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PARABOLIC EQUATIONS FOR MOTIONLESS AND MOVING INHOMOGENEOUS MEDIA AND THEIR SOLUTION

Vladimir E. Ostashev[1] **Jules Colas**[2] **Didier Dragna**[2] **D. Keith Wilson**[1]

[1] U.S. Army Engineer Research and Development Center, Hanover, NH 03755-1290

[2] Univ Lyon, École Centrale de Lyon, CNRS, INSA Lyon, LMFA, UMR5509, 69130, Ecully, France

ABSTRACT

Parabolic equations are among most popular numerical techniques in many fields of physics including atmospheric and ocean acoustics. This article considers extra-wide-angle, wide-angle, and narrow-angle parabolic equations (EWAPE, WAPE, and NAPE, respectively) that are valid for sound propagation in motionless and moving inhomogeneous media, and with arbitrary variations in the sound speed and arbitrary (subsonic) Mach numbers. Within the ranges of their applicability, these parabolic equations exactly describe the phase of the sound waves and are therefore termed the *phase-conserving* EWAPE, WAPE, and NAPE. On the other hand, WAPEs and NAPEs from the literature are valid for low Mach numbers and/or small variations in sound speed; they correctly describe the phase of a sound wave only within these approximations. Although the variations in sound speed and Mach number are often relatively small, omitting second-order small terms pertinent to these quantities can result in large cumulative phase errors for long propagation ranges. Therefore, the phase-conserving EWAPE, WAPE and NAPE can be preferable in applications. Numerical implementation of the latter two equations can be done with minimal modifications of existing PE codes.

Keywords: *Parabolic equations, motionless and moving media*

**Corresponding author*: *vladimir.ostashev@colorado.edu.*

1. INTRODUCTION

In atmospheric acoustics, the narrow-angle parabolic equation (NAPE) and wide-angle parabolic equation (WAPE) are very efficient numerical techniques that can handle many phenomena such as stratification and refraction, scattering by turbulence and other inhomogeneities, ground impedance, and propagation over slowly varying terrain [1, 2]. NAPEs and WAPEs are also used in ocean acoustics [3], nonlinear acoustics [4], and other fields such as electromagnetic and seismic wave propagation.

Section 11.2 in Ref. [2] suggests considering WAPEs in the high-frequency (short-wavelength) approximation when the derivatives of the sound speed, density, and medium velocity can be omitted. The resulting WAPEs become much simpler than the previously used equations [5, 6] and can still be used in many applications. Ostashev et al. [7] build on this approach and formulate new EWAPEs (extra-wide-angle parabolic equations or one-way equations), WAPEs, and NAPEs in the high-frequency approximation.

In particular, Ref. [7] derives the EWAPE [given by Eq. (B1)] that is applicable to arbitrary variations in the sound speed and arbitrary (subsonic) Mach numbers, but considers this equation only very briefly in Appendix B. The goal of the present article is to analyze this EWAPE and the WAPE and NAPE which can be obtained from this equation in detail. The results obtained are pertinent to sound propagation in both motionless and moving inhomogeneous media.

The EWAPE, WAPE, and NAPE considered in this article can be preferable to those used in the literature because the latter equations are valid for low Mach numbers and/or small variations in the sound speed. Although the sound-speed variations and Mach numbers are often relatively small, omitting second-order small terms pertinent to these quantities might result in significant cumulative phase errors for long propagation ranges (see Ref. [7] for details). Furthermore, within the ranges of their applicability, the EWAPE, WAPE, and NAPE considered in the present article exactly describe the phase of a sound wave and are termed the *phase-conserving* parabolic equations. On the other hand, WAPEs and NAPEs from the literature, correctly describe the phase of a sound wave only for low Mach numbers and/or small variations in the sound speed.

This article is organized as follows. Section 2 considers the phase-conserving EWAPE, WAPE, and NAPE. In Sec. 3, numerical implementation of the WAPE and NAPE is outlined and numerical results are presented. The results are summarized in Sec. 4.

2. PHASE-CONSERVING EWAPE, WAPE, AND NAPE

2.1 Parabolic equations

Reference [7] expresses the sound pressure p of a monochromatic sound wave in terms of the auxiliary function ϕ:

$$p(\mathbf{R}) = \sqrt{\frac{\varrho}{\varrho_0}}\left(1 + \frac{i}{\omega}\mathbf{v}\cdot\nabla\right)\phi(\mathbf{R}). \tag{1}$$

