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EFFECT OF APPROPRIATE ROTATIONAL INERTIA ON HIGH FREQUENCY VIBRATIONS IN TIMOSHENKO BEAM THEORY

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ABSTRACT

In this paper, a simple Timoshenko beam theory (STBT) is proposed and a radiative energy transfer model (RETM) based on STBT theory is established. The appropriate rotational inertia of Timoshenko beam theory (TBT) is discussed and the effects of two forms of rotational inertia on the high frequency vibration of a transverse vibrating beam are evaluated. Based on elastic theory and wave propagation theory, it is found that when the frequency is higher than the critical frequency, there is a second spectrum in TBT theory, which has no obvious physical significance. On the other hand, the so-called slope inertia Bresse Timoshenko (SIBT) is neither consistent nor accurate when the wavelength of the vibrating beam is close to the height of the beam. The theory cannot be used for vibration analysis above the critical frequency. By numerical analysis of the surface, the simple Timoshenko theory established in this paper and the energy model of high frequency vibration are very direct and effective. The effectiveness and simplicity of this model are verified by comparing it with several beam theory wave propagation methods.

Keywords: *Timoshenko beam theory (TBT), rotary inertia, wave velocity, RETM*

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1. INTRODUCTION

There are several beam theories that describe the dynamic behavior of beams or beam-like structures, the classical theories are Euler-Bernoulli beam theory (i.e. classical beam theory), Timoshenko beam theory (i.e. first-order shear beam theory), etc. Euler-Bernoulli beam theory (EBT) does not consider shear deformation and rotational inertia, will overestimate the frequency of the structure, so it is more suitable for thin beam structure [1]. Rayleigh beam theory takes into account the rotational inertia of the structure, which alleviates the frequency overestimation problem to some extent [2]. Pure shear beam theory takes into account the effect of shear deformation and has certain applicability to short and thick beam structures. Timoshenko beam theory (TBT) takes into account first-order shear deformation and rotational inertia, which improves the accuracy of non-slender beams and is widely used in engineering [3]. Although a great deal of research has been done on TBT in recent years [4–7], there are still some interesting issues worth exploring. First of all, the most controversial phenomenon is the existence of two spectrum TBT, two different frequencies correspond to different modes [8–10]. Some scholars believe that the second spectrum is physically meaningless [11, 12], and some scholars support the existence of the second spectrum. The second spectrum in TBT is derived from the fourth time derivative term, which is indeed of no obvious physical significance. Meanwhile, compared with other terms, the effect of the fourth time derivative term on the spectrum is very small and can be ignored. Love [13] proposed an improved Timoshenko beam by removing the fourth time derivative term, known as a truncated Bresse-Timoshenko beam [14]. It has been pointed out that the

Love beam theory, although more simple and consistent, cannot be deduced from the variation.

More recently, Elishakoff et al. [15–17] proposed a Timoshenko beam based on slope inertia (SIBT) by using deflection slope instead of bending rotation in the kinetic energy expression. The fourth time derivative term of the Timoshenko beam is eliminated while a new sixth order term is introduced. Although Elishakoff proposed that the derivation and application of SIBT has some advantages over Timoshenko beam, it is found that the variational process is inconsistent with the original displacement field form and that there is no elastic wave propagation beyond the critical frequency. In this study, the physical significance of the two spectra of TBT is no longer entangled. From the perspective of wave propagation, the beam model suitable for high-frequency vibration is established by taking the appropriate moment of inertia. Compared with other beam theories, the model is simpler in form and meets the requirements in accuracy.

