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# WEAKLY NONLINEAR PROPAGATION IN TUBES: FROM BRASS INSTRUMENTS TO NOISE PROPAGATION

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## ABSTRACT

In 2000, Ludovic Menguy and Joël Gilbert generalized the Burgers equation in order to incorporate thermoviscous losses due to boundary layers on the walls. This allowed studying the nonlinear propagation in brass instruments. Later, Joël Gilbert and colleagues were involved on the study of high-level noise sources from which emerged the study of nonlinear propagation of noise in tubes. This is the subject of the present paper. The problem is solved numerically using a fractional step method together with a convexification method. This one is suited to nonlinear propagation of acoustic signals containing a large number of pre-shocks and shocks which coalesce during the propagation. Model predictions and experimental data are compared and shown to be in a good agreement. It is shown that the Gaussianity of narrowband noise at the inlet of the tube is not conserved during nonlinear propagation.

**Keywords:** *Nonlinear propagation, Burgers equation, noise.*

## 1. INTRODUCTION

Part of Joël Gilbert's research has been devoted to the description of the nonlinear propagation of acoustic waves in uniform and nonuniform tubes of finite length using weakly nonlinear acoustic models [1, 2]. The propagation of periodic plane waves with the combined effects of nonlinearity and dissipation in the boundary layer of the tube have been described using generalized Burgers equations for forward and backward traveling waves. Numerical solutions are sought in the frequency domain with the assumption that the two waves propagating in opposite directions do not exchange energy (no interaction). Therefore, the internal sound field is described by the superposition of the two wave profiles. Results of this numerical model are used to predict the brassiness behavior of brass instruments [3]. In 2013, Joel Gilbert and colleagues were involved on the study of high-level noise sources coupled with a horn connected to a large reverberant room [4]. Part of this study was an experimental work devoted to the nonlinear propagation of narrowband noise in a uniform tube. The present study is a continuation of this initial work. The purpose of this article is to describe the nonlinear propagation of a random signal (noise) in a tube with a generalized Burgers equation (nonlinearity and dissipation due to friction in the boundary layer). An hybrid algorithm (calculations are performed in both the time and frequency domain) that includes the effects of dissipation on the propagation of finite-amplitude sound is proposed. In this particular case of propagation of a random signal with formation and interaction (coalescence) of a large number of discontinuities (shocks), the convexification method is the tool used to obtain numerical solutions to the Burgers equation.

## 2. GORVERNING EQUATION AND NUMERICAL SOLUTION

The generalized Burgers equation [5–7] used is

$$\frac{\partial p}{\partial x} - \frac{\beta}{\rho_0\, c_0^3}\, p \frac{\partial p}{\partial \tau} = -\frac{1}{R}\sqrt{\frac{l_v}{c_0}}\left[1 + \frac{\gamma - 1}{\sqrt{l_v/l_h}}\right] \frac{\partial^{1/2} p}{\partial \tau^{1/2}} \quad (1)$$

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where $p$ is the acoustic pressure, $x$ is the coordinate along which the plane wave propagates, $\tau = t - x/c_0$ is the retarded time, $c_0$ and $\rho_0$ are the adiabatic sound speed and medium density, respectively, $\beta$ is the coefficient of nonlinearity. The quantities $l_v$ and $l_h$ are the viscous and thermal characteristic lengths, $R$ is the radius of the tube and $\gamma$ is the specific heat ratio.
The right hand side of Eq. (1) with a fractional derivative of order 1/2 is an operator which models the memory effect of boundary layer friction on the walls of the tube [8–10], it is defined as

$$\frac{\partial^{1/2} p(x,\tau)}{\partial \tau^{1/2}} = \frac{1}{\sqrt{\pi\ \tau}} * p(x,\tau) \qquad (2)$$

where the symbol * denotes the convolution product.

Here, only the boundary-value problem, with $p$ prescribed as a function of $\tau$ at $x = 0$ and with the waveform evolving with $x$, is studied (Fig. 1). The input sound source is a random signal (noise) and a one-way wave propagation is considered (infinite tube).

