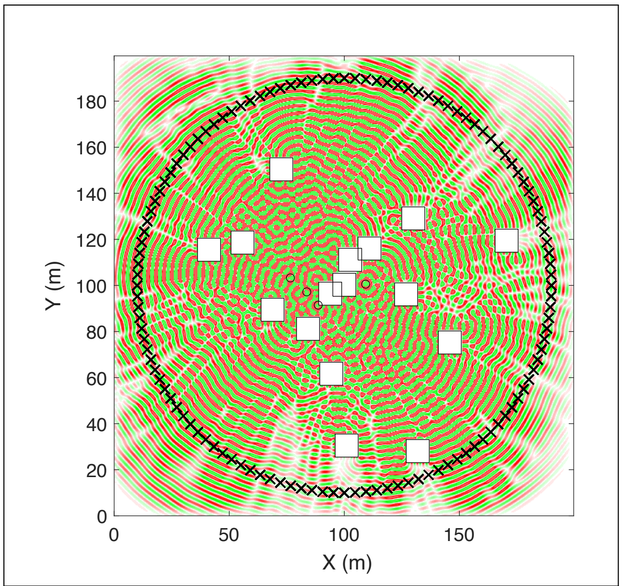

Figure 1. Two-dimensional, finite-diffrence time-domain simulation of sound propagation at 100 Hz, with 16 randomly placed "buildings" (solid squares 10 m on a side) within a 200 m by 200 m domain. Red is positive pressure; green is negative. A ring of 120 receivers (the x's) is placed at a fixed radius of 80 m from the source.

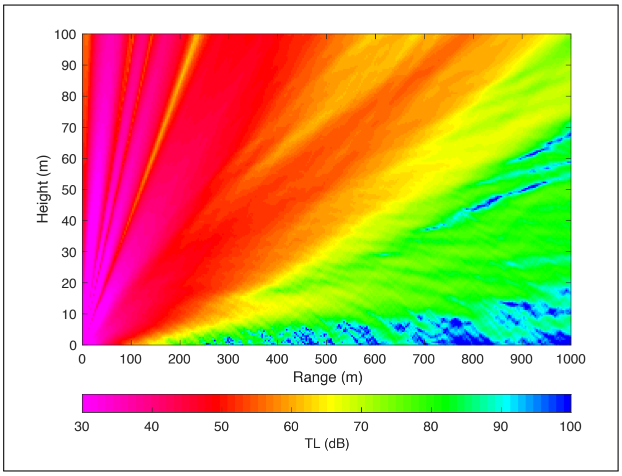

Figure 2. Parabolic equation calculation for upwind propagation in moderately windy, turbulent conditions (friction velocity of $u_* = 0.3$ m/s). The source height is 1.5 m and the frequency is 400 Hz.

cussed. The basic idea is to treat the parameters of a baseline distribution for the scattered signal as random variables, in order to account for variations in the properties of the environment. *Amplitude modulation* can be useful for modeling the intermittency of turbulence; i.e., the variability of turbulence strength in space and time. It could also be useful for modeling the impact of varations in size and density of buildings in an urban landscape. This leads to the *compound gamma* for signals at a single sensor, and the *compound variance gamma* for signals at pairs of sensors. *Phase modulation* can be useful for signals that are scattered by inhomogeneities spanning a broad range of spatial scales. By introducing phase modulation into the Rice model, the relatively large phase variance of signals scattered by large-scale turbulence can be captured.

2. SINGLE RECEIVER DISTRIBUTIONS

The framework described in this paper applies to situations in which a complex-valued harmonic signal (the complex envelope or *phasor* of a narrowband filtered signal) Z has independent, normally distributed real (X) and imaginary (Y) parts with equal variance. Indicating a normal distribution with mean ν and variance σ^2 as $\mathcal{N}(\nu, \sigma^2)$, we have $X \sim \mathcal{N}(\nu_x, \sigma^2)$ and $Y \sim \mathcal{N}(\nu_y, \sigma^2)$. The power (or intensity) is proportional to $|Z|^2 = ZZ^* = X^2 + Y^2$, where the asterisk indicates the complex conjugate. The incoherent sum of k independent samples of the power is $\zeta = \sum_{j=1}^{k} Z_j Z_j^* = \sum_{j=1}^{k}(X_j^2 + Y_j^2)$, where j is an index indicating the sample.

