DOI: 10.61782/fa.2023.0267

forum**acusticum** 2023

CONSONANCE AND DISSONANCE FOR DYADS: COMBINING COMPACTNESS AND ROUGHNESS

Isabella Masina[1,2*] **Giuseppe Lo Presti**[3] **Domenico Stanzial**[4]

[1] Department of Physics and Earth Science, Ferrara University, Italy
[2] Istituto Nazionale di Fisica Nucleare, Sez. Ferrara, Ferrara, Italy
[3] Information Technology Department, CERN, Geneva, Switzerland
[4] CNR Institute for Microelectronics and Microsystems, Bologna, Italy

ABSTRACT

There are basically two types of approaches that aim to explain on physical grounds the psychoacoustic perception of consonance and dissonance in music. One is based on the "compactness" of the waveform of the combined signal, while the other on the absence of "roughness" induced by beats. In a previous detailed study of each approach for dyads, we found that none of the associated model versions is fully satisfactory when faced to perceptual data, while a surprisingly successful agreement is found by combining the two approaches. In the present contribution, we extend our analysis by exploring how compactness models for dyads can be related to the early arguments by G.B. Benedetti.

Keywords: *consonance, dissonance, music, psychoacoustics*

1. INTRODUCTION

Explaining on physical grounds the auditory perceptions of consonance and dissonance (C&D) in music is an issue still open to scientific debate, the actual functioning of our hearing system being not fully understood. The present work aims at extending our previous work on dyads [1], by analyzing in some detail the early proposal by G.B. Benedetti [2] and its successive developments.

**Corresponding author*: *masina@fe.infn.it.*

A brief historical review is useful to understand the status of the art.

The commensurability (or coincidence) theory, proposed by G. Galilei [3] likely building on arguments by G.B. Benedetti [2] and others [4], is in fact based on dyad's waveform periodicity [1]. These ideas triggered contributions and debates from other scientists and musicians [4], including *e.g.* Euler [5]. In relation to the discoveries about higher harmonics pioneered by M. Mersenne, R. Descartes and J. Sauveur, another theory was formulated by J.P. Rameau and P. Estève, relating consonance to the largest presence of common harmonics [6, 7]. For instance, the recent model of ref. [8] belongs to this category. The main criticism against these two theories was the fact that the associated C&D indicators are discontinuous functions of the frequency ratios. Pioneering attempts to obtain experimentally a continuous C&D function where carried out by F. Foderà [9].

Yet a different approach appeared in the fall of the XIX century, when H. Helmholtz [10] suggested C&D to be related to the absence of the roughness sensation due to beats; the associated C&D indicators being naturally continuous, this approach had many followers who further refined them, including Plomp and Levelt [11] and Hutchinson and Knopoff [12, 13].

These approaches, traditionally considered as alternative and competing, gave rise to the two categories of explanations for C&D that we denote for short "compactness", including the two sub-categories of "periodicity" and "harmonicity", and "roughness".

Focusing on dyads, in ref. [1] we performed a perceptual test on the consonance of 38 dyads that can be formed using the just scale within two octaves, including some microtonal intervals; the results, normalized to the range $[0,1]$, are reported in Fig. 1.

As for the compactness approach, we proposed various indicators related to periodicity and harmonicity, showing that they are essentially equivalent; we identified among them the indicators that had been previously suggested; and we proposed a method to extend compactness models to the continuum. As a representative example, the geometric mean continuum indicator C_G^P is shown in Fig. 1. As for the roughness approach, we improved the associated indicator to include the beats of the mistuned octave and fifth; the roughness indicator $C^R_{\alpha_{85}}$ of Fig. 1 is a representative example.

We then carried out a complete analysis to assess whether a single refined model among the two categories of compactness and roughness, or rather a combination of them, provides a satisfactory explanation of the perceptual data. We found that combined models, with similar weight attributed to compactness and roughness, are highly successful when compared to the perceptual observations, and perform significantly better than the original models: this non trivial result demonstrates that compactness and roughness are both fundamental ingredients of an effective explanation of C&D [1]. As an example, the model denoted by $C^{tot}_{G,\alpha_{85}}$ in Fig. 1 is obtained by combining with equal weights the former periodicity and roughness models.

In this contribution, we extend our previous analysis to the early estimator suggested by G.B. Benedetti [2] and the indicators inspired to it, as the one suggested by J. Tenney [14, 15].