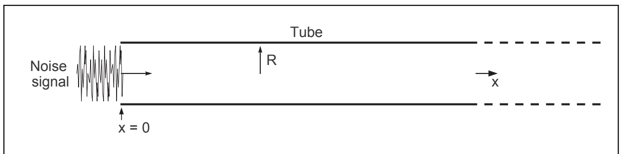


**Figure 1**. Nonlinear propagation of a noise signal in an infinite tube of radius $R$.

Here a phenomenological approach is used, it considers the two physical processes (nonlinearity and dissipation) as being independent, and therefore superposable, when the signal propagates over small distances. According to the fractional step method [11–13], Eq. (1) is decoupled in two separate equations:

$$\frac{\partial p}{\partial x} = -\chi \frac{\partial^{1/2} p}{\partial \tau^{1/2}} \text{ with } \chi = \frac{1}{R}\sqrt{\frac{l_v}{c_0}}\left[1 + \frac{\gamma - 1}{\sqrt{l_v/l_h}}\right] \qquad (3)$$

$$\frac{\partial p}{\partial x} = \frac{\beta}{\rho_0\, c_0^3}\, p\, \frac{\partial p}{\partial \tau}\ . \qquad (4)$$

These equations are solved independently, over each incremental step $\Delta x$ following the Strang splitting method [14].

The solution of Eq. (3) is expressed in the frequency domain [7]:

$$p(x,\omega) = p(0,\omega)\ e^{-\sqrt{j\omega}\chi x}\ . \qquad (5)$$

The numerical solution of the lossless Burgers Eq. (4) is obtained by using the convexification method (in the time domain) [15, 16]. The principles are detailed in Ref. [17]. This method uses the mathematical tools and properties of convex analysis. Its starting point is the Hopf-Cole solution to the Burgers equation in the inviscid limit, then the Legendre-Fenchel transform and the convex envelope construction are employed to obtain the single-value waveform with physical meaning.

## 3. COMPARAISONS

Numerical predictions of the generalized Burgers equation (Eq. 1) obtained with the fractional step method are now compared with experimental data. The waveguide is a tube with inner radius $R = 8\,mm$ (Fig. 1). Note that the propagation of plane waves is limited to frequencies lower than the eigen frequency of the first higher mode [5]:

$$f < 1.84\, \frac{c_0}{2\pi R} \simeq 12.6\ kHz, \qquad (6)$$

where $c_0 \simeq 345\ m.s^{-1}$ is the speed of sound in air.

The input sound source is a narrowband noise (center frequency $f_c = 2.5\ kHz$ and bandwidth $\Delta f = 1\ kHz$) assumed to be Gaussian and statistically stationary. The figure 2 gives the histograms of realizations and the Probability Density Function (PDF) of the acoustic pressures measured at $x = 0$ and $x = 2\ m$. The PDF with a Gaussian distribution at the inlet of the tube (x = 0) is defined by

$$P(p) = \frac{1}{\sqrt{2\pi}\ \sigma}\ e^{-\frac{(p-\mu)^2}{2\ \sigma^2}} \qquad (7)$$

with parameters $\mu = 0$ and $\sigma \simeq 1576$ Pa. At distance $x = 2\ m$, the parameters are $\mu = 0$ and $\sigma \simeq 988$ Pa. Due to nonlinear interactions the Gaussianity of the signal is not preserved during acoustic propagation in a tube as observed by Bjørnø and Gorbatov [12, 18].

Figure 3 shows the spectral content of these acoustic signals obtained with a discrete Fourier transform and a Savitzky-Golay (S-G) filter [19]. The S-G filter is based on polynomial approximation of the data in a moving window, it has two parameters: the degree of the fitted polynomial function $k_1$ (for $k_1$ = 0 and 1, a weighted moving average and a linear regression are performed respectively) and the length of the window $k_2$ (the number of data).

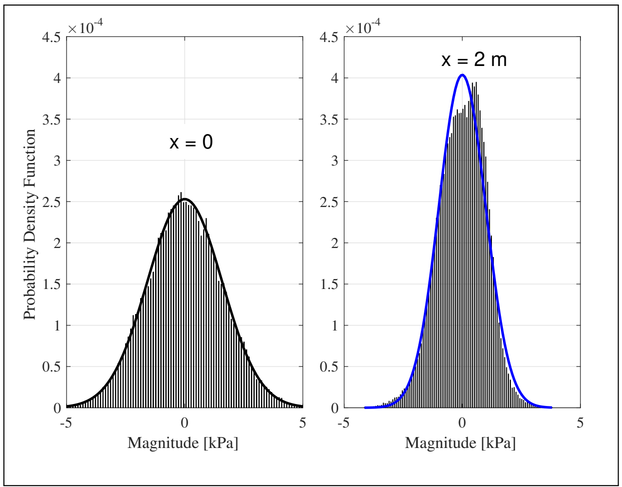


**Figure 2**. Acoustic pressure distribution (histogram bars) and the Probability Density Function (PDF), see Eq.(7). Narrowband noise with a center frequency $f_c = 2.5\ kHz$ and bandwidth $\Delta f = 1\ kHz$.
$x = 0 : \mu = 0$ and $\sigma \simeq 1576\ Pa$.
$x = 2\ m : \mu = 0$ and $\sigma \simeq 988\ Pa$.

