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GAUSSIAN SERIES SOLUTION FOR SONIC BLACK HOLES IN DUCT TERMINATIONS

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ABSTRACT

Sonic black holes (ABHs) at duct terminations slow down impinging waves by means of a set of rings separated by cavities, whose inner radii diminish following a power law profile. Energy tends to concentrate at the end of the waveguide and is dissipated by visco-thermal losses, resulting in a very low reflection coefficient. This anechoic behavior is governed by a modified Webster equation that takes into account the wave propagation inside the duct of variable cross section and wall admittance. In this paper it is shown that the generalised Webster equation can be transformed into a Helmholtz-type equation with non-constant wave number. We solve its weak form by expanding the solution in terms of Gaussian functions and show how the modal distribution within the SBH affects the occurrence and disappearance of peaks and dips in the SBH reflection coefficient.

Keywords: *sonic black holes, slow sound, waveguide, acoustic black hole, anechoic termination*

1. INTRODUCTION

Sonic black holes (SBHs) at duct terminations were originally proposed in [1] and consist of waveguides that slow down propagating acoustic waves in such a way that, in an ideal scenario, these would never reach the end of the tube resulting in zero reflection. According to [1], this could be achieved by placing a very large number of rings in the duct, separated by cavities, such that their inner radius decays from that of the uniform duct to zero at the duct termination, following a power-law profile. However, in practice the number of rings that can be built is limited and therefore most efforts to date have been devoted to describing the performance of more realistic SBHs, either by the transfer matrix method (TMM) (see e.g., [2–4], by simulations with the finite element method (FEM) [5–7] or by semi-analytical models [8].

In this work we deal with the theoretical SBH in [1], which deserves further exploration as several of its features have not yet been analyzed in detail. We show that the original Webster equation governing the SBH acoustics can be transformed into a Helmholtz equation with spatially varying wavenumber for a locally scaled pressure, and then derive its variational formulation as well as an associated eigenvalue problem to compute the SBH modes. The variational problems are solved by expanding the scaled pressure in terms of Gaussain functions. The singularity at the duct termination is avoided by considering there a residual rigid cross-section, characterised by a residual radius, which is shown to play the same role as the residual thickness in beams and plates with acoustic holes (ABHs) [9–12]. An analysis of the modes within the SBH depending on the residual radius reveals the ultimate reason for the occurrence and disappearance of peaks and dips in the reflection coefficient of the SBH.

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2. STATEMENT OF THE PROBLEM

2.1 Governing equation inside a SBH

As discussed in [1], plane wave propagation inside a radially symmetric waveguide of section $S(x)$, local radius $r(x)$, and wall impedance $Y(x)$ is governed by the generalized Webster equation,

$$\frac{\mathrm{d}^2\hat{p}}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\left(\ln S\right)\frac{\mathrm{d}\hat{p}}{\mathrm{d}x} + \left[k_0^2 + \mathrm{i}Z_0\frac{2Y}{r}k_0\right]\hat{p} = 0, \quad (1)$$

where $\hat{p}(x)$ is the acoustic pressure, $k_0 = \omega/c_0$ the wavenumber, and ω and c_0 respectively stand for the angular frequency and the speed of sound. $Z_0 = \rho_0 c_0$ is the air characteristic impedance with ρ_0 being the air density.

Introducing the locally scaled pressure $\phi = S^{1/2}\hat{p}$, we can transform Eqn. (1) into a Helmholtz equation,

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \kappa^2(x)\phi = 0, \quad (2)$$

with spatially varying wavenumber,

$$\kappa^2 = k_0^2 + \mathrm{i}Z_0\frac{2Y}{r}k_0 - \frac{1}{2}\frac{S''}{S} + \frac{1}{4}\frac{S'^2}{S^2}. \quad (3)$$

Considering the wall admittance for the ideal ring/cavity SBH (see [1,2]),

$$Y(x) = -\mathrm{i}\frac{k_0}{Z_0}\frac{R^2 - r^2}{2r}, \quad (4)$$

and the cross section $S(x) = \pi r(x)^2$, after some manipulations the wavenumber squared simplifies to

$$\kappa^2 = k_0^2\frac{R^2}{r^2} - \frac{r''}{r}. \quad (5)$$

2.2 Strong and weak formulation of the sonic black hole problem in a finite duct termination

We want to solve the problem depicted in Fig. 2 in which a plane wave propagating in a semi-infinite duct impinges on the SBH from the left. To avoid the singularity of Eqn. (1) at the origin, we consider that $r(x)$ never becomes zero and that there is a residual cross section at the SBH termination with residual radius $r_0 \equiv r(0)$. This avoids the need of introducing the somewhat artificial length imperfection in [1]. The problem is that of finding $\phi(x)$ such that,