2. BENEDETTI-INSPIRED INDICATORS

To establish our notation, we consider two tones (simple or complex, in the latter case with a harmonic spectrum) with frequencies f_1 and f_2, such that $f_1 \leq f_2$. The lower frequency f_1 is fixed (*e.g.* at middle C, like in our test), and the higher frequency varies, $f_2 = M/N\, f_1$, where M and N are integer numbers. The ratio $f_2/f_1 = M/N \geq 1$ can be written using the smallest possible integer numbers by defining

$$n = \frac{N}{\mathrm{GCD}[N,M]} \quad , \quad m = \frac{M}{\mathrm{GCD}[N,M]} \quad , \tag{1}$$

where GCD stands for the greatest common divisor, so that the integers m and n do not have prime factors in common and

$$\frac{f_2}{f_1} = \frac{m}{n} \geq 1 \ . \tag{2}$$

The Benedetti height [2] is a dissonance *estimator* given by

$$h^B = nm \ , \tag{3}$$

introduced to account for dyad's rankings, in order of increasing dissonance. It indeed accounts for rankings quite successfully, but fails when one assumes it as an *indicator*, giving absolute scores (indeed, it was not originally proposed to accomplish this): considering for instance the chromatic intervals within two octaves in the just scale, the consonant dyads are all squeezed to low h^B values, while similarly dissonant dyads are spread in a large range of h^B values.

To overcome this problem, Tenney [14] proposed to rather use a logarithmic scale, suggesting that the Benedetti-Tenney height,

$$h^{BT} = \log_2 nm \ , \tag{4}$$

might be used as reasonable dissonance indicator (to be called harmonic distance).

The Tenney choice of a logarithmic scale is arbitrary, and one could rather imagine to exploit different mathematical scalings. For instance, one may define the inverse of nm as consonance indicator. To be even more general, one might introduce the real and positive parameter α, and consider as consonance indicator

$$I^{iB} = 1/(nm)^\alpha \ . \tag{5}$$

This is fully legitimate in the spirit of Benedetti's proposal, who exploited the product nm just in connection with rankings, never attributing to it an interpretation as absolute scores for dissonance.

In order to make our analysis quantitative, we have to directly compare each model prediction with our test results [1] and evaluate the associated reduced chi square. To do this, we have first to normalize to the range $[0,1]$ the previously introduced indicators. Notice that normalized consonance and dissonance indicators are complementary to 1. In particular, the normalized Benedetti and Benedetti-Tenney consonance indicators are

$$\tilde{I}^X = 1 - \frac{h^X - h^X_{min}}{h^X_{max} - h^X_{min}} \ , \quad X = B, BT \ , \tag{6}$$

where h^X_{min} and h^X_{max} are the minimum and maximum values taken by h^X within some set of selected dyads.

