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ESTIMATING LONGER MODAL FUNCTIONS USING SHORTER VERTICAL ARRAYS IN A CLOSED-FORM MANNER

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ABSTRACT

High-resolution estimation of modal functions has been recently demonstrated. Two VLAs collect pressure data due to a monochromatic source, and feed a subspace algorithm that computes, in a fully automatic closed-form manner, modal functions at sensors depths. Estimation accuracy can be improved at will by improving the SNR, using the same limited number of sensors. In this proposal, the apparatus is improved in order to reduce by two the number of required sensors to cover a given portion of the water column, or equivalently, to double the covered portion of the water column using the same number of sensors. This is made possible by allowing the source and the hydrophones to be deployed at different depth locations. The modified algorithm continues to exhibit the same attractive features of the original one, i.e. closed-form and asymptotically (as SNR increases) unbiased estimation. At limited SNR, a moderate degradation of the estimates is observed because the collected data matrices are not symmetric anymore.

Keywords: *normal modes, shallow water, vertical horizontal arrays, subspaces, high resolution*

1. INTRODUCTION

Normal modes are a convenient means to model low-frequency acoustic propagation in shallow waters. Their estimation from the measured acoustic field is useful to conduct such tasks as geo-inversion and source localization. For this purpose, subspace techniques are very attractive because of reduced complexity and guaranteed convergence. However, they require sensing of the whole water column in order to take benefit from the orthogonality between the different modes [1–3]. This requirement has recently been relaxed by a subspace technique [4] that does not require full sensing of the water column. Two vertical linear arrays (VLA) are deployed at the same depth locations, while a mono-chromatic source is activated successively at each of the considered depth locations. By doing so, two data matrices are collected from the two VLAs. It has been proved that modal functions appear as the eigenvectors of the matrix computed as the product of one data matrix by the pseudo-inverse of the other one. Estimation accuracy can be improved by increasing the signal to noise ratio (SNR), without the need to increase the number of sensors, a property often referred to as high-resolution.

Nevertheless, the system requires a number of sensors (per VLA) at least equal to the number of modes. In this paper, we relax this condition. Unlike the original design, source depth locations are now different from VLA depth locations. This allows to cover double the initial portion of the water column using the same number of sensors, or equivalently, if one is to sense a given set of depth locations, then it would require half the number of sensors. At the same time, this results in the newly collected data matrices not being symmetric anymore. The subspace algorithm is revisited accordingly to show that the two portions of the water column (corresponding to source and VLAs depth locations, respectively) are now estimated by means of singular vector decomposition (instead of eigen vector decomposition in the original algorithm), who is slightly more computation demanding. Also because the two portions of the water column are now estimated inde-

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pendently, a scaling ambiguity occurs that we manage to solve at a negligible cost.

2. DATA MODEL

We denote $\|\cdot\|$ as the Euclidean norm; T, H, $*$, -1 and $\natural$ as -matrix transpose, trans-conjugate, conjugate, inverse and Moore pseudo-inverse, respectively. $[\mathbf{A}]_{ij}$ is entry at row i and column j of matrix $\mathbf{A}$. $\mathrm{Diag}\,(d_1, d_2, \cdots)$ denotes the diagonal matrix with $d_1, d_2, \cdots$ along the diagonal. $\mathbf{0}_{a,b}$ is the $a \times b$ zero matrix. $\mathbf{I}_a$ is the $a \times a$ identity matrix. Indexes are dropped when they can be inferred.

Where $[O, x^{(1)}, x^{(2)})$ designates the sea surface, a point in the waveguide is characterized by its coordinates r, ψ, z, where ψ is the angle counter-clockwise from $[O, x^{(1)})$ and r is the horizontal spacing between this point and the reference water column $x^{(1)} = x^{(2)} = 0$, where an acoustic source is activated at depth $z = z^S$, emitting a narrow-band signal at some frequency f_0.