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - k_0^2\phi = 0, \quad \forall\, x \in (-\infty, -L], \quad (6a)$$

$$-\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2} - \kappa^2(x)\phi = 0, \quad \forall\, x \in [-L, 0], \quad (6b)$$

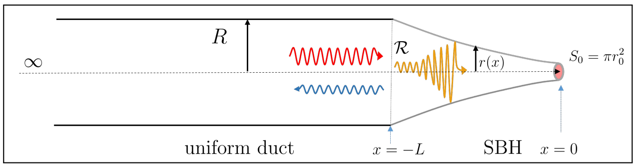

Figure 1. Schematic of the SBH problem. A wave propagating form the left enters a SBH where its slows down while its amplitude increases and its wavelength decreases. The reflection coefficient $\mathcal{R}$ characterizes its efficiency.

where L is the length of the SBH. These equations are supplemented with the following boundary conditions: a rigid surface at $x = 0$, $S_0 = \pi r_0^2$, and continuity of the acoustic pressure and the acoustic particle velocity at $x = -L$.

The solution to Eqn. (6a) is $\phi(x) = \mathrm{e}^{-\mathrm{i}k_0 x} + \mathcal{R}\mathrm{e}^{\mathrm{i}k_0 x}$ where $\mathcal{R}$ is the unknown reflection coefficient of the SBH. To solve Eqn. (6b), we first establish its variational form. As usual, this is found by multiplying Eqn. (6b) by a test function ψ^* ($*$ stands for the complex conjugate), integrating over the computational domain, and then integrating by parts to have the same order for the derivatives of the unknown scaled pressure ϕ and the test function ψ^*. This yields,

$$\int_{-L}^{0}\frac{\mathrm{d}\psi^*}{\mathrm{d}x}\frac{\mathrm{d}\phi}{\mathrm{d}x}\mathrm{d}x - \int_{-L}^{0}\kappa^2(x)\psi^*\phi\mathrm{d}x + \mathrm{i}k_0\psi^*(-L)\phi(-L) + \frac{S'}{2S}(-L)\psi^*(-L)\phi(-L) - \frac{S'}{2S}(0)\psi^*(0)\phi(0) = 2\mathrm{i}k_0\mathrm{e}^{\mathrm{i}k_0 L}\psi^*(-L). \quad (7)$$

In addition, to compute the SBH modes we can derive a variational eigenvalue problem by splitting κ^2 into its frequency and non-frequency dependent terms. We obtain,

$$-\omega^2\frac{R^2}{c_0^2}\int_{-L}^{0}\frac{1}{r^2}\psi^*\phi\mathrm{d}x + \mathrm{i}\omega\frac{1}{c_0}\psi^*(-L)\phi(-L) + \int_{-L}^{0}\frac{r''}{r}\psi^*\phi\mathrm{d}x + \int_{-L}^{0}\frac{\mathrm{d}\psi^*}{\mathrm{d}x}\frac{\mathrm{d}\phi}{\mathrm{d}x}\mathrm{d}x + \frac{S'}{2S}(-L)\psi^*(-L)\phi(-L) - \frac{S'}{2S}(0)\psi^*(0)\phi(0) = 2\mathrm{i}k_0\mathrm{e}^{\mathrm{i}k_0 L}\psi^*(-L). \quad (8)$$

We solve Eqn. (7) and Eqn. (8) by expanding ψ^* and

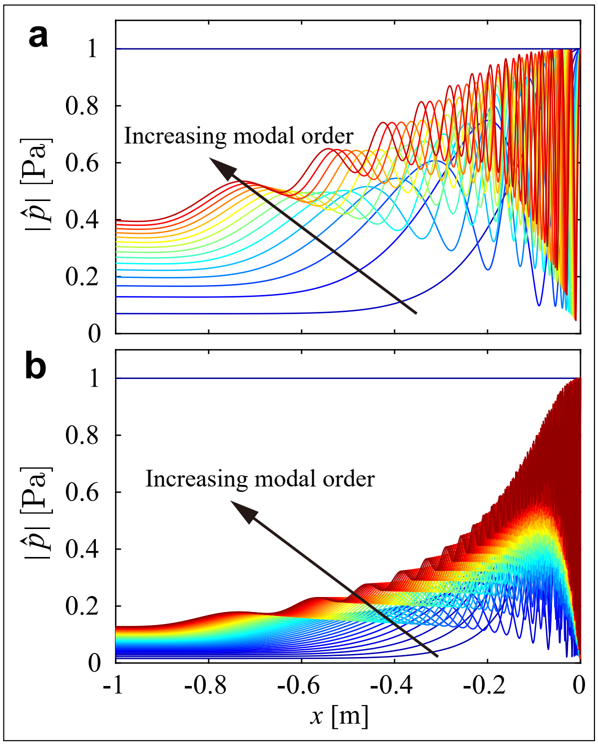

Figure 2. SBH mode shapes for (a) $r_0 = 0.01$ m and (b) $r_0 = 0.001$ m. Damping value: $\eta = 0.05$.

