



OPEN ACCESS OF A WIDE-ANGLE PARABOLIC EQUATION MODEL FOR SOUND PROPAGATION IN A MOVING ATMOSPHERE ABOVE AN ABSORBING AND ROUGH GROUND.

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ABSTRACT

Parabolic equation based methods are widely used in environmental acoustics because they can accurately model acoustic propagation of complex sound sources above a mixed ground in a refractive and scattering atmosphere. The present paper proposes an open-access model based on the wide-angle parabolic equation (WAPE) in moving medium for arbitrary Mach numbers. The WAPE is derived by an expansion of a square-root pseudo-differential operator using Padé(1,1) approximant in one-way wave equation. It is then solved by a finite-difference technique based on Crank-Nicholson method, and is valid for propagation angle up to 35° with respect to the nominal direction. The paper describes both the validation process against an analytical solution, and the platform for on-line open-access, including comments and examples of numerical predictions for end-users.

Keywords: Parabolic equation, open-access

1. INTRODUCTION

The parabolic equation (PE) approximation method is an important contribution to the modeling of wave propagation as it can solve problem of long-range wave propagation in a range-dependent environments. It was first introduced to solve electromagnetic problems [1] and was then applied in other scientific field such as ocean acoustics [2, 3], geophysics [4] or outdoor acoustics for example [5–11]. The main limitations of the PE methods are the small validity angle with respect to the nominal propagation direction and phase error due to effective sound speed approximation, which have been addressed quite recently [12–14]. However, despite the popularity of the PE methods, there are few open-access repositories allowing its sharing in the outdoor acoustics community.

Thus, this paper aims at sharing an open-access repository [15] of a wide-angle parabolic equation model based on a Padé (1,1) approximation and a numerical resolution with the Crank-Nicholson algorithm. The code is written in *Matlab* language. It differs from the version proposed in [14] by incorporating the effects of ground roughness (see Section 2.3). The paper is organised as follow: the Sec.2.1 presents the theory of the WAPE in moving and motionless medium, the Sec.2.2 presents the numerical solution of the WAPE with the Crank-Nicholson algorithm, the Sec.2.3 and Sec.2.4 review the ground effects modelling and atmosphere modelling, the Sec.3 presents a validation of the WAPE with an analytical solution for constant wind, and finally the Sec.4 gives the conclusions.

2. REVIEW OF THEORY

2.1 WAPE in moving and motionless medium

An extra-wide-angle parabolic equation (EWAPE) for sound wave propagation in a moving medium with arbitrary Mach numbers M_x have been proposed recently [14]. In a two-dimensional vertical plane (x, z) and assuming that the air density ρ_0 is a constant, Equations (27) and (39) of [14] for the sound pressure $\hat{p}(x, z)$ and for the velocity potential $\hat{\phi}(x, z)$ in the frequency domain reduce to:

$$\hat{p}(x, z) = \left(1 + \frac{iM_x}{k_0} \frac{\partial}{\partial x}\right) \hat{\phi}(x, z), \quad (1)$$

$$\left(\frac{\partial}{\partial x} - ik_0\gamma_x^2\sqrt{1+\varepsilon+\hat{\mu}} + ik_0\hat{\tau}\right) \hat{\phi}(x, z) = 0, \quad (2)$$

where $k_0 = \omega/c_0$ is the wavenumber, c_0 is the reference sound speed, $\gamma_x^2 = 1/(1 - M_x^2)$, $\varepsilon = (c_0/c)^2 - 1$ is the deviation of the refractive index, $\hat{\mu} = \frac{1}{\gamma_x^2 k_0^2} \frac{\partial^2}{\partial z^2}$, and $\hat{\tau} = M_x \gamma_x^2 \sqrt{1 + \varepsilon}$.