The primary objective of this paper is to review distributions for ζ for various scenarios. Although details of the derivations are not provided due to space limiitations, the starting point is generally the following joint probability density function (pdf) for X and Y:

$$f_{XY}(x, y|\nu, \psi, \sigma^2) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{(x-\nu_x)^2 + (y-\nu_y)^2}{2\sigma^2}\right). \quad (1)$$

Here, $\psi = \mathrm{atan2}(\nu_y, \nu_x)$. We follow the convention of indicating random variables in upper case and functional values related to those variables in lower case. The arguments of the pdf to the left of the vertical line are the values of the random variables; the arguments to the right are the parameters upon which they depend. For k independent samples, the joint pdf would be a product of pdfs in the form of Eq. (1).

2.1 Fully saturated scattering (gamma or Nakagami)

Full saturation corresponds to setting $\nu_x = \nu_y = 0$. In this case, ζ is simply the sum of the squares of $2k$ independent, normally distributed random variables with zero mean and variance σ^2. It is well known in the statistics literature that such a sum has a gamma distribution with shape factor k and scale parameter $\theta = 2\sigma^2$ [4]. The pdf for the gamma distribution is

$$f_\Gamma(\zeta|k,\theta) = \frac{1}{\Gamma(k)\theta^k}\zeta^{k-1}e^{-\zeta/\theta} \tag{2}$$

where $\Gamma(k)$ is the gamma function. When $k = 1$, the gamma distribution reduces to the exponential. The exponential and gamma distributions are familiar in the literature on wave scattering and random noise [3, 5], although they are commonly transformed from signal power to amplitude (which is proportional to $\sqrt{\zeta}$), thereby yielding the Rayleigh and Nakagami distributions [2], respectively. The mean of the gamma distribution is $k\theta = 2k\sigma^2$, whereas the variance is $k\theta^2 = 4k\sigma^4$. The scintillation index, which is often used to describe scatterred signal behavior, is defined as $S_I^2 = \langle\zeta^2\rangle/\langle\zeta\rangle^2 - 1 = \langle(\zeta - \langle\zeta\rangle)^2\rangle/\langle\zeta\rangle^2$. Hence, S_I^2 equals the variance normalized by the squared mean; for the gamma distributon, $S_I^2 = 1/k$.

Strictly speaking, Eq. (2) is called an order $d = 2k$ Erlang distribution, where d is the degrees of freedom. From a mathematical perspective, however, there is no reason why k cannot be a positive non-integer value, and there may be situations where this is useful. In particular, for unsaturated signals, k is sometimes set to a value greater than one, which can provide a reasonable model for weak scattering cases such that S_I^2 less than one [9]. But, there is a more formal approach to modeling unsaturated signals as described in the following subsection.

2.2 Unsaturated scattering (noncentral gamma or Rice)

From Eq. (1), we find the joint pdf for the amplitude $A = \sqrt{X^2+Y^2}$ and the phase $\Phi = \arctan(Y/X)$ by setting $f_{A\Phi}(a,\phi) = [|J_{a,\phi}(x,y)|f_{XY}(x,y)]_{a,\phi}$, where $J_{a,\phi}(x,y) = a$ is the Jacobian. The result is

$$f_{A\Phi}(a,\phi|\nu,\psi,\sigma^2) = \frac{a}{2\pi\sigma^2}\exp\left(\frac{2a\tilde{\nu} - a^2 - \nu^2}{2\sigma^2}\right), \tag{3}$$

where $\nu^2 = \nu_x^2 + \nu_y^2$ and $\tilde{\nu} = \nu_x \cos\phi + \nu_y \sin\phi = \nu(\cos\psi\cos\phi + \sin\psi\sin\phi) = \nu\cos(\phi - \psi)$.