ϕ as a combination of basis functions,

$$\psi^*(x) = \sum_{i=1}^{n} b_i^* \varphi_i(x), \quad \phi(x) = \sum_{i=1}^{n} a_i \varphi_i(x), \quad (9)$$

where we choose $\varphi_i(x)$ to be Gaussians as done in several of the authors previous works on ABHs for beams and plates [11–15]. Inserting Eqn. (9) into Eqn. (7) and Eqn. (8) we can obtain the expansion coefficients a_i for the scaled pressure ϕ and the eigenpairs of the problem.

3. SIMULATIONS

We consider a SBH of length $L = 1$ m, input radius $R = 0.23$ m, and order $m = 2$. We take a sound speed $c_0 = c_a(1 + \mathrm{i}\eta)$ with $c_a = 343$ m/s and $\eta = 0.05$, which is a rough approximation to thermoviscous losses. The cutoff frequency of the duct is $f_c = 1.84 c_a / 2\pi R = 445$ Hz, so we restrict our analysis to the range $[0, 400]$ Hz.

To begin with, let us focus on the distribution and shape of the modes within the SBH and their dependence on the residual radius, r_0. As observed in Fig. 2, the modes tend to concentrate strongly towards the end of the SBH. If we compare Fig. 2a for $r_0 = 0.01$ m and Fig. 2b for $r_0 = 0.001$ m, it is seen that the smaller the residual radius the higher the concentration and the greater the number of modes in the frequency range of analysis. Although not shown in the figure, it can be checked that increasing the damping, η, has a strong effect on the higher order modes of the SBH, i.e., those in Fig. 2b would be more affected than those in Fig. 2a.

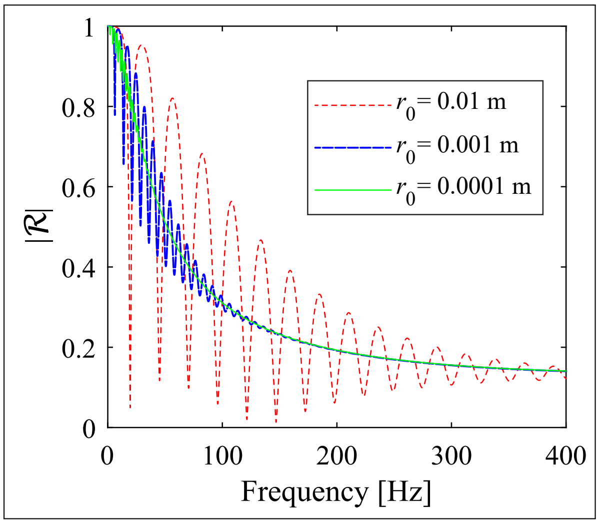

Figure 3. Absolute value of the reflection coefficient, $|\mathcal{R}|$, for various values of the residual radius r_0.

The increase and concentration of modes as r_0 decreases, and the effect of damping on the higher-order modes are ultimately responsible for the appearance and disappearance of peaks in the absolute value of the SBH reflection coefficient $|\mathcal{R}|$. The former is seen in Fig. 3, which shows $|\mathcal{R}|$ for $r_0 = \{0.01, 0.001, 0.0001\}$ m. Strong oscillations can be identified for the largest residual radius $r_0 = 0.01$ m, while they progressively decrease as r_0 becomes smaller. For $r_0 = 0.0001$ only minor peaks and dips can be identified for the lower frequencies of the spectrum.

4. CONCLUSIONS

In this work we have shown that the propagation of acoustic waves inside a SBH can be described by a Helmholtz equation with spatially varying wavenumber for a locally scaled pressure. We have derived the variational form of this equation and solve it by expanding the scaled pressure in terms of a Gaussian basis. We have also introduced a variational eigenvalue problem to calculate the SBH modes.

Simulations have revealed that the number of modes and their amplitude increase and concentrate towards the end of the SBH as we decrease the residual radius. Damping also has a strong effect on the higher order modes. This explains why the peaks and dips of the SBH reflection coefficient tend to disappear at high frequencies for small residual radii.

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