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LATENT SYMMETRIES IN ACOUSTIC WAVEGUIDE NETWORKS

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ABSTRACT

A latent symmetry is a novel type of symmetry which, in general, is not apparent from a geometric inspection of the system. Instead, it becomes visible after a suitable dimensional reduction: The so-called isospectral reduction, which is akin to an effective Hamiltonian. We show that latent symmetries in an acoustic waveguide network can lead to two interesting phenomenae: Point-wise parity of all eigenmodes, and equireflectionaly. In the latter, a geometrically asymmetric network features the same reflection from the left and from the right, just as a mirror-symmetric network would.

Keywords: Latent symmetries, waveguide networks, monomode approximation, quantum graphs

1. INTRODUCTION

Symmetries are of high importance in physics, as they dictate the fundamental form of physical laws [1, 2] and lead to precious insights such as selection rules for atoms and molecules [3, 4] or the emergence of band structures in crystals [5].

Recently, a new symmetry concept has been introduced, namely, that of *latent symmetry*. A latent symmetry is a symmetry not of the original Hamiltonian, but of an equivalent dimensionally reduced effective Hamiltonian [6, 7]. Although originally stemming from the mathematical field of graph theory, latent symmetries have been applied in the past years to physical systems as well [7–12]. In this work, we summarize the most recent results which bring the concept of latent symmetries to networks of one-dimensional waveguides. We note that such networks are also known under the term "quantum graphs"; see [13] for an introduction into this field. Besides their obvious realization in terms of acoustic waveguides (discussed here) [10, 14, 15], such quantum graphs can be realized also in the form of microwave networks [16, 17].

2. LATENT SYMMETRIES IN WAVEGUIDE NETWORKS

To illustrate the concept of latent symmetries, let us focus on the waveguide network depicted in fig. 1 (a). It consists of 14 identical waveguides with length $L = 10cm$ and width $w = 1cm$. Since we are dealing with thin waveguides ($w \ll L$), we can treat them as monomodal for sufficiently low frequencies. This allows us [10, 13–15, 18, 19] to convert the continuous problem of finding the eigenmodes ϕ (wich are solutions of the Helmholtz equation with Neumann boundary conditions) into the problem of finding the eigenvectors of the generalized matrix eigenvalue problem

$$A\,\boldsymbol{\phi} = \cos(kL) B\,\boldsymbol{\phi}\,. \tag{1}$$

Here, $\cos(kL)$ is the eigenvalue, $k = \omega/c$ denotes the wavenumber, with ω being the angular frequency and c the

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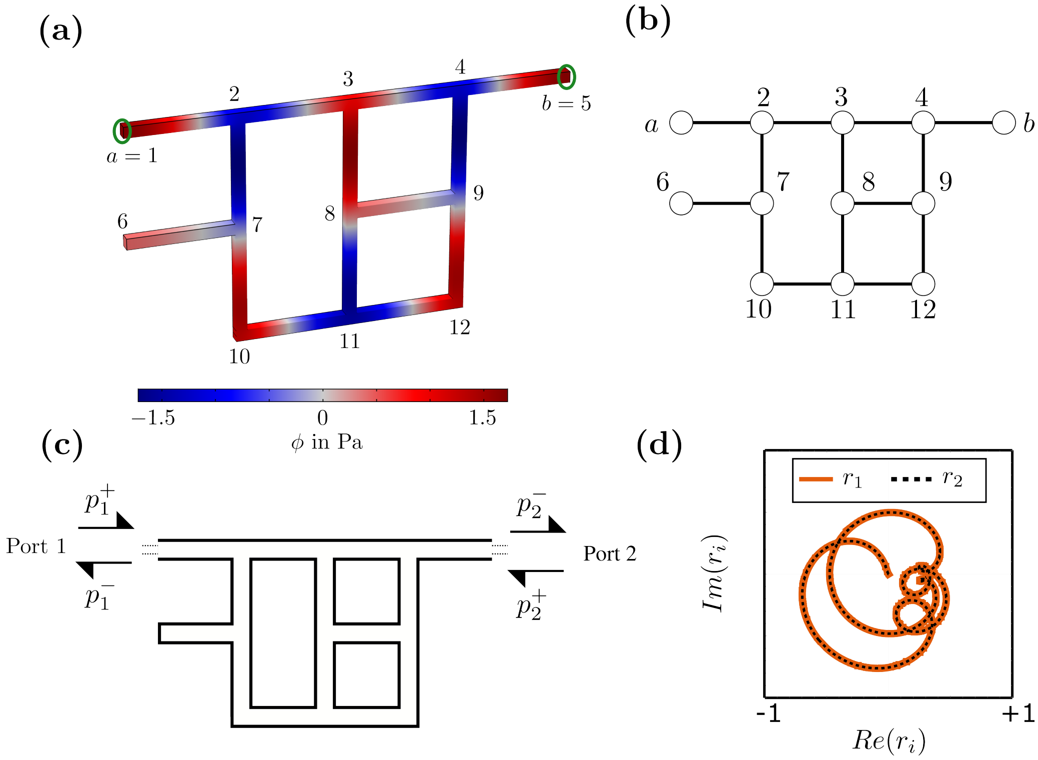