2. THEORETICAL FORMULATION

2.1 Different rotational inertia in beam theories

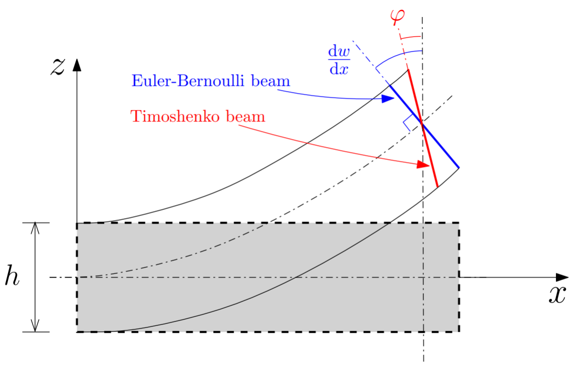

Figure 1. Timoshenko beam model

According to the Timoshenko beam theory, the axial displacement $u_x(x,z,t)$ and transverse displacement $u_z(x,z,t)$ could be expressed as follows:

$$u_z\left(x,z,t\right) = w\left(x,t\right), u_x\left(x,z,t\right) = -z\varphi\left(x,t\right). \quad (1)$$

$w(x,t)$ is the displacement components of the mid-plane along z direction and $\phi(x,t)$ is the rotation angle of the cross section. The normal strain and shear strain related to the displacements are respectively written as:

$$\left\{ \begin{array}{c} \varepsilon_{xx} \\ \gamma_{xz} \end{array} \right\} = \left\{ \begin{array}{c} \frac{\partial u_x}{\partial x} \\ \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \end{array} \right\} = \left\{ \begin{array}{c} -z\frac{\partial \varphi}{\partial x} \\ \frac{\partial w}{\partial x} - \varphi \end{array} \right\} \quad (2)$$

The vibrational energy of beam is composed of the potential energy U_e and kinetic energy T,

$$\begin{aligned} U_e &= \frac{1}{2}\int_0^L \left[D_{11}\left(\frac{\partial \varphi}{\partial x}\right)^2 + C_{13}\left(\frac{\partial w}{\partial x} - \varphi\right)^2 \right] \mathrm{d}x, \\ W_f &= \int_0^L f(x,t)w(x,t)\mathrm{d}x, \\ T &= \frac{1}{2}\int_0^L \left[I_D\left(\frac{\partial \varphi}{\partial t}\right)^2 + m\left(\frac{\partial w}{\partial t}\right)^2 \right] \mathrm{d}x, \end{aligned} \quad (3)$$

where W_f is the work provided by external forces f, which is assumed in the form of $f(x,t) = Fe^{\mathrm{i}\omega t}\delta(x - x_0)$, i is the imaginary unit, ω is circular frequency, D_{11} is the bending stiffness, C_{13} is the out of plane shear rigidity, I_D is the moment of inertia of the cross-section and m is the mass per unit length. The expressions of these parameters are as follows:

$$\begin{aligned} D_{11} &= b\int_{-h/2}^{h/2} Ez^2 dz, C_{13} = b\int_{-h/2}^{h/2} sGdz, \\ I_{\mathrm{D}} &= b\int_{-h/2}^{h/2} \rho z^2 dz, m = b\int_{-h/2}^{h/2} \rho dz. \end{aligned} \quad (4)$$

s is the correction factor, for beams with rectangular cross-sections, taking 5/6 for s is reasonable.

The Lagrangian of the bending motion of the FG beam could be written as $L = T - (U_e + U_T) + W_f$ and the motion governing equation could be obtained by applying Hamilton's principle as follows:

$$\begin{aligned} &\delta w : m\frac{\partial^2 w}{\partial t^2} - C_{13}\left(\frac{\partial^2 w}{\partial x^2} - \frac{\partial \varphi}{\partial x}\right) = Fe^{\mathrm{i}\omega t}\delta(x - x_0), \\ &\delta \varphi : D_{11}\frac{\partial^2 \varphi}{\partial x^2} - I_D\frac{\partial^2 \varphi}{\partial t^2} + C_{13}\left(\frac{\partial w}{\partial x} - \varphi\right) = 0. \end{aligned} \quad (5)$$