We call this formulation with non-zero ν_x and ν_y the *Rice model*. The amplitude distribution, or *Rice distribution*, is obtained by marginalizing (integrating) the joint pdf, Eq. (3), over the phase, with result

$$f_A(a|\nu,\sigma^2) = \frac{a}{\sigma^2}\exp\left(-\frac{a^2+\nu^2}{2\sigma^2}\right) I_0\left(\frac{a\nu}{\sigma^2}\right). \tag{4}$$

Here, I_d is the modified Bessel function of the first kind, order d. The power (amplitude squared) distribution for the Rice model is obtained by setting $f(\zeta) = [|da/d\zeta|f_A(a)]_\zeta$, with $\zeta = a^2$, leading to the order 2 *noncentral* Erlang distribution. Extension to the sum of k independent samples of the power yields the order $2k$ noncentral Erlang distribution [5]:

$$f_{N\Gamma}(\zeta|k,\nu,\sigma^2) = \frac{1}{2\sigma^2}\exp\left(-\frac{\zeta+\nu^2}{2\sigma^2}\right) \times \left(\frac{\zeta}{\nu^2}\right)^{(k-1)/2} I_{k-1}\left(\frac{\nu\sqrt{\zeta}}{\sigma^2}\right). \tag{5}$$

The mean of this distribution is $2k\sigma^2 + \nu^2$ and the variance is $4\sigma^2(k\sigma^2 + \nu^2)$. The scintillation index $S_I^2 = (k + \nu^2/\sigma^2)/(k + \nu^2/2\sigma^2)^2$. Defining the Rice factor as $R = \nu^2/2\sigma^2$ (which represents the ratio of the powers in the steady and varying parts of the signal), the mean is $2\sigma^2(R + k)$, the variance is $4\sigma^4(2R + k)$, and $S_I^2 = (2R + k)/(R + k)^2$. As discussed in connection with fully saturated signals, mathematically k can be set to a non-integer value. With this in mind, we call Eq. (5) the *noncentral* gamma distribution. It reduces to the ordinary (central) gamma distribution in the limit $\nu \to 0$. From a physical perspective, however, there does not appear to be a motivation for setting k to a non-integer value, since unsaturated scattering can be modeled using $R > 0$.

Figure 3 plots the noncentral gamma distribution for various values of R and k. The parameter $2\sigma^2$ is set to $1/(R + k)$ and ν^2 to $R/(R + k)$, thus yielding a mean of one. Increasing R and k both result in a distribution that is more peaked around the mean.

The phase distribution for the Rice model follows by marginalzing Eq. (3) over amplitude, which results in [10]

$$f_\Phi(\phi|R,\psi) = \frac{1}{2\pi}e^{-R}\Big[1 + \sqrt{R\pi}\cos(\phi - \psi) \times \exp\left(R\cos^2(\phi - \psi)\right)\operatorname{erfc}\left(-\sqrt{R}\cos(\phi - \psi)\right)\Big]. \tag{6}$$

2.3 Variable source-receiver separation

The discussion of signal distributions has so far focused on random variations driven by the spatial or temporal

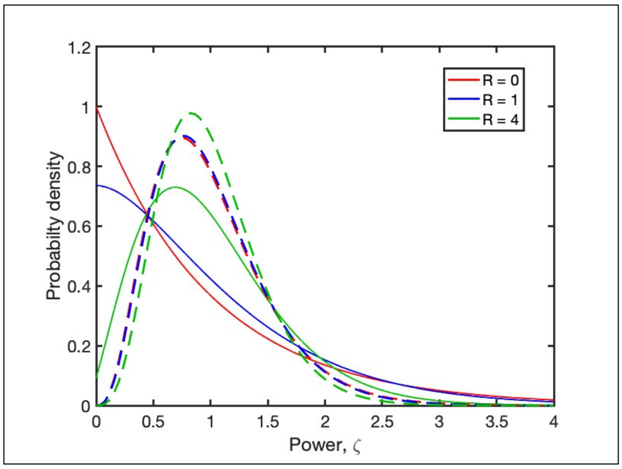

Figure 3. Noncentral gamma distributions for various values of the Rice factor R and number of incoherent samples k. Different values of R are shown as different colors. Increasing R corresponds to a stronger deterministic component to the signal. Solid lines are for $k = 1$ and dashed lines were $k = 4$.

structure of the propagation medium. But variations can also occur due to changes in the number of sources and in the propagation geometry between the source and receiver. This occurs, for example, when a microphone is situated at a roadside with passing vehicle traffic. Wilson et al. [5] examined a model where the receiver is fixed at the middle of a circle, and multiple sources are randomly distributed within the circle. The distribution of the *sound level* (logarithm of the power) was shown to be well fit by an *exponentially modfied Gaussian* (EMG) distribution [11], which represents the sum of two random variables, one exponentially distributed and the other normally distributed. The exponential part is associated with the single source that is closest to the receiver, whereas the normal part is associated with the remaining, more distant sources. This result helps to explain why sound levels in urban environments often appear to be approximately normally distributed, but with an exponential tail to the right of the peak (i.e., positive skewness).