As suggested by [14], the square-root operator in Eq.(2) can be approximated with a Padé (n, n) series expansion. The present work considers a Padé $(1,1)$ which lead to a 35° validity angle with respect to the horizontal direction x [16, 17]. Introducing the variable $\bar{\phi}$ related to the velocity potential $\hat{\phi}$ by $\hat{\phi}(x, z) = \exp(ik_0x)\bar{\phi}(x, z)$, where $\bar{\phi}$ is discretized using a Cartesian mesh of size $\Delta x = \Delta z = \lambda/10$: $\phi_m^n = \bar{\phi}((m-1)\Delta x, (n-1)\Delta z)$ with λ the wavelength. The Eq. (2) then becomes:

$$\Psi_1(x, z) \frac{\partial \bar{\phi}}{\partial x} = ik_0 \Psi_2(x, z) \bar{\phi}, \quad (3)$$

where the functions Ψ_1 and Ψ_2 are given by:

$$\Psi_m = h_{m,0} + \frac{h_{m,2}}{k_0^2} \frac{\partial^2}{\partial z^2}, \quad m = 1, 2. \quad (4)$$

The coefficients $h_{m,j}$ are defined as $h_{1,0} = 1 + b_{1,1}\varepsilon$, $h_{1,2} = b_{1,1}/\gamma_x^2$, $h_{2,0} = a_{1,1}\gamma_x^2\varepsilon - (1 + b_{1,1}\varepsilon)\tilde{\tau}$, and $h_{2,2} = a_{1,1} - b_{1,1}\tilde{\tau}/\gamma_x^2$, with $a_{1,1} = 1/2$ and $b_{1,1} = 1/4$. The function $\tilde{\tau}$ is written:

$$\tilde{\tau} = M_x \gamma_x^2 (\sqrt{1 + \varepsilon} - M_x) = \hat{\tau} - M_x^2 \gamma_x^2. \quad (5)$$

In a second step, the acoustic pressure \hat{p} can be calculated from ϕ_m^n at $x_m = m\Delta x$ and $z_n = n\Delta z$ using a second-order centered finite difference scheme [14]:

$$\hat{p}(x_m, z_n) = e^{ik_0x_m} \left[(1 - M_x)\phi_m^n + \frac{iM_x}{2k_0\Delta x} [\phi_{m+1}^n - \phi_{m-1}^n] \right], \quad (6)$$

where the starting field is defined as [17]:

$$\bar{\phi}(0, z_s) = \sqrt{ik_0(A_0 + A_2k_0^2z_s^2)} e^{-\frac{k_0^2z_s^2}{B}}, \quad (7)$$

with $A_0 = 1.3717$, $A_2 = -0.3701$, $B = 3$, $z_s = z - h_s$, and h_s is the source height.

2.2 Numerical solution of the WAPE in moving medium

The Crank-Nicholson algorithm can be used to reduce Equation (3) to a matrix system that can be solved numerically from x to $x + \Delta x$:

$$\left[\Psi_1 - \frac{ik_0\Delta x}{2} \Psi_2 \right] \bar{\phi}(x + \Delta x) = \left[\Psi_1 + \frac{ik_0\Delta x}{2} \Psi_2 \right] \bar{\phi}(x), \quad (8)$$

where the terms Ψ_1 and Ψ_2 can be written:

$$\Psi_1 = 1 + \frac{\varepsilon}{4} + \frac{1}{4k_0^2\gamma_x^2} \frac{\partial^2}{\partial z^2}, \quad (9)$$

$$\Psi_2 = \frac{\gamma_x^2\varepsilon}{2} - \left(1 + \frac{\varepsilon}{4}\right)\tilde{\tau} + \frac{(2\gamma_x^2 - \tilde{\tau})}{4k_0^2\gamma_x^2} \frac{\partial^2}{\partial z^2}. \quad (10)$$

The second derivative with respect to z is estimated using a second order finite difference scheme:

$$\left(\frac{\partial^2}{\partial z^2} \right) \phi_m^n = \frac{\phi_m^{n+1} - 2\phi_m^n + \phi_m^{n-1}}{k_0^2\Delta z^2}. \quad (11)$$