Eliminating variable w or φ in Eq. (5), the governing equations for transverse displacement is derived as follows:

$$\begin{aligned} &D_{11}\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} - \left(I_D + \frac{mD_{11}}{C_{13}}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} \\ &+ \frac{mI_D}{C_{13}}\frac{\partial^4 w}{\partial t^4} = \left(1 - \frac{I_D\omega^2}{C_{13}}\right)Fe^{\mathrm{i}\omega t}\delta(x - x_0). \end{aligned} \quad (6)$$

The form of rotational kinetic energy in TBT is consistent with the displacement field, so the governing equation can also be derived by force equilibrium, which is consistent with the variational process.

SIBT theory expresses rotational kinetic energy in the form of slope inertia,

$$T = \frac{1}{2}\int_0^L \left[I_D \left(\frac{\partial^2 w}{\partial x \partial t}\right)^2 + m\left(\frac{\partial w}{\partial t}\right)^2 \right] \mathrm{d}x, \tag{7}$$

and then the governing equation is derived as:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} - \left(I_D + \frac{mD_{11}}{C_{13}}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{I_D D_{11}}{C_{13}}\frac{\partial^6 w}{\partial x^4 \partial t^2} = (1 - \frac{I_D\omega^2}{C_{13}})Fe^{\mathrm{i}\omega t}\delta(x - x_0). \tag{8}$$

If the rotation term $I_D\frac{\partial^2 \varphi}{\partial t^2}$ in Eq. (5) is replaced by $I_D\frac{\partial^3 w}{\partial x \partial t^2}$ from the point of view of force balance, a simpler governing equation is obtained as follows:

$$D_{11}\frac{\partial^4 w}{\partial x^4} + m\frac{\partial^2 w}{\partial t^2} - \left(I_D + \frac{mD_{11}}{C_{13}}\right)\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \tag{9}$$

In fact, this rotation pattern is consistent with the rayleigh beam theory, and the governing equation is much simpler. We can call it the simple Timoshenko beam theory (STBT).

2.2 Wave propagation and energy model

The analysis of high frequency vibration is based on wave theory and energy model. Firstly, the wave propagation under several theories is compared. In the unloaded region of the beam, the dispersion equation for the free vibration of the beam could be obtained by substituting the traveling wave form as $w(x,t) = w_0 e^{\mathrm{i}(\omega t - \kappa x)}$ into the homogeneous form of Eqs. (6, 8 and 9).

$$D_{11}\kappa^4 - m\omega^2 - \left(I_D + \frac{mD_{11}}{C_{13}}\right)\kappa^2\omega^2 + \mu_1\frac{mD_{11}}{C_{13}}\omega^4 - \mu_2\frac{I_D D_{11}}{C_{13}}\kappa^4\omega^2 = 0 \tag{10}$$

When $\mu_1 = 1, \mu_2 = 0$, Eq (10) is the dispersion equation of TBT; When $\mu_1 = 0, \mu_2 = 1$, Eq (10) is the dispersion equation of SIBT; When $\mu_1 = 0, \mu_2 = 0$, Eq (10) is the dispersion equation of STBT. There is critical frequency $\omega_c = \sqrt{\frac{C_{13}}{I_D}}$ in TBT and SIBT, which determines the propagation of waves in both theories. The roots of Eq. (10) are as follows:

$$\kappa_1 = \sqrt{N_1}, \kappa_2 = -\sqrt{N_1}, \kappa_3 = \sqrt{N_2}, \kappa_4 = -\sqrt{N_2}, \tag{11}$$

where

$$N_{1,2} = \frac{\left(I_D + \frac{mD_{11}}{C_{13}}\right)\omega^2 \pm \Delta}{2\left(D_{11} - \mu_2\frac{I_D D_{11}}{C_{13}}\omega^2\right)}, \tag{11a}$$