A hydrophone placed at range r and depth z^R collects, up to an unknown noise contribution, a signal [5]

$$x(z^S, z^R, r) = b_s e^{j\frac{\pi}{4}} \sum_{m=1}^{M} \phi_m(z^R)\phi_m(z^S)\frac{e^{-j\kappa_m r}}{\sqrt{\kappa_m r}}$$

where b_s is an unknown complex amplitude. Environment parameters κ_m and $\phi_m(z), m = 1, \cdots, M$ correspond to the m-th modal function and the attached wavenumber, not in any particular order. The above is better written in matrix notation as

$$b_s e^{j\frac{\pi}{4}}[\phi_1(z^R), \cdots, \phi_M(z^R)]\mathbf{A}(r)[\phi_1(z^S), \cdots, \phi_M(z^S)]^T$$

where, in $\mathbf{A}(r) \hat{=} \mathrm{Diag}\,[A_1(r), \cdots, A_M(r)]$, we have $A_m(r) \hat{=} e^{-\kappa_m r}/\sqrt{\kappa_m r}$.

Let's have a first VLA at range R_1 and a second VLA at range R_2. Each is made of P sensors placed at the same depth locations $z_1^R, \cdots, z_P^R$ to which we attach $\Phi_R \hat{=} \begin{bmatrix} \phi_1(z_1^R) & \cdots & \phi_M(z_1^R) \\ \vdots & & \\ \phi_1(z_P^R) & \cdots & \phi_M(z_P^R) \end{bmatrix}$ showing all propagating modes at all sensed VLA depths. When the source is at depth z^S, then the i-th VLA outputs the P-dimensional $b_s e^{j\frac{\pi}{4}}\Phi_R \mathbf{A}(R_i)[\phi_1(z^S), \cdots, \phi_M(z^S)]^T$.

Let's imagine that we place the source successively at depths $z_1^S, \cdots, z_Q^S$, hence, collecting two data matrices $\mathbf{X}(R_1)$ and $\mathbf{X}(R_2)$ from both VLAs, given by

$$\mathbf{X}(R_i) = b_s e^{j\frac{\pi}{4}}\Phi_R \mathbf{A}(R_i)\Phi_S^T \quad (1)$$

where Φ_S is defined similarly as Φ_R and shows all propagating modes at the considered source depths. In order to cover a maximum of depth locations, we make sure to have $\{z_1^S, \cdots, z_Q^S\} \cap \{z_1^R, \cdots, z_P^R\} = \emptyset$. As regularly assumed [3, 6, 7], if both P and Q are $\leq M$, then tall Φ_S and Φ_R are full column rank with probability one. Their respective real-valued pseudo-inverses are denoted by $\Phi_S^\natural$ and $\Phi_R^\natural$.

3. ALGORITHM DEVELOPMENT

Sampled modal functions appear as columns of Φ_S and Φ_R, at different depth locations however. Our objective is to determine the set of parameters $\{\phi_m(z_p^R), \phi_m(z_q^Q); m = 1, \cdots, M; p = 1, \cdots, P; q = 1, \cdots, Q\}$, as well as wavenumbers $\kappa_1, \cdots, \kappa_M$. To derive our subspace algorithm, we notice that rank-deficient $\mathbf{X}(R_1)$ and $\mathbf{X}(R_2)$ have each rank M. Thanks to the special common structure (1) of data matrices $\mathbf{X}(R_1)$ and $\mathbf{X}(R_2)$, we do have

$$\mathbf{X}^\natural(R_k) = \Phi_S^{\natural,T}\mathbf{A}^{-1}(R_k)\Phi_R^\natural \quad (2)$$

thanks to which we are able to write

$$\begin{aligned} \mathbf{D}_R \;&\hat{=}\; \mathbf{X}(R_2)\mathbf{X}^\natural(R_1) = \sqrt{R_1/R_2}\,\Phi_R \mathbf{C}\Phi_R^\natural \\ \mathbf{D}_S \;&\hat{=}\; \left(\mathbf{X}^\natural(R_2)\mathbf{X}(R_1)\right)^T = \sqrt{R_2/R_1}\,\Phi_S \mathbf{C}\Phi_S^\natural \end{aligned}$$

where $\mathbf{C} \hat{=} \mathrm{Diag}\left(e^{\kappa_1(R_1-R_2)}, \cdots, e^{\kappa_M(R_1-R_2)}\right)$. On one hand, $\mathbf{D}_R$ infers about eigen modes at the receiver side i.e., Φ_R. On the other side, $\mathbf{D}_S$ infers about eigen modes at the source side i. e., Φ_S. At last, we adopt the substitution variables $\kappa_m^\dagger \hat{=} (R_1 - R_2)\kappa_m$, mainly to avoid manipulation of the typically very low variables κ_m.