The numerical scheme associated with the Crank-Nicholson algorithm for the WAPE method is thus:

$$M_1\phi_{m+1}^n = M_2\phi_m^n, \quad (12)$$

where the matrices M_1 and M_2 are given by:

$$M_1\phi_m^n = \left[1 + \frac{\varepsilon_m^n}{4} - \frac{ik_0\Delta x}{2} \left(\frac{(\gamma_x^2)_m^n \varepsilon_m^n}{2} - (1 + \frac{\varepsilon_m^n}{4})\tilde{\tau}_m^n \right) \right] \phi_m^n + \left[\frac{2 - ik_0\Delta x(2(\gamma_x^2)_m^n - \tilde{\tau}_m^n)}{8k_0^2(\gamma_x^2)_m^n} \right] \frac{\phi_m^{n+1} - 2\phi_m^n + \phi_m^{n-1}}{\Delta z^2}, \quad (13)$$

$$M_2\phi_m^n = \left[1 + \frac{\varepsilon_m^n}{4} - \frac{ik_0\Delta x}{2} \left(\frac{(\gamma_x^2)_m^n \varepsilon_m^n}{2} - (1 + \frac{\varepsilon_m^n}{4})\tilde{\tau}_m^n \right) - \frac{2 - ik_0\Delta x(2(\gamma_x^2)_m^n - \tilde{\tau}_m^n)}{4k_0^2(\gamma_x^2)_m^n \Delta z^2} \right] \phi_m^n + \left[\frac{2 - ik_0\Delta x(2(\gamma_x^2)_m^n - \tilde{\tau}_m^n)}{8k_0^2(\gamma_x^2)_m^n \Delta z^2} \right] (\phi_m^{n+1} + \phi_m^{n-1}). \quad (14)$$

The matrix M_1 in Equation (13) is tridiagonal with diagonal elements

$$b_n = \left[1 + \frac{\varepsilon}{4} - \frac{ik_0\Delta x}{2} \left(\frac{\gamma_x^2\varepsilon}{2} - (1 + \frac{\varepsilon}{4})\tilde{\tau} \right) - \frac{2 - ik_0\Delta x(2\gamma_x^2 - \tilde{\tau})}{4k_0^2\gamma_x^2\Delta z^2} \right], \quad (15)$$

and off-diagonal elements

$$a_n = c_n = \left[\frac{2 - ik_0\Delta x(2\gamma_x^2 - \tilde{\tau})}{8k_0^2\gamma_x^2\Delta z^2} \right]. \quad (16)$$

Similarly, the matrix M_2 in Equation (14) is tridiagonal with diagonal elements

$$e_n = \left[1 + \frac{\varepsilon}{4} + \frac{ik_0\Delta x}{2} \left(\frac{\gamma_x^2\varepsilon}{2} - \left(1 + \frac{\varepsilon}{4}\right)\tilde{\tau} \right) - \frac{2 + ik_0\Delta x(2\gamma_x^2 - \tilde{\tau})}{4k_0^2\gamma_x^2\Delta z^2} \right], \quad (17)$$

and off-diagonal elements

$$d_n = f_n = \left[\frac{2 + ik_0\Delta x(2\gamma_x^2 - \tilde{\tau})}{8k_0^2\gamma_x^2\Delta z^2} \right]. \quad (18)$$

The boundary condition at $z = 0$ ($n = 1$) written with respect to the admittance β_{eff} can be obtained by using the centered second order scheme at the fictitious point $z = -\Delta z$:

$$\frac{\phi_m^2 - \phi_m^0}{2\Delta z} + ik_0\beta_{\text{eff}}\phi_m^1 = 0. \quad (19)$$

The first lines of the matrices M_1 and M_2 are changed accordingly, with modified coefficients:

$$c_{1g} = 2c_1, \quad b_{1g} = b_1 + 2ik_0\Delta z\beta_{\text{eff}}c_1, \\ f_{1g} = 2f_1, \quad e_{1g} = e_1 + 2ik_0\Delta z\beta_{\text{eff}}f_1.$$

The expression of β_{eff} is detailed in the next section.