Δ is the discriminant of the roots. There are two wave propagation modes in TBT. When frequency $f < f_c$, κ_1 and κ_2 are propagation waves with opposite propagation directions, κ_3 and κ_4 are evanescent waves. When $f > f_c$, all waves are propagating waves. In SIBT, When $f < f_c$, κ_1 and κ_2 are propagation waves with opposite propagation directions, κ_3 and κ_4 are evanescent waves. When $f > f_c$, all waves are evanescent waves. Therefore, SIBT is not suitable for high frequency vibrations. In STBT, the wave propagates all the time and is not limited by the critical frequency. The phase velocity c_p and group velocity c_g can be obtained from the dispersion relation of Eq (10), which are the key parameters for building the energy model.

$$c_p = \frac{\omega}{\kappa}, c_g = \frac{\partial \omega}{\partial \kappa}. \tag{12}$$

The energy methods commonly used in high-frequency vibration include statistical energy analysis (SEA), energy flow analysis (EFA), energy finite element method (EFEM) and radiative energy transfer method (RETM). This paper takes RETM as an example to illustrate the differences between several beam theories in high frequency analysis.

The main variables in RETM are energy density W and energy flow intensity I, and the energy balance relationship could be established as follows:

$$\frac{dI}{dx} + P_{diss} = P_{in}, \tag{13}$$

where P_{in} is the input power provided by the lateral load and P_{diss} is the dissipated power density related to damping, which is the same as in SEA with $P_{diss} = \eta\omega W$, where η is the hysteretic damping loss factor of structures.

From the equivalent relationship between the energy propagation velocity and the group velocity,

$$I_i = c_{gi}W_i, \frac{d(c_{gi}W_i)}{dx} + \eta\omega W_i = P_{in,i}. \tag{14}$$

The subscript i represents the different types of waves, and the particular solution of Eq. (14), which is the free-field solution of the wave, can be written as follows:

$$W_i = P_{in,i} G_i (S, R), \tag{15}$$

where $G_i(S, R)$ is the kernel function of the energy density, which represents the energy of the receiving point R caused by the unit excitation in the free field at the point S, it could be expressed as follows:

$$G_i (S, R) = \frac{e^{-m_i r(S,R)}}{2c_{gi}}, \tag{16}$$

where $r(S, R)$ represents the distance between the excitation source S and the receiving point R, and $m_i = \eta\omega/c_{gi}$ is the energy attenuation coefficient related to damping. Similarly, the energy intensity could be expressed as follows:

$$\mathbf{I}_i = P_{in,i} \mathbf{H}_i (S, R), \; \mathbf{H}_i (S, R) = \frac{e^{-m_i r(S,R)}}{2} \mathbf{n} (S, R), \tag{17}$$

where $\mathbf{n}$ represents the unit vector between the source S and the receiving point R, which could be expressed as a sign function as $\mathbf{n}(S, R) = \mathrm{sign}(R - S)$.

The energy of the entire vibration field could be considered to be composed of the direct field and the reflected field. The energy of the reflected field is replaced by fictive sources set at the boundary. One of the advantages of this model is that the Huygens principle could be directly applied, that is, the most general field arises from the superposition of the direct field produced by the actual source $P_{in,i}$ located at S in the domain Ω, and the diffracted field produced by the fictive source σ_i located at P on $\partial\Omega$. All these considerations are summarised by the following relationships:

$$\begin{aligned} W_i (R) &= \int_\Omega P_{in,i} (S) G_i dS + \int_{\partial\Omega} \sigma_i (P) G_i dP, \\ \mathbf{I}_i (R) &= \int_\Omega P_{in,i} (S) \mathbf{H}_i dS + \int_{\partial\Omega} \sigma_i (P) \mathbf{H}_i dP. \end{aligned} \tag{18}$$

Thus, the energy of any receiving point on the beam could be expressed as $W (R) = W_\alpha (R) + W_\beta (R), I (R) = I_\alpha (R) + I_\beta (R)$.