3. MULTIPLE RECEIVER DISTRIBUTIONS

This section considers extensions of the Erlang (gamma) distribution to two or more receivers, for fully saturated signals. Results are not presently known for unsaturated signals.

3.1 Variance gamma

We next consider the product $\zeta = \sum_{j=1}^{k} Z_j (Z_j')^*$ where Z_j and Z_j' represent samples at two different points, which may be separated in space and time. This complex product represents, for example, a complex covariance or a cross spectral density estimate between a pair of receivers.

Suppose the two signals are fully saturated but with differing variances, i.e. $\langle X^2 \rangle = \langle Y^2 \rangle = \sigma^2$ and $\langle (X')^2 \rangle = \langle (Y')^2 \rangle = (\sigma')^2$. The correlation coefficient ρ is defined such that $\rho = \langle XX' \rangle / (\sigma\sigma') = \langle YY' \rangle / (\sigma\sigma')$. Since the phase is uniformly distributed in full saturation, the cross products $\langle XY \rangle$ and $\langle X'Y' \rangle$ must be zero. Based on these assumptions, Wilson et al. [12] showed that the real and imaginary parts of the two-point product, $\zeta_r = \sum_{j=1}^{k} (X_j X_j' + Y_j Y_j')$ and $\zeta_i = \sum_{j=1}^{k} (Y_j X_j' - X_j Y_j')$, respectively, are both described by variance gamma (VΓ) distributions. Namely, $\zeta_r \sim \mathrm{V}\Gamma\,(k, \rho, 2\sigma\sigma')$ and $\zeta_i \sim \mathrm{V}\Gamma\left(k, 0, 2\sigma\sigma'\sqrt{1-\rho^2}\right)$, where

$$f_{\mathrm{V}\Gamma}(\zeta|k,\rho,\theta) = \frac{2}{\sqrt{\pi}\Gamma(k)} \frac{|\zeta|^{k-\frac{1}{2}}}{\sqrt{1-\rho^2}\theta^{k+\frac{1}{2}}} e^{\frac{2\rho\zeta}{\theta(1-\rho^2)}} \times K_{k-\frac{1}{2}}\left(\frac{2|\zeta|}{\theta(1-\rho^2)}\right), \quad (7)$$

in which $K_\nu(x)$ is the modified Bessel function of the second kind, order ν. The mean of the VΓ distribution is $k\rho\theta$ and the variance is $k(1+\rho^2)\theta^2/2$. Using the asymptotic expansion for the modified Bessel function, it can be readily shown that the VΓ distribution for the real part reduces to the gamma in the limit $\rho \to 1$, whereas the imaginary part vanishes in this limit. Figure 4 plots the VΓ distribution for various values of ρ, with θ and k both set to one. The solid lines are for the real part, dashed lines are for the imaginary part.

3.2 Complex Wishart and matrix gamma

Next consider the case of N receivers. Let $Z_{nj} = X_{nj} + iY_{nj}$, where Z_{nj} is the jth sample of the complex signal at receiver n. Furthermore, define vectors of the signals for the sample j as $\mathbf{Z}_k = [Z_{1j}, Z_{2j}, \ldots, Z_{Nj}]$, and similarly for $\mathbf{X}_j$ and $\mathbf{Y}_j$. We then define