2.3 Ground impedance model

The domain is bounded by a ground with an acoustic impedance condition at $z = 0$. The acoustic properties of the ground are taken into account by an effective admittance model [18] that considers sound absorption by pores and sound scattering by the surface roughness. The implementation of this model in the parabolic equation method has been validated [10]. The effective admittance β_{eff} is defined as:

$$\beta_{\text{eff}} = \beta + \beta_{\text{rough}} = \frac{1}{Z} + \beta_{\text{rough}}, \quad (20)$$

where β is the acoustic admittance of the ground, Z is the acoustic impedance of the ground and β_{rough} is the average effect of surface roughness on admittance. For example, the impedance Z can be calculated using Miki's impedance model [19]:

$$\frac{Z}{Z_0} = 1 + 6.17 \left(\frac{\rho_0 f}{a_{\text{fr}}} \right)^{-0.632} + i9.44 \left(\frac{\rho_0 f}{a_{\text{fr}}} \right)^{-0.632} \quad (21)$$

$$\frac{k}{k_0} = 1 + 8.73 \left(\frac{\rho_0 f}{a_{\text{fr}}} \right)^{-0.618} + i12.76 \left(\frac{\rho_0 f}{a_{\text{fr}}} \right)^{-0.618}, \quad (22)$$

where $Z_0 = \rho_0 c_0$ is the specific impedance of air, ρ_0 is the density of air, k_0 is the wavenumber in the air, k is the wavenumber for sound waves in the ground, $f = \omega/2\pi$ is the frequency of the sound and a_{fr} is the ground airflow resistivity ($\text{kN}\cdot\text{s}\cdot\text{m}^{-4}$). This model has a frequency validity range defined by the relation $f > 0.01 a_{\text{fr}}/\rho_0$ [20].

The expression for β_{rough} corresponds to a 2D rough surface with a small and slowly-varying roughness [21], valid under the condition $|k_0\zeta \cos \theta_i| < 1$ and $|\partial\zeta/\partial x| < 1$. The term ζ (m) is the roughness height profile of the ground and θ_i is the angle between the incident wave and the line perpendicular to the ground surface. The parameter β_{rough} is given by:

$$\beta_{\text{rough}}(\kappa) = \int_{-\infty}^{+\infty} \frac{d\kappa'}{k_0 k_z(\kappa')} (k_0^2 - \kappa\kappa') W(\kappa - \kappa'), \quad (23)$$

where $\kappa = k_0 \sin \theta_i$ is the x component of the wavenumber, κ' is the integration variable, $k_z(\kappa) = \sqrt{k_0^2 - \kappa^2}$ is the z component of the wavenumber, and W is the roughness spectrum of the ground, corresponding to the Fourier transform of the autocorrelation function of the surface height profile. Considering that the probability density of the ground roughness heights is a normal distribution, W is defined as:

$$W(k) = \frac{\sigma_h^2 l_c}{2\sqrt{\pi}} e^{-\frac{k^2 l_c^2}{4}}, \quad (24)$$

where σ_h is the standard deviation of the ground roughness heights and l_c is the correlation length of the horizontal variations of the ground.

The Eq.(23) can be rewritten and numerically solved with the following formulation [22]:

$$a = \sum_{s=\pm 1} \int_0^{\sqrt{k_0}} \frac{[k_0^2 + s\kappa(k_0 - u^2)]^2}{k_0\sqrt{-u^2 + 2k_0}} \times W(\kappa + s[k_0 - u^2]) du \quad (25)$$

$$b = - \sum_{s=\pm 1} \int_0^{+\infty} \frac{[k_0^2 + s\kappa\sqrt{k_0^2 + u^2}]^2}{k_0\sqrt{k_0^2 + u^2}} \times W\left(\kappa + s\sqrt{k_0^2 + u^2}\right) du. \quad (26)$$