In a practical scenario, we collect data matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$, noise-corrupted versions of $\mathbf{X}_1$ and $\mathbf{X}_2$, respectively. The algorithm is executed as follows:

1. Collect $P \times Q$ matrices $\mathbf{Y}_1$ and $\mathbf{Y}_2$ from first and second VLA, respectively.
2. For $m = 1, \cdots, M$, let σ_m be the largest singular value of $\mathbf{Y}_1$. Let u_m and v_m be, respectively, the associated left and right unit-norm singular vector. Compute $\mathbf{Y}_1^\natural$ as $\sum_{m=1}^{M} \frac{1}{\sigma_m} v_m u_m^H$.
3. Compute $\mathbf{Y}_2^\natural$ in a similar fashion.
4. Compute $\mathbf{D}_R \hat{=} \mathbf{Y}_2 \mathbf{Y}_1^\natural$ and $\mathbf{D}_S \hat{=} \left(\mathbf{Y}_2^\natural \mathbf{Y}_1\right)^T$.

5. Perform EVD of $\mathbf{D}_R$ to obtain eigen values λ_m^R and associated unit-norm eigen vector $\mathbf{w}_m^R$ for $m = 1, \cdots, M$.
6. Perform EVD of $\mathbf{D}_S$ to obtain the similarly-defined λ_m^S and $\mathbf{w}_m^S$, for $m = 1, \cdots, M$.
7. Compute $\lambda_m \hat{=} (\lambda_m^R + \lambda_m^S)/2$ for $m = 1, \cdots, M$.
8. Estimate $\kappa_m^\dagger$ as $\arg(\lambda_m)$, selected in $[0, 2\pi]$.
9. Estimate $\left[\phi_m(z_1^R), \cdots, \phi_m(z_P^R)\right]^T$ by $\mathbf{w}_m^R$ and $\left[\phi_m(z_1^S), \cdots, \phi_m(z_Q^S)\right]^T$ by $\mathbf{w}_m^S$.

4. RESOLVING THE AMBIGUITY

Under noise-free observation, we have $\left[\phi_m(z_1^R), \cdots, \phi_m(z_P^R)\right]^T = c_m^R \mathbf{w}_m^R$, and $\left[\phi_m(z_1^S), \cdots, \phi_m(z_Q^S)\right]^T = c_m^S \mathbf{w}_m^S$ where c_m^R and c_m^S are unknown indeterminacies. In other words, using the same number of sensors, we estimate two portions of the water column but each with a different unknown scaling. This level of ambiguity is not acceptable and a countermeasure is subsequently developed. For instance, we impose that one depth location be common to both VLA and source. Let it be the first one, so that $z_1^S = z_1^R = z_1$ and, for all m, $\phi_m(z_1) = c_m^R \left[\mathbf{w}_m^R\right]_1 = c_m^S \left[\mathbf{w}_m^S\right]_1$, or also $c_m^R = c_m^S \left[\mathbf{w}_m^S\right]_1 / \left[\mathbf{w}_m^R\right]_1$, leading to

$$\left[\phi_m(z_1), \phi_m(z_2^R), \cdots, \phi_m(z_P^R)\right]^T = c_m^S \frac{\left[\mathbf{w}_m^S\right]_1}{\left[\mathbf{w}_m^R\right]_1} \mathbf{w}_m^R$$
$$\left[\phi_m(z_1), \phi_m(z_2^S), \cdots, \phi_m(z_Q^S)\right]^T = c_m^S \mathbf{w}_m^S$$

Hence, the m-th mode (sampled at all depths) verifies $\left[\phi_m(z_1), \phi_m(z_2^S), \cdots, \phi_m(z_Q^S), \phi_m(z_2^R), \cdots, \phi_m(z_P^R)\right]^T$ $= c_m^S \left[\left[\mathbf{w}_m^S\right]_1, \left[\mathbf{w}_m^S\right]_2, \cdots, \left[\mathbf{w}_m^S\right]_Q, \frac{\left[\mathbf{w}_m^S\right]_1 \left[\mathbf{w}_m^R\right]_2}{\left[\mathbf{w}_m^R\right]_1}, \cdots, \frac{\left[\mathbf{w}_m^S\right]_1 \left[\mathbf{w}_m^R\right]_P}{\left[\mathbf{w}_m^R\right]_1} \right]^T \hat{=} c_m^S \mathbf{w}_m^{SR}$, where $\mathbf{w}_m^{SR}$ is computed by the algorithm and unknown c_m^S accounts for an acceptable (scale) ambiguity.