$$\mathbf{S} = \sum_{j=1}^{k} \mathbf{Z}_j \tilde{\mathbf{Z}}_j. \quad (8)$$

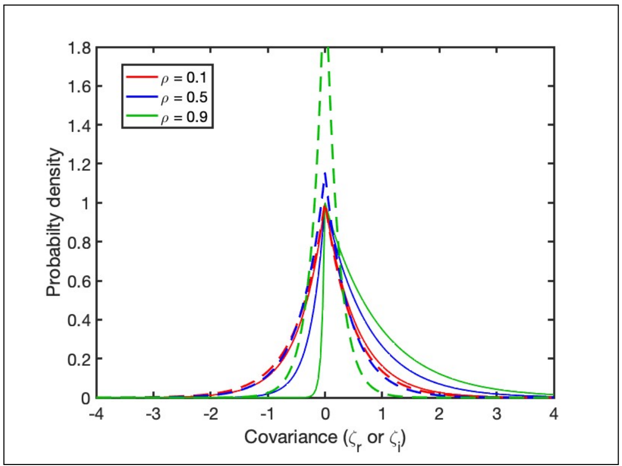

Figure 4. Variance gamma distributions for various values of the correlation coefficient ρ with $k = \theta = 1$. Solid lines are for the real part and dashed liines are for the imaginary part.

The matrix $\mathbf{S}$, which is $N \times N$, generalizes the variable ζ to the multivariate case. In the literature, $\mathbf{S}$ is often termed the *scatter matrix*, although the usage of the term *scatter* in this context differs from its usage in wave scattering.

When $\mathbf{X}_k$ and $\mathbf{Y}_k$ are given by multivariate normal distributions, $\mathbf{S}$ has a *complex Wishart distribution*, as first derived by Goodman [13]. Maiwald and Kraus [14] provide an accessible introduction from the perspective of signal processing. The complex Wishart distribution has the following form:

$$f_{\mathrm{CW}}(\mathbf{S}|k, \mathbf{V}) = \frac{|\mathbf{S}|^{k-N}}{\Gamma_N(k)|\mathbf{V}|^k} \exp\left[-\mathrm{tr}(\mathbf{V}^{-1}\mathbf{S})\right], \quad (9)$$

where

$$\Gamma_N(k) = \pi^{N(N-1)/2} \prod_{n=1}^{N} \Gamma(k - n + 1) \quad (10)$$

and $\mathbf{V} = \langle \mathbf{Z}\tilde{\mathbf{Z}} \rangle$ is the $N \times N$ covariance matrix.

The complex Wishart distribution is a multivariate generalization of the Erlang distribution. Although it is defined for integer k, in principle k can be set to a non-integer value in Eq. (9), just as k in the Erlang can be set to a non-integer value, thus leading to a gamma distribution. For this reason, we call the non-integer generalization of Eq. (9) the *matrix gamma* distribution [15]. As with the single variate gamma distribution, non-integer k values larger than one may provide a useful approximation for weak scattering.

Due to the statistical assumptions underlying the derivation of the matrix gamma distribution, the marginals of the diagonal elements must be gamma-distributed, whereas the off-diagonal elements are VΓ-distributed.

4. MODULATED DISTRIBUTIONS

In this section, we consider distributions resulting when parameters of the previously described distributions are randomly varied, or *modulated*. This can be a useful modeling approach when a larger-scale process randomly perturbs a smaller scale scattering process. This may happen, in particular, when environmental variability occurs on scales much larger than the size of the inhomogeneities that dominate the scattering, as charactereized by the Fresnel zone. The environmental variability may impact the amplitude or phase of the signal, or both.

The starting point for modeling modulated distributions is a joint pdf of the amplitude and phase of the signal, such as Eq. (3). Let us write the joint pdf as $f_{A\Phi}(a, \phi|\eta, \xi)$, where η represents one or more fixed parameters, whereas ξ represents one or more parameters that vary in response to changes in the environment. In Eq. (3), for example, the parameters ν, θ, and σ^2 would each be assigned to one of these sets. We conceptualize the parameters ξ as a random variable Ξ, for which a new pdf $f_\Xi(\xi|\gamma)$ is specified, in which γ represents one or more statistical parameters upon which Ξ depends. Based on this approach the integral for the marginalized pdf is

$$f_{A\Phi}(a, \phi|\eta, \gamma) = \int f_{A\Phi}(a, \phi|\eta, \xi) f_\Xi(\xi|\gamma)\, d\xi. \quad (11)$$

This is known as a compound pdf integral.