2.4 Atmospheric effects

The refraction effect is considered through the wind vertical profile $U(z)$ and temperature vertical profile $T(z)$:

$$U(z) = U_{\text{ref}} \left(\frac{z}{z_{\text{ref}}} \right)^{\alpha}, \quad (27)$$

$$T(z) = T_0 + a_T \ln \frac{z}{z_0}, \quad (28)$$

where U_{ref} (m/s) is the wind speed at height z_{ref} above the ground level, z (m) is the height above the ground, α is the wind shear factor, T_0 (K) is the air temperature at the ground surface, a_T (K/m) is a refraction coefficient that determine the shape of the temperature profile, and $z_0 = 0.13h_v$ (m) is the roughness height that depends on vegetation height h_v (m). The Mach number in the direction of propagation is thus defined as:

$$M_x = U(z) \cos \theta / \sqrt{\Gamma R_g T(z)} \quad (29)$$

where $\Gamma = 1.41$ is the heat ratio, $R_g = 286.7$ is the perfect gas constant and θ is the angle between the wind direction and source-receiver direction (rad).

Atmospheric absorption is considered in accordance with the standard [23], which depends on air temperature T (K), atmospheric pressure p_{atm} (Pa) and relative humidity h_r (%).

Atmospheric turbulence scattering effect can be approximated by adding a scattering contribution $\text{SPL}_{\text{scatter}}$ to the free field attenuation term in refracting atmosphere that neglect turbulent scattering $\text{SPL}_{\text{noscatter}}$, as proposed in the *Harmonoise* project [24, 25]. Although this approach does not consider complex phase effects related to atmospheric turbulence, it has the advantage of being fast in computing duration and it simulates realistic sound pressure levels in the interference pattern regions and in the shadow zone. The attenuation term Δ_L is thus given by:

$$\Delta_L = 10 \log_{10} \left(10^{\frac{\text{SPL}_{\text{noscatter}}}{10}} + 10^{\frac{\text{SPL}_{\text{scatter}}}{10}} \right), \quad (30)$$

with:

$$\begin{aligned} \text{SPL}_{\text{scatter}} = & 25 + 10 \log_{10} \gamma_T \\ & + 3 \log_{10} \frac{\omega}{1000} + 10 \log_{10} \frac{r}{100}, \end{aligned} \quad (31)$$

where r (m) the source-receiver distance, and γ_T a measure of turbulence strength [25].

3. VALIDATION OF WAPE MODEL AGAINST AN ANALYTICAL SOLUTION

Following [14], the implementation is validated against an analytical solution for uniformly moving medium (constant wind vertical profile), in presence of a perfectly flat and rigid ground.

Figure 1 presents results for frequencies $f = 50$ Hz, 250 Hz and 1000 Hz with a point source located at $z = 80$ m, a Mach number of $M_x = 0.05$, and a receiver height $z_r = 2$ m. This geometrical configuration has been chosen to present a comparison with interference patterns. Results show that the PE method do not predict sound pressure accurately in close range up to 150 m, which is due to the angular validity of the method. There is a good agreement between the WAPE and the analytical solution for distances above 150 m, which is consistent with the results of [14].

4. CONCLUSION

This paper presented an open-access repository for a wide-angle parabolic equation model for sound propagation in a moving atmosphere above an absorbing and rough ground [15]. The theory of the WAPE in moving medium has been reviewed, the numerical solution of the WAPE has been given according to the Crank-Nicholson algorithm, ground impedance model and the modelling of atmospheric effects have been detailed. A validation of the WAPE model against an analytical solution for constant wind is finally proposed, where results show an excellent agreement between the two formulations in the far field (distances above 150 m).