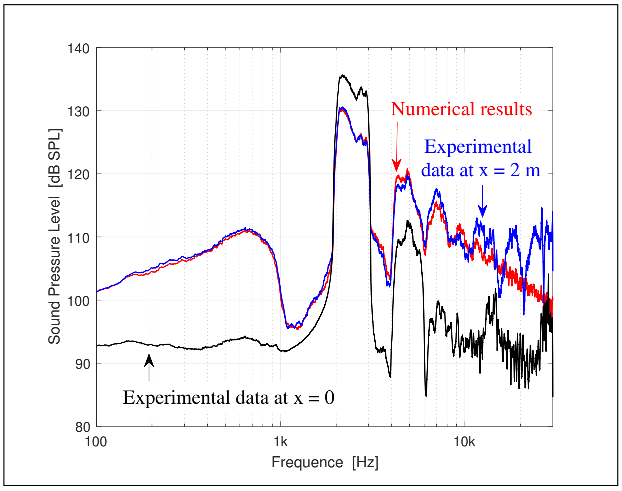


**Figure 3**. Sound Pressure Levels measured at the distance $x = 0$ and $x = 2\ m$. Narrowband noise with a center frequency $f_c = 2.5\ kHz$ and bandwidth $\Delta f = 1\ kHz$. Numerical results are obtained with the fractional step method. The value of the incremental step $\Delta x = x/N$ with $N = 3$. The three curves are obtained with a Savitzky-Golay smoothing filter (with parameters : $k_1 = 1$ and $k_2 = 171$).

This representative example shows that nonlinear processes are present with the transfer of energy from the central band of the spectrum to both lower and higher frequencies [15, 20]. The first process is the steepening of the waveform leading to shock formation, and consequently to the energy transfer to high frequencies. The second process is the coalescence of the shocks causing the growth of the time scale characteristic of the waveform and the corresponding energy transfer in the low frequency range.

The agreement between the numerical and experimental data is good up to the frequency of 12 $kHz$ (global error close to 2.5%). Beyond that, the differences can be attributed to the limitations of the plane wave model (in tube of radius $R = 8\ mm$, its predictions are limited to frequencies below $13\,kHz$) and the microphone conditioner.

The proposed numerical approach is an alternative to predict the nonlinear propagation of random signals for which analytical predictions are limited [21–23]. The predictions of this numerical method can be used in nonlinear propagation problems concerning noise generators such as high-speed jets [24–27].

## 4. ACKNOWLEDGMENTS

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## 5. REFERENCES

[1] L. Menguy and J. Gilbert, "Weakly nonlinear gas oscillations in air-filled tubes; solutions and expériments," *Acta Acustica*, vol. 86, pp. 798–810, 2000.

[2] J. Gilbert, L. Menguy, and M. Campbell, "A simulation tool for brassiness studies," *J. Acoust. Soc. Am.*, vol. 123, no. 54, pp. 1854–185, 2008.

[3] M. Campbell, J. Gilbert, and A. Myers, *The Science of Brass Instruments*. Springer Cham, 2021.

[4] C. Kameni, *Étude et optimisation d'une source sonore pour la sollicitation acoustique fort niveau des satellites*. Ph.D. dissertation, Le Mans University, Le Mans, France, 2017.

[5] A. D. Pierce, *Acoustics - An Introduction to Its Physical Principles and Applications*. New York: Acoustical Society of America, 1989.

[6] M. Bruneau, *Fondamentals of acoustics*. London: ISTE, 2006.

[7] A. Chaigne and J. Kergomard, *Acoustics of musical instruments*. New York: Springer-Verlag, 2016.

[8] N. Sugimoto, "Burgers equation with a fractional derivative; hereditary effects on nonlinear acoustic waves," *J. Fluid Mech.*, vol. 225, pp. 631–653, 1991.