Therefore, wave number and group velocity are important parameters of the energy equation and its kernel function. TBT and SIBT are limited by the critical frequency, so it is necessary to establish different energy models at different frequencies in the energy model, which is less convenient than STBT.

3. VALIDITY AND DISCUSSION

3.1 Wave propergation in different beam theories

In the case of homogeneous aluminum beams, the geometrical parameters of the beam are adopted as follows: $b = 0.1$ m, $h = 0.1$ m and $L = 2$ m. Fig. 2 compares wave number curves with frequency in different beam theories, where "EB" represents Euler-Bernoulli beam theory, "Rayleigh" represents Rayleigh beam theory, "Shear" represents pure shear beam theory, "TBT1" and "TBT2" represent two types of waves in Tiomoshenko beam theory, "STBT" represents simple TBT theory. "R-SIBT" and "I-SIBT" represent the real and imaginary parts of the wave number in SIBT theory, respectively.

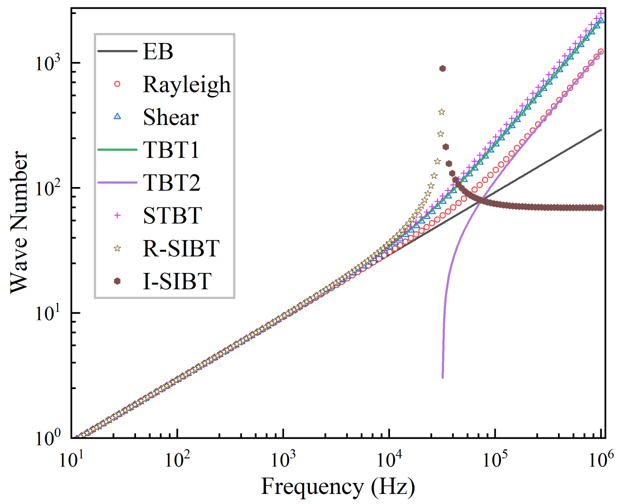

Figure 2. Dispersion curves of different beam theories.

In EB theory, the wave number increases linearly with frequency, TBT1 increases nonlinearly with frequency approximately after the critical frequency, and TBT2 appears after the critical frequency and increases rapidly and eventually increases to a value less than TBT1. The wave number in Shear theory is almost the same as TBT1, and the wave number in Rayleigh theory is the same as TBT2 after the critical frequency. The wave number of STBT theory is similar to and slightly larger than TBT1, and the wave number of SIBT is close to infinity near the critical frequency, after which it no longer propagates into evanescent wave. Thus, the TBT theory seems to be a combination of Shear theory and Rayleigh theory with two waves fitting, and SIBT theory does not seem to be applicable to

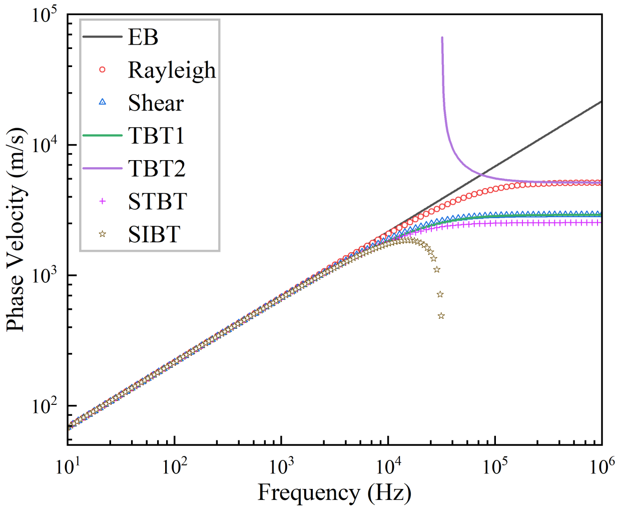

Figure 3. Phase velocity of waves with different beam theories.

high frequency analysis because it does not have propagating waves at very high frequencies. STBT theory seems to be the effect of superposition of two waves in TBT theory.