5. REFERENCES

- [1] M. A. Leontovich and V. A. Fock, "Solution of the problem of propagation of electromagnetic waves along the earth's surface by the method of parabolic equation," *J. Phys. Ussr*, vol. 10, no. 1, pp. 13–23, 1946.
- [2] F. D. Tappert, "The parabolic approximation method," in *Wave Propagation and Underwater Acoustics* (J. B. Keller and J. S. Papadakis, eds.), Lecture Notes in Physics, pp. 224–287, Berlin, Heidelberg: Springer, 1977.
- [3] M. D. Collins, "Applications and time-domain solution of higher-order parabolic equations in underwater acoustics," *The Journal of the Acoustical Society of America*, vol. 86, pp. 1097–1102, Sept. 1989.
- [4] J. F. Claerbout, *Fundamentals of geophysical data processing*, vol. 274. mcgraw-hill ed., 1976.
- [5] M. J. White and K. E. Gilbert, "Application of the parabolic equation to the outdoor propagation of sound," *Applied Acoustics*, vol. 27, pp. 227–238, Jan. 1989.
- [6] J. N. Craddock and M. J. White, "Sound propagation over a surface with varying impedance: A parabolic equation approach," *The Journal of the Acoustical Society of America*, vol. 91, pp. 3184–3191, June 1992.
- [7] P. Blanc-Benon, L. Dallois, and D. Juvé, "Long range sound propagation in a turbulent atmosphere within the parabolic approximation," *Acta Acustica united with Acustica*, vol. 87, no. 6, pp. 659–669, 2001.
- [8] D. Heimann and E. M. Salomons, "Testing meteorological classifications for the prediction of long-term average sound levels," *Applied Acoustics*, vol. 65, pp. 925–950, Oct. 2004.
- [9] E. Barlas, W. J. Zhu, W. Z. Shen, M. Kelly, and S. J. Andersen, "Effects of wind turbine wake on atmospheric sound propagation," *Applied Acoustics*, vol. 122, pp. 51–61, July 2017.
- [10] B. Kayser, B. Gauvreau, and D. Ecoti re, "Sensitivity analysis of a parabolic equation model to ground impedance and surface roughness for wind turbine noise," *The Journal of the Acoustical Society of America*, vol. 146, pp. 3222–3231, Nov. 2019.

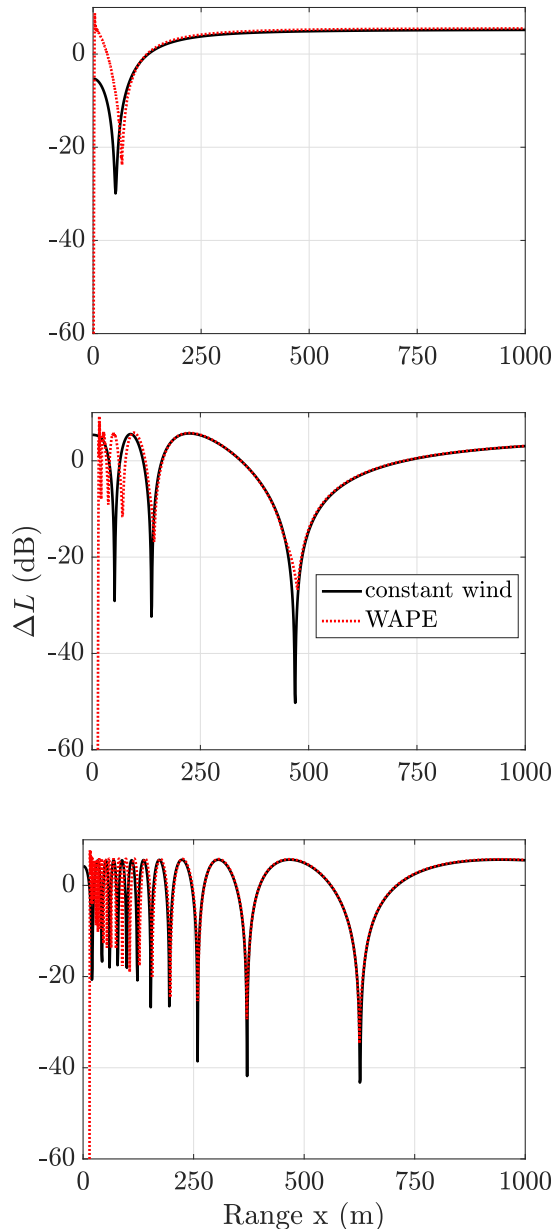


Figure 1. Comparison of sound pressure relative to free field ΔL calculated with the analytical formulation (black line) and the parabolic equation solution (red line) for $f = 50$ Hz (top), $f = 250$ Hz (middle) and $f = 1000$ Hz (bottom).