Here, $\mathbf{R} = (x, y, z)$ are the Cartesian coordinates, $\mathbf{v}(\mathbf{R})$ is the medium velocity, $\varrho(\mathbf{R})$ and ϱ_0 are the density and its reference value, and ω is the frequency of the sound wave. The auxiliary function satisfies the convective Helmholtz equation, Eq. (28) in Ref. [7]. Starting with this equation and omitting the derivatives of the sound speed c and medium velocity $\mathbf{v}$, the following EWAPE is derived

$$\left(\frac{\partial}{\partial x} + ik_c\hat{\tau} - ik_c\gamma_x^2\sqrt{1+\hat{\mu}+\hat{\chi}}\right)\phi = 0. \tag{2}$$

This equation describes sound propagation in the positive direction of the x-axis, i.e., it is valid for the angles $\theta < 90°$ between the direction of sound propagation and the x-axis. In Eq. (2), the following notations are used

$$k_c = \frac{\omega}{c}, \quad M_x = \frac{v_x}{c}, \quad \gamma_x = \frac{1}{\sqrt{1-M_x^2}},$$
$$\hat{m}_\perp = \frac{i}{\omega}\mathbf{v}_\perp\cdot\nabla_\perp, \hat{\tau} = M_x\gamma_x^2\left(1+\hat{m}_\perp\right),$$
$$\hat{\mu} = \frac{1}{k_c^2\gamma_x^2}\nabla_\perp^2, \quad \hat{\chi} = 2\hat{m}_\perp + \hat{m}_\perp^2. \tag{3}$$

Here, k_c is the wavenumber in a motionless medium, v_x and $\mathbf{v}_\perp = (v_y, v_z)$ are the velocity components in the direction of the x-axis and in the transverse plane (y, z), respectively, and M_x and γ_x are the Mach number and Lorentz factor pertinent to v_x. Furthermore, the operators $\nabla_\perp = (\partial/\partial y, \partial/\partial z)$, $\hat{m}_\perp$, $\hat{\tau}$, $\hat{\mu}$, and $\hat{\chi}$ act on the transverse coordinates y and z.

The EWAPE [Eq. (2)] is derived by applying the high-frequency approximation to the linearized equations of fluid dynamics and omitting the derivatives of the ambient quantities. Other than that this equation is valid for arbitrary variations in the sound speed and arbitrary (subsonic) Mach numbers $M = v/c$. Equation (2) coincides with Eq. (B1) in Ref. [7], where it is considered only briefly. The main goal of the present article is to consider Eq. (2) in detail and to formulate the WAPE and NAPE starting with this equation.

In the literature, WAPEs are usually derived from the corresponding EWAPEs by approximating a square-root pseudo-differential operator with the Padé (n,n) series. Applying this approach to the EWAPE given by Eq. (2) yields

$$\left[\frac{\partial}{\partial x} - \frac{i\omega}{c_{\mathrm{eff}}} - \frac{M_x\gamma_x^2}{c}\mathbf{v}_\perp\cdot\nabla_\perp - ik_c\gamma_x^2\sum_{j=1}^{n}\frac{a_{j,n}(\hat{\mu}+\hat{\chi})}{1+b_{j,n}(\hat{\mu}+\hat{\chi})}\right]\phi = 0. \tag{4}$$

In this equation, $c_{\mathrm{eff}} = c + v_x$ is the effective sound speed which is introduced as a convenient notation rather than an approximation. Furthermore, the coefficients $a_{j,n}$ and $b_{j,n}$ are given by

$$a_{j,n} = \frac{2}{2n+1}\sin^2\frac{j\pi}{2n+1}, \quad b_{j,n} = \cos^2\frac{j\pi}{2n+1}. \tag{5}$$

Equation (4) is the desired WAPE valid for the arbitrary variations in the sound speed and Mach numbers. In the Padé (1,1) approximation, $n = 1$ so that only the first term in the series in Eq. (4) is retained. In this approximation, the WAPE is valid for the propagation angles $\theta \lesssim 35°$ [3, 7]. Numerical implementation of this WAPE is considered in Sec. 3.

The NAPE can be derived from Eq. (2) by approximating the square-root operator with the Taylor series and keeping terms of order θ and θ^2. The result is

$$\left(\frac{\partial}{\partial x} - \frac{i}{2k_c}\Delta_\perp - \frac{i\omega}{c_{\mathrm{eff}}} + \frac{1}{c_{\mathrm{eff}}}\mathbf{v}_\perp\cdot\nabla_\perp\right)\phi = 0. \tag{6}$$

This equation is the desired NAPE valid for arbitrary variations in the sound speed and Mach numbers. Equation (6) is applicable for the propagation angles $\theta \lesssim 20°$ [3, 7]. Section 3 considers numerical implementation of this equation.

2.2 Analysis

The EWAPE, WAPE, and NAPE from Sec. 2.1 are valid for arbitrary variations in the sound speed and arbitrary Mach numbers. On the other hand, in the literature, many EWAPEs and all WAPEs and NAPEs are valid for low Mach numbers and/or small variations in the sound speed.