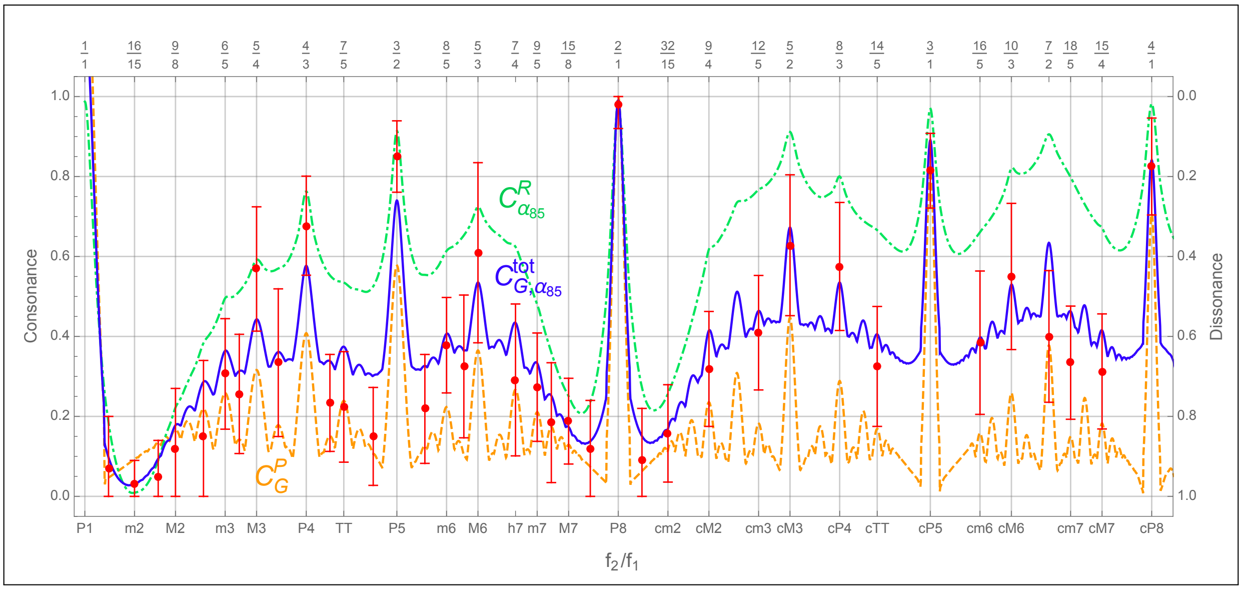

Figure 1. Results of the test on the consonance of 38 dyads [1]; the dashed and dot-dashed curves show the predictions of the representative periodicity model C_G^P and of the representative roughness model $C^R_{\alpha_{85}}$. The solid curve is the model $C^{tot}_{G,\alpha_{85}}$, obtained by combining the previous two with equal weights.

The indicator I^{iB} is instead associated to the following normalized consonance indicator, which we call inverted Benedetti indicator for short,

$$\tilde{I}^{iB} = \frac{I^{iB} - I^{iB}_{min}}{I^{iB}_{max} - I^{iB}_{min}} \ , \tag{7}$$

where I^{iB}_{min} and I^{iB}_{max} are the minimum and maximum values taken by I^{iB} for some set of chosen dyads.

Here we take the 38 dyads within two octaves studied in ref. [1]. The value $\tilde{I}^{B,BT,iB} = 1$ is thus reached only by the octave, $f_2/f_1 = 2$. A good model is expected to have a reduced chi square smaller than or about 1. The value of the reduced chi square for the Benedetti and Benedetti-Tenney consonance indicators turn out to be respectively equal to 20.5 and 1.8. The latter value, together with the reduced chi square of $\tilde{I}^{iB}$ as a function of α, is shown in Fig. 2. We see that for $0.3 < \alpha < 0.6$, $\tilde{I}^{iB}$ performs better than $\tilde{I}^{TB}$; in particular, $\tilde{I}^{iB}$ has a minimum at $\alpha \approx 0.4$, where the reduced chi square equals 0.7.

We now discuss how the above Benedetti-inspired indicators are related to the periodicity and harmonicity models discussed in ref. [1].

3. COMPACTNESS MODELS

Periodicity models relate the perception of C&D to the compactness of the waveform of the dyadic signal, according to the criterion that the shorter is the period of the missing fundamental (*i.e.* the fundamental bass in music theory) with respect to the periods of the two tones, the more the interval is consonant. Harmonicity models are instead based on the compactness of the dyad's harmonic structure, that is on the hypothesis that consonance increases with the number of coincident harmonics between the two tones.

As argued in ref. [1], these two model categories give practically the same indicators, so that they can be considered two subcategories of the compactness approach. The basic result of the latter is that the most consonant intervals are those whose frequency ratios involve the smallest integer numbers, all other intervals sharing more or less the same degree of dissonance.

