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ANATOMY OF PIERCE'S WAVE EQUATION

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ABSTRACT

The construction of a sound wave propagation operator in the presence of a base flow is addressed here from an acoustic potential function expressed in a very general form, as proposed by Pierce [1]. In particular the velocity field is not obtained by taking the simple gradient of this potential scalar function. Among all the possible formulations, we show that it is possible to choose a self-adjoint form already used by the authors in aeroacoustics [2, 3]. This formulation, called Pierce's wave equation, allows us to establish from a variational principle, properties of acoustic energy conservation [4] together with a high frequency approximation which does not present any singularity due to the presence of caustics [5]. Geometrical acoustics equations are also retrieved [6].

Keywords: *self-adjoint wave operator, energy conservation, stability, geometrical acoustics*

1. INTRODUCTION

The motivation that led to this work is to propose a wave equation able to accurately describe propagation effects in the presence of an arbitrary shear base flow without coupling with the vorticity mode. The linearization of Euler equations, which describes without approximation the propagation through a base flow, does not allow to decouple the acoustic solution from hydrodynamic instabilities.

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Following the approach proposed by Pierce [1], a general definition of the acoustic scalar potential ϕ can be introduced leading to an associated wave equation. Among all these equations, a self-adjoint operator is selected to ensure the conservation of acoustic energy [7]. The coupling with the instability modes becomes formally impossible by construction with this building principle. It remains to verify that the chosen operator provides a good approximation of propagation effects. Pierce's equation has been finally selected,

$$D_0^2\phi - \nabla\cdot(a_0^2\nabla\phi) + (\nabla\cdot\boldsymbol{u}_0)D_0\phi = 0 \qquad (1)$$

where $D_0 = \partial_t + \boldsymbol{u}_0\cdot\nabla$ is the material derivative along the mean flow, $\boldsymbol{u}_0(\boldsymbol{x})$ is the velocity of the base flow and $a_0(\boldsymbol{x})$ the speed of sound. The primitive acoustic variables are determined from the acoustic potential, namely for the velocity and pressure,

$$\boldsymbol{u} = (A/\rho_0)\,\nabla\phi \qquad p = -A\,D_0\phi \qquad (2)$$

where $A = p_0^{(\gamma-1)/\gamma}$. For a parallel base flow [2], the previous expressions can be simplified by noting that the base flow satisfies $\nabla p_0 = 0$ and $\nabla\cdot\boldsymbol{u}_0 = 0$. For simplicity, but noting that this assumption is very often verified, we assume in all that follows that $\nabla p_0 \simeq 0$ for the reconstruction (2).

2. ENERGY CONSERVATION

Recasting Eq. (1) as $\mathcal{L}_0\,\phi = 0$, the self-adjointness property of $\mathcal{L}_0$ ensures the existence of a Lagrangian density Λ

$$\mathbb{L} = \frac{1}{2} < \phi, \mathcal{L}_0\,\phi >= \iint \Lambda\; d\boldsymbol{x}\,dt$$

which is found to be such that

$$-2\Lambda = (D_0\phi)^2 - a_0^2\,(\nabla\phi)^2 \qquad (3)$$

The variational principle for $\Lambda = \Lambda(\partial_t\phi, \nabla\phi, \boldsymbol{x})$ implies that there is no variation over space-time of the Lagrangian, $\delta\mathbb{L} = 0$. In particular, the zero variation in time leads to the energy conservation equation

$$\frac{\partial \mathcal{E}_a}{\partial t} + \nabla\cdot\boldsymbol{F}_a = 0 \qquad (4)$$

where $\mathcal{E}_a$ is the acoustic energy density and $\boldsymbol{F}_a$ flux, given by

$$\mathcal{E}_a = \partial_t\phi\frac{\partial\Lambda}{\partial(\partial_t\phi)} - \Lambda \qquad \boldsymbol{F}_a = \partial_t\phi\frac{\partial\Lambda}{\partial(\nabla\phi)} \qquad (5)$$

Eqs. (3) and (5) allow to retrieve in a very elegant way the general energy equation derived by Morfey [4]. More generally, the variational principle can also be formulated by introducing the space-time differentiation $\dot{\nabla} = (\partial_t, \nabla)^T$ as follows,

$$\dot{\nabla}\cdot\overline{\overline{\boldsymbol{T}}} = 0 \qquad (6)$$

where $\overline{\overline{\boldsymbol{T}}}$ is the fourth-order stress energy tensor. Eq. (4) corresponds to the first scalar equation of system (6).

3. GEOMETRICAL ACOUSTICS

Pierce's equation (1) can also be recast in the form of a divergence by using $\dot{\nabla}$, namely

$$\dot{\nabla}\cdot\left(\overline{\overline{\boldsymbol{E}}}\cdot\dot{\nabla}\phi\right) = 0 \qquad (7)$$

where the $\overline{\overline{\boldsymbol{E}}}$ tensor reads

$$\overline{\overline{\boldsymbol{E}}} = \begin{pmatrix} 1 & \boldsymbol{u}_0^T \\ \boldsymbol{u}_0 & \boldsymbol{u}_0\otimes\boldsymbol{u}_0 - a_0^2\overline{\overline{\boldsymbol{I}}} \end{pmatrix} \qquad (8)$$

and $\overline{\overline{\boldsymbol{I}}}$ is the space identity tensor. By considering a local plane wave for the acoustic potential, that is $\phi = Ae^{i(\boldsymbol{k}\cdot\boldsymbol{x}-\omega t)}$ in Eq. (7), the real part of this expression provides the dispersion relation of acoustic waves whereas the imaginary part provides the acoustic energy conservation. This compact form (7) allows furthermore to determine the associated Hamiltonian H_0 expressed in the physical space by introducing a suitable metric [8], and to finally derive ray-tracing equations [5, 6].

4. CONCLUDING REMARKS

Some properties of self-adjoint Pierce's Eq. (1) have been briefly emphasized here. The use of a scalar potential (2) prevents any coupling with the vorticity while taking into account sound propagation effects in presence of a base flow. The variation principle provides a clear statement about the acoustic energy conservation. A consistant formulation of geometrical-acoustics approximation is also found.

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