To further investigate the ranges of applicability of these parabolic equations, they are compared here to the geometrical acoustics equations. Specifically, starting with the EWAPE [Eq. (2)], WAPE [Eq. (4)], and NAPE [Eq. (6)], the eikonal equations can be derived which describe the phase of a sound wave propagating in an inhomogeneous medium. It can be shown that these eikonal equations coincide with the eikonal equation known in geometrical acoustics, within the ranges of applicability of the EWAPE, WAPE, and NAPE (i.e., for $\theta < 90°$, $\theta \lesssim 35°$, and $\theta \lesssim 20°$, respectively). Therefore, the parabolic equations from Sec. 2.1 are termed the *phase-conserving* EWAPE, WAPE, and NAPE. On the other hand, using a similar approach, it can be shown that WAPEs and NAPEs from the literature correctly describe the phase of a sound wave only for low Mach numbers and/or small variations in the sound speed.

The ranges of applicability of EWAPEs, WAPEs, and NAPEs can also be investigated by comparing them with the exact equations for sound propagation in a stratified moving medium, e.g., Eqs. (2.63) and (2.64) in Ref. [2]. It can be shown that withing the ranges of their applicability, the phase-conserving EWAPE [Eq. (2)], WAPE [Eq. (4)], and NAPE [Eq. (6)] coincide with the high-frequency approximation of the exact equations. On the other hand, WAPEs and NAPEs from the literature coincide with the exact equations only for low Mach numbers and/or small variations in the sound speed.

3. NUMERICAL IMPLEMENTATION

3.1 WAPE and NAPE

In this section, numerical implementation of the phase-conserving WAPE [Eq. (4)] and NAPE [Eq. (6)] is presented.

We will consider 2D sound propagation in the vertical (x, z) plane and assume that $v_z = 0$. In this case, the operators in Eq. (3) simplify significantly. We also express the auxiliary function in the form

$$\phi(x,z) = \hat{\phi}(x,z)e^{ik_0 x}, \qquad (7)$$

where $\hat{\phi}$ is the complex amplitude and $k_0 = \omega/c_0$ and c_0 are the reference wavenumber and sound speed.

Substituting with Eq. (7), the phase-conserving WAPE [Eq. (4)] in the Padé (1,1) approximation can be written as

$$\left(h_{1,0} + \frac{h_{1,2}}{k_0^2}\frac{\partial^2}{\partial z^2}\right)\frac{\partial\hat{\phi}}{\partial x} = ik_0\left(h_{2,0} + \frac{h_{2,2}}{k_0^2}\frac{\partial^2}{\partial z^2}\right)\hat{\phi}, \qquad (8)$$

where the coefficients $h_{n,m}$ are given by

$$h_{1,0} = 1, \quad h_{1,2} = \frac{b_{1,1}}{n^2\gamma_x^2}, \quad h_{2,0} = \frac{c_0}{c_{\text{eff}}} - 1,$$
$$h_{2,2} = \frac{a_{1,1}}{n} + \frac{b_{1,1}h_{2,0}}{n^2\gamma_x^2}. \qquad (9)$$

Here, $n = c_0/c$ is the refraction index in a motionless medium and $a_{1,1} = 1/2$ and $b_{1,1} = 1/4$. Equation (8) can be efficiently solved numerically using the Crank-Nicholson approach as described in Ref. [7] and Sec. 11.2.2 of Ref. [2]

Substituting with Eq. (7), the phase-conserving NAPE [Eq. (6)] can be written as Eq. (8) but with different coefficients:

$$h_{1,0} = 1, \quad h_{1,2} = 0,$$
$$h_{2,0} = \frac{c_0}{c_{\text{eff}}} - 1, \quad h_{2,2} = \frac{1}{2n}. \qquad (10)$$

The NAPE solution proceeds similarly to that of Eq. (8).

3.2 Numerical results

Let a point source be located above rigid ground in a uniformly moving medium. The sound pressure due to the source is a sum of the direct and ground reflected waves. Section VI B in Ref. [7] provides analytical formulas for the auxiliary functions ϕ corresponding to these waves. These results and Eq. (1) enable us to calculate the relative sound pressure level

$$\Delta L(x,z) = 20\log_{10}\left(\frac{|p(x,z)|}{|p_0(x,z)|}\right). \qquad (11)$$