[9] B. Lombard, D. Matignon, and Y. L. Gorrec, "A fractional burgers equation arising in nonlinear acoustics: theory and numerics," in *Proc. of 9th IFAC Symposium on Nonlinear Control Systems*, (Toulouse, France), pp. 406–411, 2013.

[10] B. Lombard and D. Matignon, "Diffusive approximation of a time-fractional burger's equation in nonlinear acoustics," *SIAM J. Appl. Math.*, vol. 76, no. 5, pp. 1765—-1791, 2016.

[11] R. O. Cleveland, M. F. Hamilton, and D. T. Blackstock, "Time-domain modeling of finite-amplitude sound in relaxing fluids," *J. Acoust. Soc. Am.*, vol. 99, no. 6, pp. 3312–3318, 1996.

[12] M. F. Hamilton and D. T. Blackstock, *Nonlinear Acoustics*. San Diego: Academic Press, 1998.

[13] A. Loubeau and F. Coulouvrat, "Effects of meteorological variability on sonic boom propagation from hypersonic aircraft," *AIAA J.*, vol. 47, no. 11, pp. 2632–2641, 2009.

[14] G. Strang, "On the construction and comparison of difference schemes," *SIAM J. Numer. Anal.*, vol. 5, no. 3, pp. 506—-517, 1968.

[15] S. N. Gurbatov, 0leg V. Rudenko, and A. I. Saichev, *Waves and Structures in Nonlinear Nondispersive Media : General Theory and Applications to Nonlinear Acoustics*. Springer-Verlag, 2011.

[16] W. A. Woyczyński, *Burgers-KPZ Turbulence*. Berlin: Springer-Verlag, 1998.

[17] P. Béquin, "Burgers equation and convexification method: Propagation of wave packets and narrowband noise," *J. Acoust. Soc. Am.*, vol. 151, no. 2, pp. 1223–1234, 2022.

[18] L. Bjørnø and S. N. Gurbatov, "Evolution of universal high-frequency asymptotic forms of the spectrum in the propagation of high-intensity acoustic noise," *Soviet Physics - Acoustics*, vol. 31, pp. 179–182., 1985.

[19] A. Savitzky and M. J. E. Golay, "Smoothing and differentiation of data by simplified least squares procedures," *Analytical Chemistry*, vol. 36, no. 8, pp. 1627—-1639, 1964.

[20] K. A. Naugolnykh and L. A. Ostrovsky, *Nonlinear Wave Processes in Acoustics*. Cambridge, UK: Cambridge University Press, 1998.

[21] J. F. Scott, "The nonlinear propagation of acoustic noise," *Proc. R. Soc. Lond. A, Math. Phys. Sci.*, vol. 383, pp. 55—-70, 1982.

[22] D. G. Crighton and S. Bashforth, "Nonlinear propagation of broad band jet noise," in *Proc. of AIAA 6th AEROACOUSTICS CONFERENCE*, (Hartford, USA), pp. 406–411, 1980.

[23] G. P. Howell and C. L. Morfey, "Non-linear propagation of broadband noise signals," *J. Sound Vib.*, vol. 114, no. 2, pp. 189–201, 1987.

[24] K. L. Gee, V. W. Sparrow, M. M. James, J. M. Downing, C. M. Hobbs, T. B. Gabrielson, and A. A. Atchley, "The role of nonlinear effects in the propagation of noise from high-power jet aircraft," *J. Acoust. Soc. Am.*, vol. 123, no. 6, pp. 4082–4093, 2008.

[25] P. Menounou and A. N. Athanasiadis, "A generalized statistical burgers equation to predict the evolution of the power spectral density of high-intensity noise in atmosphere," *J. Acoust. Soc. Am.*, vol. 126, no. 3, pp. 983–994, 2009.

[26] M. R. Shepherd, K. L. Gee, and M. S. Wochner, "Short-range shock formation and coalescence in numerical simulation of broadband noise propagation," *J. Acoust. Soc. Am.*, vol. 126, no. 6, pp. 2886–2893, 2009.

[27] W. J. Baars, C. E. Tinney, M. S. Wochner, and M. F. Hamilton, "On cumulative nonlinear acoustic waveform distortions from high-speed jets," *J. Fluid Mech.*, vol. 749, pp. 331–366, 2014.