For performance evaluation, and in order to accommodate scaling indeterminacy, we will measure estimation accuracy by means of a normalized Mean Square Error (MSE) [4, 8], one that ranges between 0 (exact estimates of the modal functions, up to an unknown multiplicative factor) and 1 (exact and estimated modal functions are orthogonal).

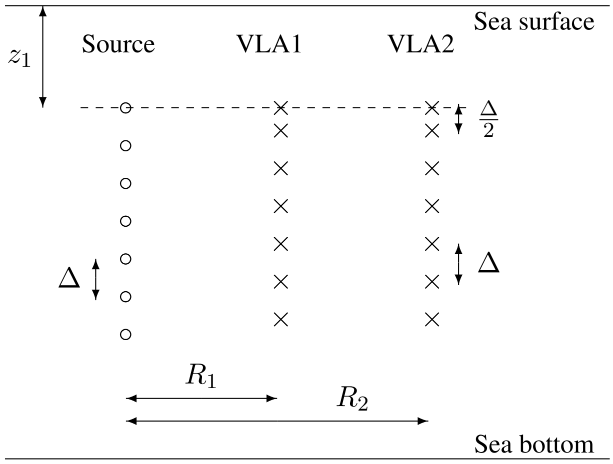

Figure 1. Illustration of source and sensor locations.

5. SIMULATIONS

In order to test the algorithm, we consider a Pekeris waveguide with a water column sound speed of 1500 [m/s] and a bottom half-space with a compressional speed of 1800 [m/s] and a density $\rho = 1.8$ [Kg/m^3]. Data is generated using KRAKEN normal mode propagation model in a 100 [m] depth waveguide.

Having chosen $P = Q$, source and VLAs are deployed as shown in Fig. 1: Source is activated at depth positions $z_1, z_1 + \Delta, \cdots, z_1 + (P-1)\Delta$, while VLA sensors are deployed at depth positions $z_1, z_1 + \Delta/2, z_1 + 3\Delta/2, \cdots, z_1 + \Delta/2 + (P-2)\Delta$, where $\Delta = 1.6$ [m]. On one hand, we adjust reference depth z_1 such that the center of the array is always at depth 50 [m] i.e., in the middle of the water column, while array length is made to vary as shown by the vertical axes of Fig. 2 and Fig. 3. On the other hand, the first VLA is maintained at a range $R_1 = 200$ [m] from the source, while the range of the second VLA is made to vary as shown by the horizontal axes of Fig. 2 and Fig. 3.

The above-generated acoustic field is corrupted by a randomly generated zero-mean complex-valued circular Additive White Gaussian Noise (AWGN) with a standard deviation equal to 10^{-6}. The normalized MSE on modal functions is averaged over 1000 Monte Carlo runs and reported as the Averaged Normalized MSE (ANMSE). Measured ANMSE is shown in [dB] in Fig. 2 and Fig. 3, for a source emitting at 50 [Hz] and 100 [Hz], corresponding to $M = 4$ and $M = 9$ propagating modes, respectively.

We can see that estimation accuracy increases with the array length (i.e., with the number of sensors, deployed every $\Delta = 1.6$ [m]). Having assumed range-independent propagation, inter-VLA spacing has an effect only at some disadvantageous specific range values for which, we presume, matrix $\mathbf{C}$ is close to be singular, a phenomenon that happens more often at larger frequencies.

