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COMPUTATIONAL MODEL ORDER REDUCTION FOR SYSTEMS WITH RADIATION DAMPING

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ABSTRACT

Radiation of waves in an open domain is a common case for many applications in acoustics. Depending on the system, the damping effect introduced may not be negligible. When using the finite element method to model such problems, appropriate techniques like absorbing boundary conditions or perfectly matched layers can be applied to achieve free field radiation despite the domain truncation.

We show how the resulting quadratic eigenvalue problem of an acoustic system with free radiation boundary conditions can be solved. The resulting eigenmodes are classified into physical modes and computational modes due to the domain truncation. A reduced-order model is constructed by projection into a modal subspace. We suggest a selection criterion for modes to include in the basis using a similarity criterion with respect to selected full model solutions.

We apply the model order reduction to a simple 2D example of a Helmholtz oscillator. Both the accuracy and performance of the strategy are evaluated, showing highly accurate results. Due to the high computational effort for solving the eigenvalue problem, significant performance improvements can only be expected from the reduction technique if many evaluations of the reduced order model are required.

Keywords: *model order reduction, open domain, free radiation, quadratic eigenvalue problem.*

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1. INTRODUCTION

Vibrating structures radiate sound waves into the environment, leading to a damping effect. The importance of this radiation damping ranges from negligible for typical steel structures in vibrating air to substantial for lightweight MEMS structures in water. When using the finite element method to model such vibroacoustic problems, appropriate techniques like absorbing boundary conditions (ABCs) or perfectly matched layers can be used to enable free field radiation at the domain boundary. While this works well in time and frequency domain problems, the solution of the free oscillation eigenvalue problem to determine the natural modes of the system including the effects of radiation damping is less straightforward. Computing those eigensolutions is desirable since they give insight into the system behavior and they can serve as a projection basis to generate reduced order models.

While projection-based reduced order models are an established technique for conservative systems, e.g. mechanical or acoustical systems with low proportional damping, considerably less work has been done for systems with substantial damping effects. The relevant eigenvalue problem (EVP) for systems with radiation damping becomes a quadratic one. A common approach here is to linearize the quadratic EVP, and use the modes of the resulting generalised EVP [1–3]. Damping due to free field radiation is also not often analyzed in an eigenproblem setting. It can, however, be treated similarly to damping due to an absorbing mechanical material [4] or interface damping [5].

In this work we focus on the solution of the free oscillation eigenvalue problem for an acoustic system with open domain. The system is modelled by the finite element method using the open source software openCFS [6].

Model order reduction using a modal basis constructed from eigenmodes of the quadratic eigenvalue problem is conducted and compared to the full system is the frequency domain on the example of a Helmholtz oscillator.

2. GOVERNING EQUATIONS AND FEM FORMULATION

We consider systems governed by the acoustic wave equation in a domain Ω, which is appropriate for compressible, inviscid fluids. Formulating it in terms of the acoustic pressure p one obtains the following weak form

$$\int_{\Omega} p' \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} \, \mathrm{d}\Omega + \int_{\Gamma_\mathrm{r}} p' \frac{1}{c} \frac{\partial p}{\partial t} \, \mathrm{d}\Gamma + \int_{\Omega} \nabla p' \cdot \nabla p \, \mathrm{d}\Omega = \int_{\Omega_\mathrm{e}} p' p_\mathrm{e} \, \mathrm{d}\Omega \,, \quad (1)$$

where p' denotes the test function and c is the speed of sound. The absorbing boundary condition at Γ_r is obtained from the radiation condition, and p_e denotes a known volume source.

Discretization with finite elements and application of a Galerkin procedure yields a discretized system of the form

$$\boldsymbol{M}\ddot{\boldsymbol{p}}(t) + \boldsymbol{C}\dot{\boldsymbol{p}}(t) + \boldsymbol{K}\boldsymbol{p}(t) = \boldsymbol{f}(t) \quad (2)$$

with the stiffness matrix $\boldsymbol{K}$, the damping matrix $\boldsymbol{C}$, the mass matrix $\boldsymbol{M}$, and the forcing vector $\boldsymbol{f}(t)$. The pressure degrees of freedom are contained in the unknown vector $\boldsymbol{p}(t)$ and a dot denotes a time derivative. Assuming harmonic forcing of the form $\boldsymbol{f}(t) = \Re\{\hat{\boldsymbol{f}} e^{j\omega t}\}$ one can obtain the frequency domain solution of eq. (2) by solving

$$\left(-\omega^2 \boldsymbol{M} + j\omega \boldsymbol{C} + \boldsymbol{K}\right) \hat{\boldsymbol{p}} = \hat{\boldsymbol{f}} \,. \quad (3)$$