3.1 Periodicity models

The frequency of the dyad's missing fundamental, f_0, is

$$f_0 = \frac{f_1}{n} = \frac{f_2}{m} \ . \tag{8}$$

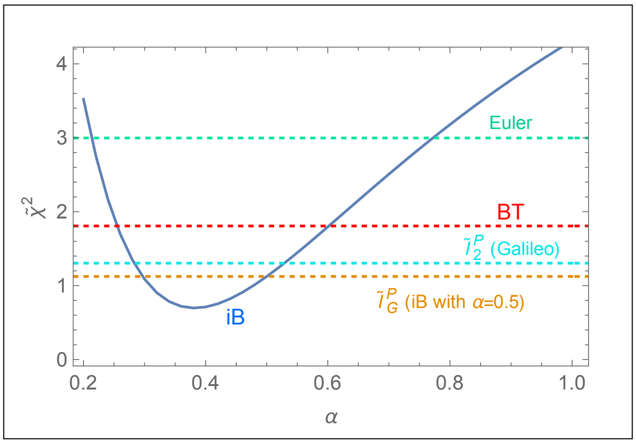

Figure 2. Reduced chi square as a function of α for the inverse Benedetti model $\tilde{I}^{iB}$ (iB, solid line). From top to bottom, we show for comparison the reduced chi square for the models by Euler [5] and Benedetti-Tenney (BT), the Galilei inspired $\tilde{I}_2^P$ model and the geometric mean model $\tilde{I}_G^P$ (which is equal to $\tilde{I}^{iB}$ with $\alpha = 0.5$).

The basic idea of the periodicity approach is that the higher is f_0 with respect to the composing frequencies f_1 and f_2, the higher is the degree of consonance. Since there are various ways to establish this comparison, there are accordingly many periodicity indicators [1] .

Comparing f_0 with f_1 and f_2, one obtains as periodicity indicators respectively

$$I_1^P = \frac{f_0}{f_1} = \frac{1}{n} \quad , \quad I_2^P = \frac{f_0}{f_2} = \frac{1}{m} \quad . \tag{9}$$

The second indicator is better than the first in accounting for the test results, and it corresponds to the fraction of ”concordant pulses” proposed *in nuce* as a consonance indicator by G. Galilei [3].

Other indicators can be obtained by considering mean values of f_1 and f_2. For the arithmetic, geometric and harmonic means, $f_A = (f_1 + f_2)/2$, $f_G = \sqrt{f_1 f_2}$ and $f_H = f_A^2/f_G$, we have

$$I_A^P = \frac{f_0}{f_A} = \frac{2}{n+m} \quad , \quad I_G^P = \frac{f_0}{f_G} = \frac{1}{\sqrt{nm}} \quad ,$$
$$I_H^P = \frac{f_0}{f_H} = \frac{n+m}{2nm} \quad . \tag{10}$$

It can be seen that these three indicators give quite similar predictions, intermediate with respect to those of I_1^P and I_2^P. Notice that I_G^P is equal to the inverted Benedetti indicator I^{iB} with $\alpha = 0.5$.

More in general, any function that can be written as a combination of the indicators above, can be seen as a consonance indicator related to periodicity. A consonance indicator based on periodicity is thus some function that increases when m and n are as small as possible.

To study the reduced chi square for the above periodicity indicators, we first normalize them to the range $[0, 1]$,

$$\tilde{I}_X^P(f_2/f_1) = I_X^P(f_2/f_1)/I_X^P(2) \quad , \tag{11}$$

where $X = 1, 2, A, G, H$. As an example, the reduced chi square of the normalized consonance indicators associated to $\tilde{I}_2^P$ and $\tilde{I}_G^P$ are respectively equal to 1.3 and 1.1, and they are shown in Fig. 2.

3.2 Harmonicity models

Let us denote the harmonic series of f_1 and f_2 with $\{n_1 f_1\}$ and $\{n_2 f_2\}$, where n_1 and n_2 are integers, starting from 1. A coincidence in the harmonics happens if the relation $n_1 f_1 = n_2 f_2$ is fulfilled for some n_1 and n_2. In the case it is, the lowest coincidence of the harmonics happens for the smallest possible values of n_1 and n_2, to be denoted by $n_1^{c_1}$ and $n_2^{c_1}$. Since the relation above implies

$$\frac{n_1}{n_2} = \frac{f_2}{f_1} = \frac{m}{n} \quad , \tag{12}$$

where m and n are already as small as possible, the first coincidence happens for $n_1^{c_1} = m$ and $n_2^{c_1} = n$. The second coincidence happens for $n_1^{c_2} = 2m$ and $n_2^{c_2} = 2n$, the third for $n_1^{c_3} = 3m$ and $n_2^{c_3} = 3n$, and so on.

A consonance indicator based on harmonicity is thus some function that increases when m and n are as small as possible. As this is precisely the same request as in the case of periodicity indicators, the two approaches are basically equivalent.