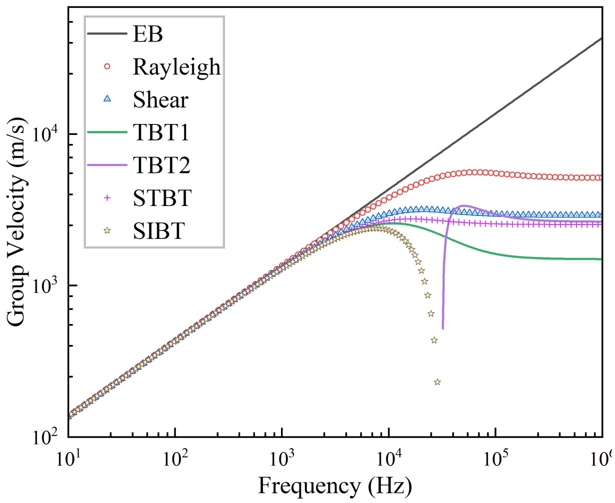

Figure 4. Group velocity of waves with different beam theories.

Fig. 3 shows the phase velocity curves of waves in different beam theories. TBT1 and Shear theory are still very close, but the phase velocity of TBT2 is close to infinity near the critical frequency, which is an anomaly and the reason why many researchers question the physical significance of the two spectra of TBT theory. After the critical frequency TBT2 is very close to the phase velocity of Rayleigh. STBT is close to TBT1, but slightly smaller than TBT1 at high frequencies. And SIBT decreases sharply to zero after increasing frequency.

Fig. 4 shows the group velocity curves of waves in different beam theories. The group velocity of EB theory can increase infinitely with frequency, which shows that the theory is not perfect. In SIBT, a wave group cannot propagate at higher frequencies, and the theory cannot be used for high frequency analysis. The group velocity of Shear and Rayleigh theory is greater than that of TBT theory at higher frequencies. The group velocity of STBT theory is close to TBT1 at low frequencies and TBT2 at high frequencies.

3.2 Energy response

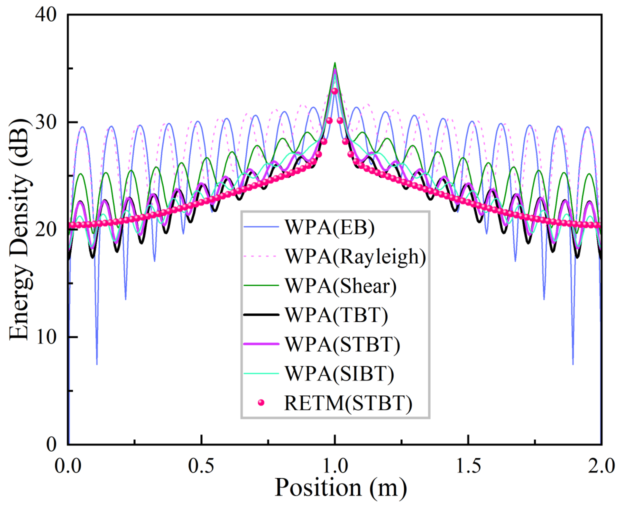

Figure 5. Energy response of different beam theories, $f = 10\,kHz, \eta = 0.1$.

In Fig. 5, the energy responses of several beam theoretical wave propagation solutions are calculated and compared with the RETM solutions of STBT theory. EB theory neglects shear deformation and rotational inertia, leading to higher energy estimation than other theories. Although Rayleigh theory considers the effect of rotational inertia, it has not been significantly improved compared with EB beam theory. Although Shear theory only considers shear deformation, the energy response has been greatly improved compared with EB and Rayleigh theory. The energy responses of the three beam theories

(TBT,STBT,SIBT) considering both shear and rotation effects are similar. The RETM solution of STBT theory is a smooth curve without fluctuation, which is in good agreement with the results of TBT and STBT.