3. MODEL ORDER REDUCTION

The suggested model order reduction procedure is based on the eigenmodes of the system eq. (2). The corresponding (right) quadratic eigenvalue problem (QEVP) is

$$\left(\lambda_\mathrm{i}^2 \boldsymbol{M} + \lambda_\mathrm{i} \boldsymbol{C} + \boldsymbol{K}\right) \boldsymbol{v}_\mathrm{i} = \boldsymbol{0} \,, \quad (4)$$

with the eigenvalues λ_i and the (right) eigenvectors $\boldsymbol{v}_i$, which, if not stated differently, are just referred to as eigenvectors. Typically the eigensolution of an subcritically damped system with n degrees of freedom consists of n complex-conjugated eigenvalue pairs λ_i, λ_i^* and eigenvector pairs $\boldsymbol{v}_i, \boldsymbol{v}_i^*$ [3]. The two eigenvectors in each pair are linearly dependent [2]. In continuous open domain problems the wave number k_i in direction i of an open end becomes a continuous function of the frequency $k_i = k_i(\omega)$ [7]. Therefore, the eigenvalues also become a function of frequency. As we will see in section 4 this yields several computational eigenvalues due to the discretization applied to eq. (1) and the truncation of the open domain.

Based on the modal expansion theorem [8] the physical coordinates $\boldsymbol{p}$ can be represented as a linear combination of selected mode shapes (eigenvectors) $\boldsymbol{v}_i$ and their respective modal coordinates q_i. Selecting a subset of $m \ll n$ modes we approximate the solution by

$$\boldsymbol{p} \approx \sum_{i=1}^{m} \boldsymbol{v}_i q_i = \hat{\boldsymbol{V}} \hat{\boldsymbol{q}} \,. \quad (5)$$

where the reduced modal basis is contained in the mode shape matrix $\boldsymbol{V} = \left[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\right]$ and the generalized modal coordinates are collected in the modal coordinate vector $\boldsymbol{q} = \left[q_1, \ldots, q_m\right]^T$, with $(\,)^T$ denoting the transpose of a vector.

Qu [3] describes the general modal reduction of quadratic systems by first linearising (e.g. companion linearisation [2]) and then projecting into the modal subspace. It is suggested that the projection is done by applying the reduced modal expansion to the linearised system and then multiplying from the left with the complex-conjugate transpose of the reduced modal basis built from left eigenvectors of the adjoint (left) eigenvalue problem. Since the sets of left and right eigenvectors coincide for QEVPs with real-valued symmetric matrices $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ [2] eq. (3) is transformed to the modal harmonic system by just using $\boldsymbol{V}$ and $\boldsymbol{V}^*$ with $(\,)^*$ denoting the complex-conjugate transpose, yielding

$$\boldsymbol{V}^* \left(-\omega^2 \boldsymbol{M} + j\omega \boldsymbol{C} + \boldsymbol{K}\right) \boldsymbol{V} \boldsymbol{q} = \boldsymbol{V}^* \boldsymbol{f} \,. \quad (6)$$

However, a non-conservative quadratic system cannot be decoupled by a non-singular linear transformation [1] unless a linearization is applied beforehand [2,3] or only proportional damping is present and mode shapes of the conservative system are used [2]. Therefore errors are introduced when using a truncated modal basis, since q_i may depend on an omitted q_j. Under the assumption that all modes with high influence on the solution are included in the reduced model, this effect may be negligible.

Due to the large amount of computational modes the selection of a suitable modal basis $\boldsymbol{V}$ becomes tedious.