- [11] T. Van Renterghem, K. V. Horoshenkov, J. A. Parry, and D. P. Williams, “Statistical analysis of sound level predictions in refracting and turbulent atmospheres,” *Applied Acoustics*, vol. 185, p. 108426, Jan. 2022.
- [12] V. E. Ostashev, M. B. Muhlestein, and D. K. Wilson, “Extra-wide-angle parabolic equations in motionless and moving media,” *The Journal of the Acoustical Society of America*, vol. 145, pp. 1031–1047, Feb. 2019.
- [13] V. E. Ostashev, D. K. Wilson, M. Muhlestein, M. Shaw, M. E. Swearingen, and S. McComas, “Extra-wide-angle parabolic equation for wave propagation in inhomogeneous media,” *The Journal of the Acoustical Society of America*, vol. 146, pp. 3035–3036, Oct. 2019.
- [14] V. E. Ostashev, D. K. Wilson, and M. B. Muhlestein, “Wave and extra-wide-angle parabolic equations for sound propagation in a moving atmosphere,” *The Journal of the Acoustical Society of America*, vol. 147, pp. 3969–3984, June 2020.
- [15] B. Kayser, “Wide-angle parabolic equation model,” 2023. <https://github.com/bkayser13/WAPE/>.
- [16] M. D. Collins, “A split-step Padé solution for the parabolic equation method,” *The Journal of the Acoustical Society of America*, vol. 93, pp. 1736–1742, Apr. 1993.
- [17] E. M. Salomons, *Computational Atmospheric Acoustics*. Netherlands: Kluwer Academic, kluwer academic ed., 2001.
- [18] F. G. Bass and I. M. Fuks, *Wave Scattering from Statistically Rough Surfaces: International Series in Natural Philosophy*. Elsevier, 1979.
- [19] Y. Miki, “Acoustical properties of porous materials-Modifications of Delany-Bazley models-,” *Journal of the Acoustical Society of Japan (E)*, vol. 11, no. 1, pp. 19–24, 1990.
- [20] R. Kirby, “On the modification of Delany and Bazley formulae,” *Applied Acoustics*, vol. 86, pp. 47–49, Dec. 2014.
- [21] Y. Brelet and C. Bourlier, “Bistatic Scattering from a Sea-Like One-Dimensional Rough Surface with the Perturbation Theory in HF-VHF Band,” in *IGARSS 2008 - 2008 IEEE International Geoscience and Remote Sensing Symposium*, vol. 4, pp. IV – 1137–IV – 1140, July 2008.
- [22] Y. Brelet and C. Bourlier, “SPM Numerical Results from an Effective Surface Impedance for a One-Dimensional Perfectly-Conducting Rough Sea Surface,” *Progress In Electromagnetics Research*, vol. 81, pp. 413–436, 2008.
- [23] I. 9613-1:1993, “Acoustics — Sound attenuation in free field — Part 1: atmospheric absorption calculation,” tech. rep., International Standards Organization, 1993.
- [24] D. van Maercke and J. Defrance, “Development of an Analytical Model for Outdoor Sound Propagation Within the Harmonoise Project,” *Acta Acustica united with Acustica*, vol. 93, pp. 201–212, Mar. 2007.
- [25] E. Salomons, D. van Maercke, J. Defrance, and F. de Roo, “The Harmonoise Sound Propagation Model,” *Acta Acustica united with Acustica*, vol. 97, pp. 62–74, Jan. 2011.