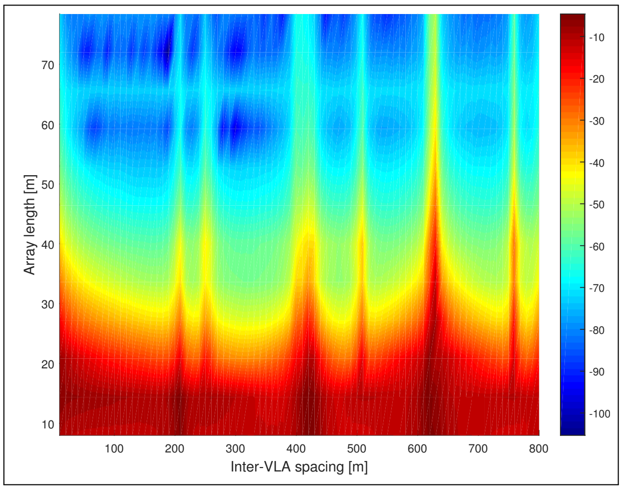

Figure 2. ANMSE results [dB] for varying VLAs length and spacing, for a mono-chromatic source emitting at 50 [Hz].

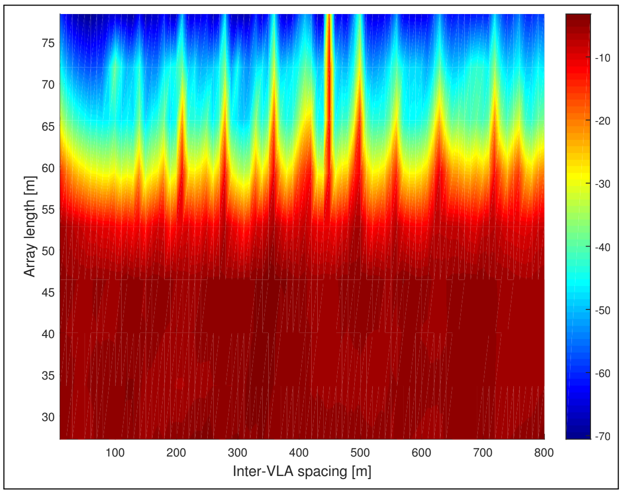

Figure 3. ANMSE results [dB] for varying VLAs length and spacing, for a mono-chromatic source emitting at 100 [Hz].

6. CONCLUSION

A recently proposed subspace algorithm allows to accurately estimate sampled modal functions using a limited number of sensors. The apparatus includes a monochromatic source and two VLAs deployed at the same depth locations. We revisit this algorithm to allow source and VLAs to be deployed at different depth locations and, consequently, manage to double the portion of the water column being sensed. The side effect is an aggravated scaling ambiguity against which a suitable countermeasure has been elaborated. The modified algorithm continues to enjoy the same attractive features of the original algorithm, including a search-free fully automatic operation.

7. REFERENCES

[1] S. N. Wolf, D. K. Cooper, and B. J. Orchard, “Environmentally adaptive signal processing in shallow water,” in *Proc. of Oceans*, pp. I–99–I–104, 1993.

[2] P. Roux, D. Cassereau, and A. Roux, “A high-resolution algorithm for wave number estimation using holographic array processing,” *J. Acoust. Soc. Am.*, vol. 115, no. 3, pp. 1059–1067, 2004.

[3] T. B. Neilsen and E. K. Westwood, “Extraction of acoustic normal mode depth functions using vertical line array data,” *J. Acoust. Soc. Am.*, vol. 111, no. 2, pp. 748–756, 2002.

[4] H. Gazzah and S. M. Jesus, “Closed-form estimation of normal modes from a partially sampled water column,” *IEEE J. Oceans Eng.*, vol. 45, no. 4, pp. 1574–1582, 2020.

[5] J. Tabrikian and H. Messer, “Three dimensional source localization in a waveguide,” *IEEE Trans. Signal Process.*, vol. 44, no. 1, pp. 1–13, 1996.

[6] H. M. Chouhan and G. V. Anand, “Normal mode wave-number estimation using a towed array,” *J. Acoust. Soc. Am.*, vol. 93, no. 4, pp. 1807–1814, 1993.

[7] J. V. Candy and E. J. Sullivan, “Model-based passive ranging,” *J. Acoust. Soc. Am.*, vol. 85, no. 6, pp. 2472–2478, 1989.

[8] D. R. Morgan, J. Benesty, and M. M. Sondhi, “On the evaluation of estimated impulse responses,” *IEEE Signal Processing Letters*, vol. 5, no. 7, pp. 174–176, 1998.