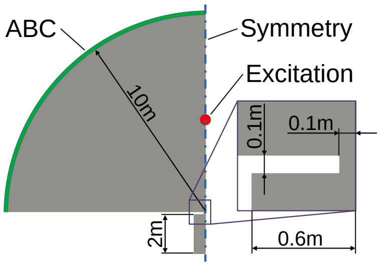

Figure 1: 2D Helmholtz resonator with exploited symmetry

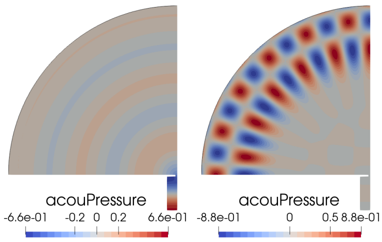

Figure 2: Second physical cavity mode (left) and typical computational mode (right)

One way of supporting this process is the comparison of the mode shapes $\boldsymbol{v}_i$ to the pressure field $\boldsymbol{p}_k = \boldsymbol{p}(\omega_k)$ obtained from the full mode (3) at a few frequencies ω_k and selecting the ones with the highest correlation. The modal assurance criterion (MAC) compares the shape of these two vectors by

$$0 \leq \mathrm{MAC}(\boldsymbol{p}_k, \boldsymbol{v}_i) = \frac{\left(\boldsymbol{p}_k^* \boldsymbol{v}_i\right)^2}{\left(\boldsymbol{p}_k^* \boldsymbol{p}_k\right)\left(\boldsymbol{v}_i^* \boldsymbol{v}_i\right)} \leq 1\,, \qquad (7)$$

where a larger value indicates a more similar shape [9].

4. APPLICATION EXAMPLE

We apply the suggested modal reduction procedure to a 2D Helmholtz resonator, where the symmetry in the vertical direction is exploited (fig. 1). The acoustic fluid considered has a density of $1000\,\mathrm{kg\,m^{-3}}$ and a compression modulus of $300\,\mathrm{Pa}$ yielding a speed of sound of $17.32\,\mathrm{m\,s^{-1}}$.

The QEVP was solved with the FEAST algorithm [10,11]. The resulting eigenvalue spectrum is displayed in fig. 3, illustrating how the discretization and truncation of

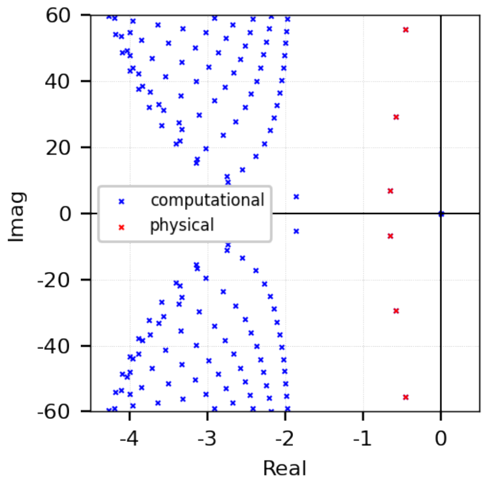

Figure 3: Eigenvalue spectrum of the problem.

the frequency-dependent eigenvalue problem of the continuous system results in many complex-conjugated computational modes. Additionally, super-critically damped computational modes along the real axis are obtained, which are located outside of the area covered in fig. 3. Furthermore, we can easily distinguish the physical cavity (Helmholz-resonator) modes by their lower damping. Figure 2 illustrates the difference in the pressure field between physical and computational modes. Cavity modes show high amplitudes within the cavity (trapped) and traveling waves in the radial direction. Computational modes feature high amplitudes outside of the cavity, especially close to the ABC, and can also be of higher order in the azimuthal direction. These modes describe possible wave propagation through the open domain (upper half-space).

The modes included in the reduction basis were selected based on their correlation to the solution of the full system (3) at certain frequency points according to the MAC (7). Figure 4 shows the MAC of the mode shapes and the harmonic solutions at the cavity natural frequencies. All modes with a MAC larger than a certain threshold are included in the reduced basis.

5. RESULTS

Figure 5 shows the frequency response of the full and the reduced models with the spatially averaged pressure amplitude across the cavity and the open domain. Figure 6 shows the correlation (MAC) of their pressure field.

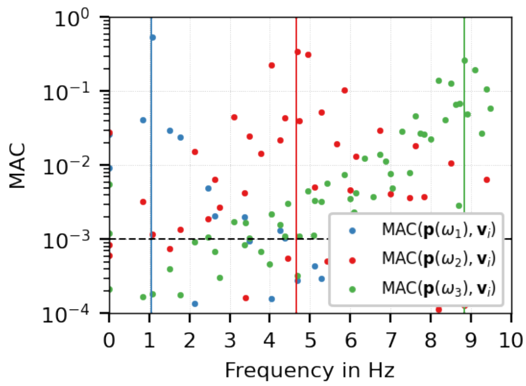

Figure 4: Correlation of mode shapes and harmonic solution at the cavity natural frequencies.