A couple of explicit examples of harmonicity indicators are for instance the functions

$$\frac{1}{2}\left(\frac{1}{n_1^{c_1}}+\frac{1}{n_2^{c_1}}\right)=\frac{m+n}{2mn} \quad , \quad \frac{1}{n_1^{c_1}}\frac{1}{n_2^{c_1}}=\frac{1}{mn} \ , \quad (13)$$

where the first is equal to the harmonic mean periodicity indicator, I_H^P, while the second is equal to the inverse Benedetti indicator I^{iB} with $\alpha = 1$.

One might guess that the similarity of the harmonic content of the two sounds, hence the associated indicator, should also depend on the amplitudes of the harmonics, namely on timbre. Denoting the weight function for the contribution of the n_i-th harmonics by w_{n_i} with $i = 1, 2$, one can build an indicator along these lines as

$$I_w^H = \sum_{n_1,n_2=1,2,...} w_{n_1}\, w_{n_2}\, \delta(n_1 f_1 - n_2 f_2) \ , \quad (14)$$

where δ is the Dirac delta function. In particular, using $w_{n_i} = 1/n_i^{\alpha}$, where α is a positive real number, one has

$$I_\alpha^H = \frac{1}{(mn)^\alpha}\left(1+\frac{1}{4^\alpha}+\frac{1}{9^\alpha}+...\right) \ . \quad (15)$$

The numerical factor in parenthesis disappears upon normalization, so that this harmonicity indicator turns out to be precisely equal to the normalized inverse Benedetti indicator, $\tilde{I}_\alpha^H = \tilde{I}^{iB}$.

3.3 Extension to continuum

We now discuss how to extend the compactness indicators to the ”continuum” (namely to non integer numbers N and M), implementing the effect of the frequency discrimination limen (DL) [16] of the hearing system [1].

Suppose that m and n take all integer values from 1 up to 50, for instance. We select the k ratios of the type m/n falling in the interval $[1, 4]$, and we denote them by x_i, with $i = 1, .., k$. The associated normalized consonance indicator, $\tilde{I}_X^P(x_i)$, is thus defined only for the k discrete values x_i. Our aim is to extend the indicator to any value of the x domain in the interval $[0, 4]$, thus turning it more ”physical”. The ear has a DL of about 3 Hz at the frequency of middle C (or C_4), which increases up to 6 Hz two octaves above [16]. The effect of the DL can be implemented by smoothing the peaks with a Gaussian characterized by a standard deviation equal to the DL at frequency f_2, $\sigma = f_{DL}(f_2)/f_1$. The DL turns out to be about $1/30$ of the critical bandwidth (CB) [17], as derived by Zwicker et al. [16]. As the CB is frequency dependent, we anchor for definiteness f_1 to C_4. The extension to the continuum can be made by calculating the distance $|x - x_i|$, for all x_i such that $|x - x_i| < 2\,\sigma(x)$, and evaluating the corresponding value for the periodicity consonance indicator $C_X^P(x)$, defined as

$$C_X^P(x) = \mathrm{Max}_i\, \tilde{I}_X^P(x_i)\, e^{-\frac{(x-x_i)^2}{2\,\sigma(x)^2}} \ . \quad (16)$$

The result is a continuous function, with smoothed peaks such that, within (beyond) about 3 (6) Hz from a peak, the consonance function does not (might) change significantly.

As an example, the geometric mean continuum indicator C_G^P of Fig. 1 has been obtained following this procedure.

4. THE COMBINED EFFECT OF COMPACTNESS AND ROUGHNESS

As discussed in ref. [1], despite the many good features, roughness models do not reproduce the data points in a satisfactory way. It seems that some other ingredient has to be introduced: indeed, the focus of roughness models is to assign penalties, rather than prizes. On the contrary, the compactness models provide essentially prizes for the simplicity of the waveform of the signal, but do not assign increasing penalties to increasingly non simple ratios of dyad’s frequencies. A complete model of C&D should both give prizes for the compactness of the signal and penalties for the presence of beats.

We thus proposed [1] to directly combine the two approaches, simply summing two representative indicators in each category. The combined consonance indicator is given by the weighted sum of a compactness model, of the periodicity or harmonicity type, and a roughness one:

$$C_{X,Y}^{tot} = \frac{F\, C_X^{P/H} + (1-F)\, C_Y^R}{N_{X,Y}} \ , \quad (17)$$

where X and Y specify the particular compactness and roughness models respectively, F is the fractional contribution of periodicty/harmonicity with respect to roughness, and $N_{X,Y}$ is a normalization factor. For large (small) values of F, the combined model is dominated by the compactness (roughness) constituent model. Of course it is not obvious that this automatically corresponds to a better model than its constituents; only the comparison with the perceptual data can reveal whether this is the case.