4.1 Amplitude modulation

Amplitude modulation can be caused by random variations in the environment that lead to varying strength of the scattering, such as intermittent variations in turbulent activity, or variations in the size and density of buildings. For fully saturated scattering, it would be natural to modulate the scale parameter $\theta = 2\sigma^2$ in a gamma distribution. Jakeman and Pusey [7], and Andrews and Phillips [8], used a second gamma pdf for the modulation, resulting in the *K-distribution*. Alternatiively, the *inverse gamma* (IΓ) distribution, which is the Bayesian conjugate prior of the gamma, can be used for the modulation. This results in a relatively simpler pdf called the *compound gamma* (CΓ) distribution [16]. The CΓ is found by solving the following integral, which is based on Eq. (11):

$$f_{\mathrm{C}\Gamma}(\zeta|k,\alpha,\beta) = \int_0^\infty f_\Gamma(\zeta|k,\theta) f_{\mathrm{I}\Gamma}(\theta|\alpha,\beta)d\theta. \quad (12)$$

Details of the derivation can be found in Ref. [16]. The result is

$$f_{\mathrm{C}\Gamma}(\zeta|k,\alpha,\beta) = \frac{\Gamma(\alpha+k)}{\Gamma(\alpha)\Gamma(k)} \frac{(\zeta/\beta)^{k-1}}{\beta(1+\zeta/\beta)^{\alpha+k}}. \quad (13)$$

Here, α and β are the shape and scale parameters of the IΓ pdf, which are referred to as the *hyperparameters* in this context. The mean of the CΓ is $k\beta/(\alpha-1)$ (for $\alpha > 1$) and the variance is $\beta^2 k(k+\alpha-1)/[(\alpha-1)^2(\alpha-2)]$ (for $\alpha > 2$). Increasing α corresponds to *decreasing* scattering intermittency; the limit $\alpha \to \infty$ corresponds to the ordinary gamma distribution. Figure 5 plots the CΓ distribution for various values of α and k. The scale parameter β is set to $(\alpha-1)/k$, thus producing a mean of one.

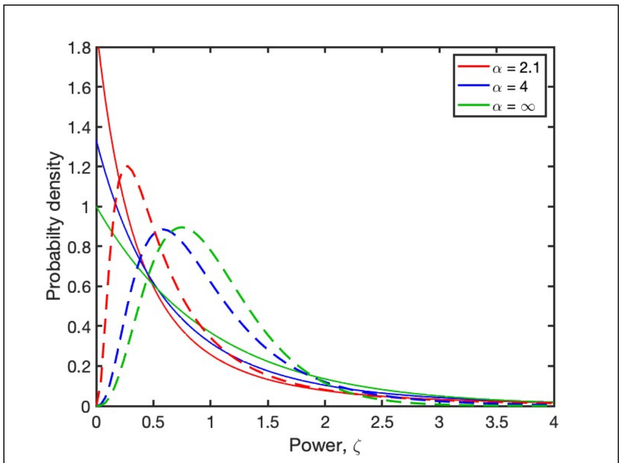

Figure 5. Compound gamma distributions for various values of the shape factor α and number of incoherent samples k. Different values of α are shown as different colors. Increasing α corresponds to *decreasing* scattering intermittency. Solid lines are for $k = 1$ and dashed lines are $k = 4$.

For radio-frequency scattering, Gurvich and Kukharets [17] modulated θ in an exponential pdf (the gamma pdf with $k = 1$) with a log-normal distribution. This approach was adopted by Wilson et al. [18] for acoustic scattering. Churnside and Clifford [19] modulated a Rice distribution with a log-normal distribution to model optical scintillations. The log-normal pdf is a natural choice because it is commonly used to representat intermittent turbulence. However, for weak intermittency, the log-normal, gamma, and inverse gamma distributions are all well approximated by a normal distribution and hence effectively equivalent. Comparisons of the compound gamma distribution to realistic simulations of sound propagation through the atmosphere, using parabolic equation (PE) methods, demonstrate excellent agreement in a range of scenarios involving turbulent scattering, refraction, and ground reflections [12]. A similar PE study for outdoor sound propagation was also performed recently by Renterghem et al. [20], although a direct comparison is not possible because the data were presented as sound levels (logarithms of ζ).