Figure 1. A latently symmetric waveguide network, realized with thin, square-shaped acoustic waveguides of length $L = 0.1m$ and side length $w = 10mm$. (a) The 10th mode of the acoustic network with closed ends. (b) shows the corresponding discrete model (graph). (c) When opening the two points a, b and connecting external waveguides to them, we can describe the system as a two-port device. (d) shows a comparison of the reflection coefficients $r_i(f)$ of the two ports in the complex plane for the case of lossy waveguides (modelled by a complex wavenumber k). As can be seen, we have broadband equireflectionality, that is, $r_1 = r_2$.

sound velocity in air, and with the N-dimensional eigenvector $\boldsymbol{\phi}$ denoting the values of the eigenmode ϕ at the N endpoints of waveguides (see inset of fig. 1(a) for details). The matrix A describes the topology of the setup, with $A_{i,j} = 1$ if the endpoints i, j are connected by a waveguide, and $A_{i,j} = 0$ otherwise. The matrix B is diagonal, with $B_{i,i} = \sum_j A_{i,j}$. For the waveguide network of fig. 1 (a), its matrix A is pictorially represented in fig. 1 (b).

By applying the transformation $\mathbf{y} = \sqrt{B}\boldsymbol{\phi}$ to eq. (1), we arrive at the "Hamiltonian" $H = \sqrt{B}^{-1} A \sqrt{B}^{-1}$ and the classic eigenvalue problem

$$H\mathbf{x} = \lambda\mathbf{x} \qquad (2)$$

with $\lambda = \cos(kL)$ and $\mathbf{x} = \sqrt{B}\boldsymbol{\phi}$. This formulation of the problem allows us to apply the theory of latent symmetries onto the problem [6,9,10,20].

A system posseses a latent reflection symmetry over a set of two sites $S = a, b$ if its effective Hamiltonian $\widetilde{H}_S(\lambda) = H_{SS} + H_{S\overline{S}} \left(\lambda \mathbb{1} - H_{\overline{SS}}\right)^{-1} H_{\overline{S}S}$ commutes with the reflection matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, where $\overline{S}$ is the complenent of S, and $H_{SS}, H_{\overline{SS}}$, etc. denote submatrices of H. The consequence of such a latent symmetry is that all eigenvectors $\mathbf{x}$ feature parity on the two sites S. When the Hamiltonian H describes an acoustic waveguide network, as is the case here, then such a latent symmetry translates to the statement that all low-frequency eigenmodes $\boldsymbol{\phi}$ feature pointwise parity on a and b [10].

Let us now come back to our waveguide network of fig. 1 (a). As can be easily shown, the corresponding Hamiltonian H is indeed latently reflection symmetric for $S = \{a, b\}$. This can also be seen from the wavefield depicted in fig. 1 (a), which corresponds to the 10-th eigenmode of our latently symmetric network and which clearly has positve point-wise parity on the two points a, b.

So far, we have seen that a latently symmetric waveguide network has point-wise parity for the (low-frequency) eigenmodes. However, such a symmetry likewise has a strong impact on the scattering properties of the network. To this end, let us open the network at the two points a, b (see fig. 1 (c)) by connecting monomodal waveguides. The scattering problem can then be described using the scattering matrix S as [12]

$$\begin{pmatrix} \psi_1^- \\ \psi_2^- \end{pmatrix} = \begin{pmatrix} r_1 & t \\ t & r_2 \end{pmatrix} \begin{pmatrix} \psi_1^+ \\ \psi_2^+ \end{pmatrix} = S \begin{pmatrix} \psi_1^+ \\ \psi_2^+ \end{pmatrix}, \quad (3)$$

where the ouput and input waves are denoted, respectively, by $(\psi_1^-, \psi_2^-)^T$ and $(\psi_1^+, \psi_2^+)^T$. The off-diagonal elements $S_{1,2}$ and $S_{2,1}$ denote the transmission coefficients; due to reciprocity they are equal in this case. The diagonal elements $S_{1,1}$ and $S_{2,2}$ denote the respective reflection coefficients of the two ports $1, 2$.

Now, since the eigenmodes of the network all feature parity on the points a, b, it can be shown—by using the Green's function of the closed setup [10,12]—that the two reflection coefficients are equal, $r_1 = r_2$ for all frequencies (of course, only as long as the monomode approximation holds). This broadband *equireflectionality* is interesting, as one would normally expect it only from a geometrically symmetric waveguide network. We note that, since all waveguides are identical, this result even holds when we introduce viscothermal losses (here modelled by a komplex wavenumber k), as can be seen in fig. 1 (d).

3. CONCLUSIONS

Latent symmetries are a recently introduced concept that give a new perspective on a class of seemingly asymmetric systems. Here, we have discussed the concept in terms of acoustic waveguide networks, where latent symmetry leads to point-wise parity of eigenmodes and broadband equireflectionality.

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