When only physical modes are included in the reduction basis, only the solution in the cavity at frequencies close to the cavity's natural frequencies is modeled well. For a high threshold of 0.1 (small reduced basis) the dynamics outside the cavity are improved, but overall the reduced model is still not accurate for frequencies further away from the natural frequencies. A decrease to 0.01 (larger reduced basis) greatly decreases the deviation from the full model. A threshold of 0.001 (largest modal basis) produces a reduced system that matches the full system almost perfectly with a slight decrease in MAC above 50 Hz. However, this may be due to the fact that only modes up to a damped eigenfrequency of $\omega_{\mathrm{d},i} < 60\,\mathrm{s}^{-1}$ were used for the reduced models. The improvements in model accuracy by using computational modes illustrate their importance in the dynamics, especially outside the cavity and at frequencies other than the natural frequencies.

The performance of the model reduction technique was evaluated by comparing the single-core CPU times and calculating a break-even-point, from whereon the application of the reduced model brings time savings when performing a frequency domain analysis. The calculation of the eigenvalues and eigenmodes took on average 253.2 s and projecting the system into the modal subspace around 21.6 s. Considering a mean time of 0.62 s per frequency step for solving the full system, around 450 frequency steps were necessary until there were improvements in CPU time with the utilized setup. This is because the evaluation of the eigenmodes and projection of the full system into the modal subspace is rather time costly. After the break-even-point the reduced model clearly outperforms the full model with a mean CPU time of 0.127 s per frequency step, hence five times faster than solving the full model.

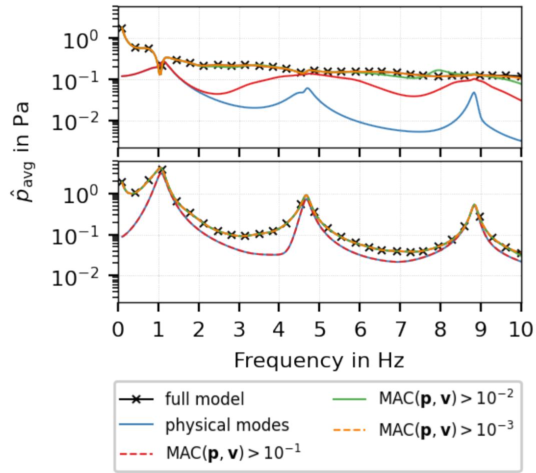

Figure 5: Frequency response of average pressure $\hat{p}_{\mathrm{avg}}$ in the open domain (top) and cavity (bottom) for the full model and different reduced models.

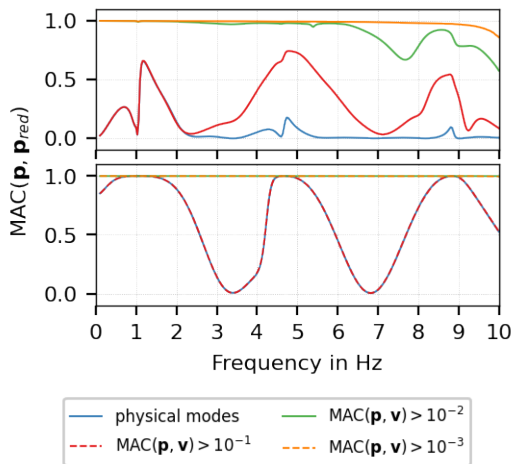

Figure 6: Similarity of pressure fields (top: open domain, bottom: cavity) for different reduced models as compared to the full model.

6. CONCLUSION

The presented workflow of model reduction by projection into a modal subspace is applicable to open-domain acoustics. By selecting a reduction basis consisting of eigenmodes based on their similarity to the harmonic solution at certain frequencies a reduced order model that yields an excellent approximation of the full model could be generated. The degrees of freedom were reduced from over 30,000 physical to merely 109, or less, modal ones. This comes at the cost of validity at the frequency range the reduced model is designed for. While the reduced order model is cheap to evaluate for a single frequency point, its generation, especially the computation of the eigenmodes is computationally expensive. Therefore this method seems only useful if a quite finely resolved frequency domain solution is desired.

7. REFERENCES

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