In the same manner as the gamma distribution, the parameter $\theta = 2\sigma\sigma'$ in the VΓ distribution can be modulated with an IΓ distribution. This results in the compound variance gamma (CVΓ) distribution [21]:

$$\begin{aligned} f_{\mathrm{CV}\Gamma}(\zeta|k,\rho,\alpha,\beta) &= \frac{4^k \alpha\beta^\alpha (1-\rho^2)^{k+\alpha}|\zeta|^{2k-1}}{((1-\rho^2)\beta + 2|\zeta| - 2\rho\zeta))^{2k+\alpha}} \\ &\times \frac{\Gamma(2k+\alpha)}{\Gamma(k+\alpha+1)\Gamma(k)} F\Big(2k+\alpha, k, k+\alpha+1, \\ &\qquad \frac{(1-\rho^2)\beta - 2|\zeta| - 2\rho\zeta)}{(1-\rho^2)\beta + 2|\zeta| - 2\rho\zeta)}\Big). \quad (14) \end{aligned}$$

where F is the Gauss hypergeometric function. The mean of the CVΓ is $(\rho k\beta)/(\alpha-1)$ and the variance is $\{k\beta^2\left[(1+\rho^2)(\alpha-1)+2\rho^2 k\right]\}/\{2(\alpha-1)^2(\alpha-2)\}$.

4.2 Phase modulation

The phase distribution predicted by the basic Rice model, Eq. (6), significantly underpredicts the phase variance observed for sound propagation through atmospheric turbulence [10]. The likely reason is that turbulence in the atmospheric boundary layer spans a very broad range of spatial scales, from centimeters to hundreds of meters. In such scenarios, the large-scale turbulence drives strong phase variations while having relatively little impact on amplitude. To address this shortcoming, the basic Rice model can be extended to include a random modulation in the signal phase.

The large-scale phase modulation can be represented with a pdf $f_\Psi(\psi|\gamma_\psi)$, where γ_ψ indicates one or more statistical parameters upon which Ψ depends. Based on this approach, the compound pdf for the overall process is

$$f_{A\Phi}(a,\phi|\nu,\sigma^2,\gamma_\psi) = \int f_{A\Phi}(a,\phi|\nu,\psi,\sigma^2) f_\Psi(\psi|\gamma_\psi) d\psi, \quad (15)$$

where Ψ is the randomized phase angle averaged over the long time scale. The von Mises distribution is a convenient and flexible choice for $f_\Psi(\psi|\gamma_\psi)$. It is given by

$$f_{\mathrm{VM}}(\psi|\kappa) = \frac{1}{2\pi I_0(\kappa)} \exp(\kappa\cos\psi). \quad (16)$$

Here κ is a measure of concentration: the larger the value of κ, the smaller the angular variance. When $\kappa \ll 1$, the von Mises distribution becomes uniform over the interval $(-\pi,\pi]$. The phase distribution for the signal is found by marginalizing Eq. (15) over amplitude, with result [10]

$$f_\Phi(\phi|\nu,\sigma^2,\kappa) = \frac{\exp\left(-\frac{\nu^2}{2\sigma^2}\right)}{2\pi\sigma^2 I_0(\kappa)} \times \int_0^\infty a e^{-\frac{a^2}{2\sigma^2}} I_0\left(\sqrt{\kappa^2 + \frac{2\kappa\nu a\cos\phi}{\sigma^2} + \frac{\nu^2 a^2}{\sigma^4}}\right) da. \quad (17)$$

Although no analytical solution to this integral is avalable, it can be readily evaluated numerically [10]. Note that the phase modulation does not impact amplitude distribution.

5. CONCLUSION

Many different statistical distributions for the amplitude, intensity, and phase of randomly scattered signals have been proposed and studied in recent decades. Research on this topic a spans a variety of applications in acoustics, optics, radio-frequency communications, and other fields. This paper endeavored to provide a broad overview that helps to draw the connections among some of the commonly used various distributions. It also endeavored to address significant gaps with regard to the best approaches for modeling the impacts of environmental variability and for the distributions at multiple sensors. In particular, several extensions to the gamma distribution were discussed: the *compound gamma* for signals that have been scattered with randomly varying strength, the *variance gamma* for the complex products (covariances and cross spectra) between pairs of sensors, and the *compound variance gamma* for signals at pairs of sensors with varying scattering strength. Furthermore, the *phase-modulated Rice* was described for signals scattered by inhomogeneities spanning a broad range of spatial scales as occurs in atmospheric turbulence.

6. ACKNOWLEDGMENTS

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