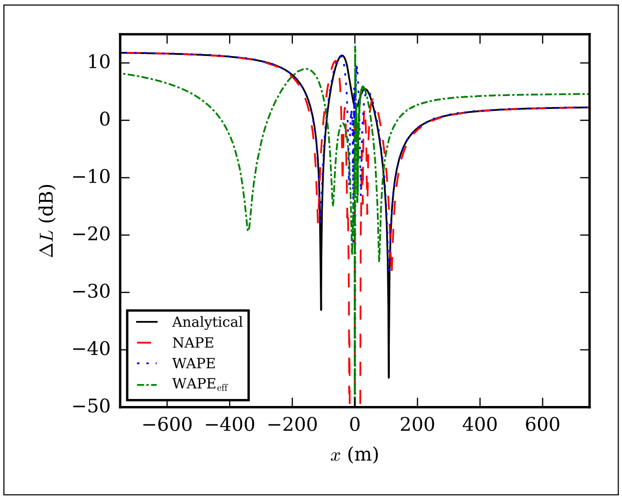

Figure 1. Relative sound pressure level ΔL due to a point source above rigid ground in a uniformly moving medium versus the propagation range x. The curves correspond to the analytical solution and numerical results obtained with the phase-conserving WAPE and NAPE and the WAPE based on the effective sound speed approximation.

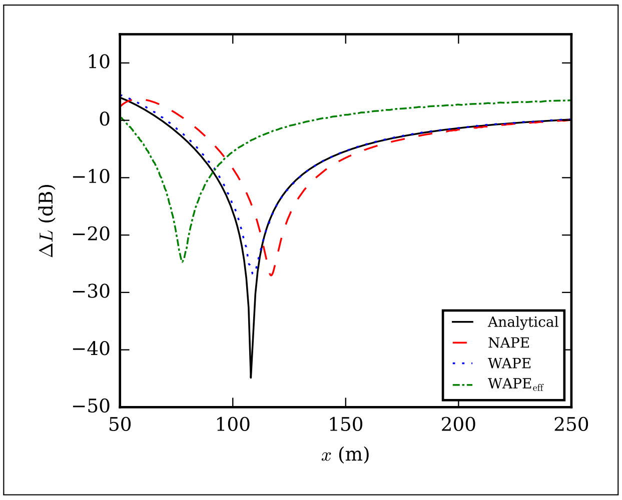

Figure 2. Same as in Fig. 1 but for $50\,\text{m} \leq x \leq 250\,\text{m}$.

Here, p_0 is the sound pressure due the point source in free space.

The analytical solution for ΔL versus the propagation range x is depicted as the solid curve in Fig. 1. The results correspond to the source located at $(x_s = 0\,\text{m}.z_s = 50\,\text{m})$, the receiver at 2 m above the ground, the sound frequency 100 Hz, and the medium moving in the positive direction of the x-axis with the Mach number $M_x = 0.5$. In Fig. 1, x varies from -1 km to 1 km. Figure 2 is an insert from Fig. 1 for $50\,\text{m} \leq x \leq 250\,\text{m}$.

Figures 1 and 2 also depict ΔL calculated with the phase-conserving WAPE [Eqs. (8) and (9)] and NAPE [Eqs. (8) and (10)]. To correctly predict locations of the interference maxima and minima, a range step in the direction of the x-axis is 1/40 of the sound wavelength. It follows from these figures that the WAPE results are close to the analytical solution for the propagation ranges $|x| \gtrsim 120$ m, when the angle θ becomes relatively small. The relative sound pressure level calculated with the NAPE is close to the analytical solution and WAPE results for $|x| \gtrsim 200$ m.

Interestingly, some of the coefficients $h_{i,j}$ pertinent to the phase-conserving WAPE and NAPE differ [compare Eqs. (9) and (10)]. Nevertheless, the WAPE and NAPE solutions are close for $|x| \gtrsim 200$ m, as they should be.

Figures 1 and 2 also depict the relative sound pressure level ΔL obtained with the Padé (1,1) approximation of the WAPE based on the effective sound speed approximation [e.g., Eq. (56) from Ref. [7], formulated for the sound pressure p]. The results significantly deviate from the analytical solution and those obtained with the phase-conserving WAPE and NAPE.

4. CONCLUSIONS

This article considered the *phase-conserving* EWAPE [Eq. (2)], WAPE [Eq. (4)], and NAPE [Eq. (6)]. These equations are derived in the high-frequency approximation and provided that the derivatives of the ambient quantities can be omitted. Within the ranges of their applicability, the EWAPE, WAPE, and NAPE exactly describe the phase of sound waves and are valid for arbitrary variations in the sound speed and arbitrary (subsonic) Mach numbers. These equations also correctly describe sound propagation in a stratified moving medium. Numerical implementation of the phase-conserving WAPE and NAPE can be done with minimal modifications of the existing codes. The phase-conserving EWAPE, WAPE, and NAPE are pertinent for sound propagation in both motionless and moving inhomogeneous media.

WAPEs and NAPEs from the literature correctly describe the phase of a sound wave only for low Mach numbers and/or small variations in the sound speed. Although these quantities are often relatively small, omitting the corresponding second-order small terms can result in significant phase errors for long propagation ranges. Therefore, it is preferable to use the phase-conserving EWAPE, WAPE, and NAPE in practical applications.

5. ACKNOWLEDGMENTS

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