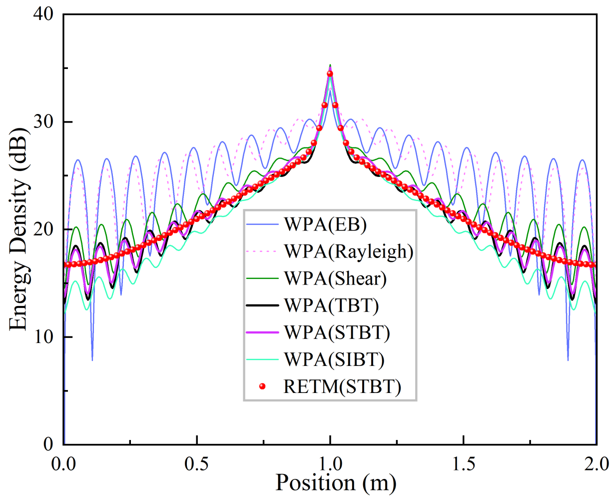

Figure 6. Energy response of different beam theories, $f = 10\ kHz, \eta = 0.15$.

Fig. 6 is a comparison of energy responses of several beam theories after damping increases to 0.15. The results of STBT and TBT are very close, and the RETM solution of STBT is also in good agreement with them. However, SIBT attenuates greatly with the increase of damping, and the energy response is much lower than that of STBT and TBT theory, indicating that SIBT theory is more sensitive to damping at high frequencies, which can be seen from the fact that its wave number turns into pure imaginary number at high frequencies. Therefore, SIBT theory is not robust in high frequency calculation even if the frequency does not reach the critical frequency.

Fig. 7 shows the results of different theoretical energy responses when the frequency is set at 20kHz. The RETM solution of STBT is still in good agreement with the WPA solution of STBT and TBT, but SIBT, due to the false wave number after frequency increase, does not propagate the wave, leading to the energy response deviating from the actual value. EB,Rayleigh and Shear theories do not consider enough shear deformation or rotational inertia in the initial assumptions. So the result is always higher than the other three theories.

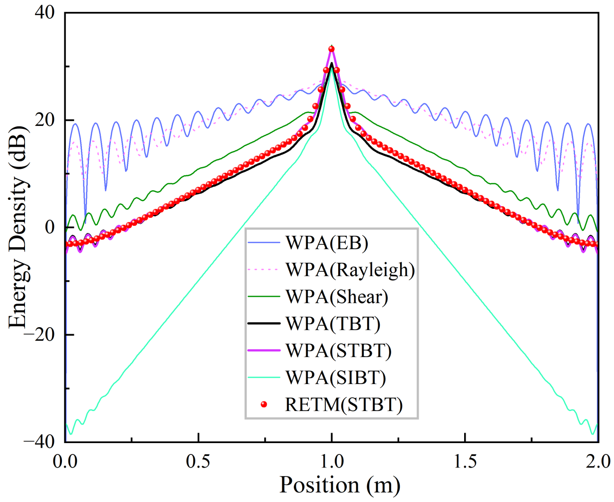

Figure 7. Energy response of different beam theories, $f = 20\ kHz, \eta = 0.15$.

4. CONCLUSION

From the above theoretical comparison and numerical analysis, the following conclusions can be drawn: TBT theory does exist the second spectrum phenomenon which is difficult to explain physically; Although the theoretical calculation process of TBT is complicated, the results of energy response are reliable. STBT theory does not have the second spectrum, and it is very close to the results of TBT theory. RETM based on STBT theory is a simple and accurate method, which can be used in the high frequency analysis of beam structure; SIBT theory is not suitable for high-frequency analysis of beam structures because of its theoretical defects

5. ACKNOWLEDGMENTS

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