We considered all possible combinations of the compactness models, with the best performing roughness models. In all the considered cases, it turns out that: the reduced chi square of the combined model has a minimum which is significantly smaller than the chi square of the constituent models; the inclusion of the mistuned octave and fifth reproduces the data slightly better, and is characterized by a lower value of F. The more pronounced minimum is found for the combined models having C^P_G, C^P_A and C^P_2 as constituent models, together with $C^R_{\alpha_{85}}$. For these three best performing models, including the effect of the mistuned octave and fifth, the minimum of the reduced chi square is found at $F \approx 58\%$ and is slightly smaller than 0.3, signaling an impressive agreement between such theoretical models and perceptual observations.

In particular, we recall that C^P_G is precisely the inverse Benedetti indicator with $\alpha = 0.5$; Fig. 1 shows the combination of C^P_G and $C^R_{\alpha_{85}}$ with equal weights, that is the combination with $F = 0.5$. The agreement with the perceptual data of our test is evident [1].

5. CONCLUSIONS

In this contribution we reviewed the results of our previous work [1] where we showed that combining the compactness and roughness approaches allows to obtain a phenomenologically satisfactory explanation of C&D based on physical grounds, that is based on the features associated to compactness (of the period or of the harmonic spectrum) and roughness present in the dyadic waveform.

In particular, we discussed here how the early proposal of the Benedetti's estimator [2] can be used to construct consonance indicators, including the one suggested by J. Tenney [14, 15], and how these indicators are related to those associated to the compactness approach.

6. ACKNOWLEDGMENTS

We would like to thank P. Barbieri for useful suggestions.

7. REFERENCES

[1] I. Masina, G. Lo Presti, and D. Stanzial, "Dyad's consonance and dissonance: combining the compactness and roughness approaches," *Eur. Phys. J. Plus*, vol. 137, p. 1254, 2022.

[2] G. B. Benedetti, "Diversarum speculationum mathematicarum et physicarum liber," *Torino*, 1585.

[3] G. Galilei, "Discorsi e dimostrazioni matematiche intorno a due nuove scienze," *Leida*, 1638.

[4] P. Barbieri, "Galileo's coincidence theory of consonances, from Nicomachus to Sauveur," *Recercare*, vol. 13, pp. 201–232, 2001.

[5] L. Euler, "Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae," *Petropoli*, 1739.

[6] J. Rameau, "Génération harmonique," *Paris*, 1737.

[7] P. Estève, "Nouvelle découverte du principe de l'harmonie," *Paris*, 1752.

[8] K. Gill and D. Purves, "A biological rationale for musical scales," *PLoS One*, vol. Dec 3;4(12):e8144, 2009.

[9] P. Barbieri, "La nascita delle teorie 'continue' della consonanza. La ignorata curva di Draghetti e Foderà, poi di Helmholtz (1771-1837)," *Acta Musicologica*, vol. 74 (1), pp. 55–75, 2002.

[10] H. von Helmholtz, "On the Sensation of Tone," 1863.

[11] R. Plomp and W. J. Levelt, "Tonal consonance and critical bandwidth," *Journal of the Acoustical Society of America*, vol. 38(4), pp. 548–560, 1965.

[12] W. Hutchinson and L. Knopoff, "The acoustic component of western consonance," *Interface*, vol. 7(1), pp. 1–29, 1978.

[13] W. Hutchinson and L. Knopoff, "The significance of the acoustic component of consonance in western triads," *Journal of Musicological Research*, vol. 3:1-2, pp. 5–22, 1979.

[14] J. Tenney, "John cage and the theory of harmony," *Soundings*, vol. 13, 1984.

[15] A. Wand, "On the conception and measure of consonance," *Leonardo Music Journal*, vol. 22, pp. 73–78, 2012.

[16] E. Zwicker, G. Flottorp, and S. S. Stevens, "Critical band width in loudness summation," *Journal of the Acoustical Society of America*, vol. 29, p. 548, 1957.

[17] H. Fletcher, "Auditory Patterns," *Rev. Mod. Phys.*, vol. 12, pp. 47–65, 1940.