DOI: 10.61782/fa.2023.1087

forum acusticum 2023

# ACOUSTIC EIGENVALUE ESTIMATION USING THE MATRIX PENCIL METHOD AND RAYLEIGH QUOTIENT ITERATION

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## ABSTRACT

The acoustic eigenvalues of an enclosure quantify its resonant frequencies and damping coefficients. This information can be used to determine the acoustic characteristics of a room. Various methods for estimating eigenvalues from a measured room impulse response can be found in the literature. Of these methods, the matrix pencil method is considered in this work. The matrix pencil method provides acceptable eigenvalue estimates, albeit alongside several spurious eigenvalues. In this work, the appearance of spurious eigenvalues is demonstrated by analyzing an analytic impulse response. Additionally, the use of the well-known Rayleigh quotient iteration algorithm to search for eigenvalues close to informed initial guesses and, thus, to minimize the estimation of spurious eigenvalues is presented. The approach is verified using finite element eigensolutions of a room.

**Keywords:** *Eigenvalue estimation, Matrix pencil method, Rayleigh quotient iteration*.

## 1. INTRODUCTION

In many scientific fields, there is often a need to estimate the eigenvalues of some system under test. The eigenvalues of physical systems can be used to quantify the resonant frequencies and damping coefficients of the modes of those systems and are, therefore, widely used for engineering applications. A few examples include the analysis of power systems [1], the development of surface acoustic wave devices [2], and the analysis of biomedical signals [3].

Eigenvalue analysis has also found applications in the field of acoustics, for example, adaptive beamforming [4], musical instrument analysis [5] and room mode parameter estimation [6]. In the context of room acoustics, the room mode parameters are resonant frequencies, damping coefficients, and complex-valued spatially dependent modal amplitudes. Mäkivirta *et al.* [7, 8] presented loudspeaker equalization techniques to modify modal decay rates in a room. The techniques require the estimation of eigenvalues, *i.e.*, resonant frequencies, and damping coefficients. Another application of eigenvalue estimation is to enable the estimation of the locally reacting surface impedance of an acoustically absorbing sample in a reverberant chamber, as described by Prinn *et al.* [9]. A necessary requirement for this impedance estimation method to provide good estimates is the availability of precise estimates of the resonant frequencies and damping coefficients of the chamber containing the sample to be measured.

Precisely estimating eigenvalues from measured signals is a challenging task, and there is a long history of research into this problem. Examples of, perhaps, the most commonly known signal processing techniques that have been developed to estimate these parameters are: MUSIC [10], ESPRIT [11], and the generalized pencil of function method [12], which is also referred to as the Matrix Pencil Method (MPM). Interestingly, it can be

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shown that these three approaches are all Prony-like methods [13, 14]. In this work, only the MPM is considered, which has found application in many fields. For example, Laroche [5] used the MPM to synthesize the admittance curves of a guitar bridge and analyze the beats of a piano tone. This was achieved by identifying modal parameters from measured signals and reconstructing the signals using the estimated parameters. More recently, Haddad and Noga [15] considered the MPM for applications in speech processing, and a quantum version of the MPM has been proposed [16], which has the potential to reduce computation times.

In this work, we are motivated by the potential of improving the approach proposed by the current authors [9] to estimate the locally reacting surface impedance at low frequencies in reverberant environments. Thus, the problem considered here is the estimation of the resonant frequencies and damping coefficients of room modes. While this problem can be partially addressed by applying the MPM, the MPM provides spurious estimates. Differentiating between valid and spurious estimates is a challenging task. This problem is mitigated here by using initial guesses of the room's eigenvalues and the well-known Rayleigh quotient iteration algorithm to minimize the appearance of spurious estimates. In the following, we refer to the approach as the Rayleigh Quotient Matrix Pencil Method (RQMPM).

The remainder of this paper is structured as follows. In Sec. 2, the MPM is briefly reviewed. In Sec. 3, the matrix pencil is re-derived, and the Rayleigh quotient iteration algorithm is presented. In Sec. 4, the RQMPM is verified and validated using a test case with both simulated and measured data. This paper is concluded in Sec. 5.

## 2. EIGENVALUE ESTIMATION

In this section, the MPM is introduced, and an example of the spurious eigenvalue estimates that we are trying to remove from the solution set is given.

### 2.1 Matrix Pencil Method

We begin by presenting Prony's model [17], which decomposes a time-dependent signal into a sum of damped exponential functions, as follows:

$$y(t) = \sum_{j=1}^{\infty} \frac{a_j}{2} \left( \mathrm{e}^{\mathrm{i}\phi_j} \mathrm{e}^{\lambda_j^+ t} + \mathrm{e}^{-\mathrm{i}\phi_j} \mathrm{e}^{\lambda_j^- t} \right) . \qquad (1)$$

In this model, $a_j$ is an amplitude component, $\phi_j$ is a phase term, $\lambda_j^{\pm} = \pm \mathrm{i}2\pi f_j - \sigma_j$ is a complex-valued eigenvalue, with resonant frequency $f_j$ and damping coefficient $\sigma_j$, and $t$ is time. In practice, the number of eigenvalues used in the sum is truncated, resulting in an approximate description of the signal, denoted here by $h$. Changing the summation index, we can write:

$$h(t) = \sum_{m=1}^{M} a_m \mathrm{e}^{\lambda_m t} , \qquad (2)$$

where $a_m$ is an $m$th complex-valued amplitude, which includes magnitude and phase, and $\lambda_m = \mathrm{i}2\pi f_m - \sigma_m$. The signal is sampled at discrete time intervals, $t_n$, such that $h(t_n) = h_n$. The maximum number of samples is denoted by $N$. Choosing a uniform sampling interval implies a time step $\Delta t = 1/f_s$, with sampling frequency $f_s$.

The discretized time signal, $\mathbf{h}$, is used to construct a matrix pencil. The MPM [12] requires the solution of the following generalized eigenvalue problem for all eigenvalues $\gamma_m$ with $m \in \{1, 2, \ldots, M\}$:

$$\det \left( \mathbf{H}_1^{\dagger} \mathbf{H}_2 - \gamma_m \mathbf{I} \right) = 0 , \qquad (3)$$

where superscript † indicates use of the Moore-Penrose pseudoinverse, $\mathbf{I}$ is the identity matrix, and $\mathbf{H}_1$ and $\mathbf{H}_2$ are Hankel matrices given by:

$$\mathbf{H}_1 = \begin{bmatrix} h_1 & h_2 & \ldots & h_p \\ h_2 & h_3 & & h_{p+1} \\ \vdots & & \ddots & \vdots \\ h_{N-p} & h_{N-p+1} & \ldots & h_{N-1} \end{bmatrix}_{(N-p)\times p} , \qquad (4)$$

and

$$\mathbf{H}_2 = \begin{bmatrix} h_2 & \ldots & h_p & h_{p+1} \\ h_3 & & h_{p+1} & h_{p+2} \\ \vdots & \ddots & & \vdots \\ h_{N-p+1} & \ldots & h_{N-1} & h_N \end{bmatrix}_{(N-p)\times p} . \qquad (5)$$

When generating the matrices, pencil parameter $p$ is an integer chosen such that $M \leq p \leq (N-M)$. Typically, $p = \mathrm{round}(N/2)$. Since $\gamma_m = \mathrm{e}^{\lambda_m \Delta t}$, the eigenvalues of the system that generates the signal can be obtained from

$$\lambda_m = \mathrm{i}\frac{\arg(\gamma_m)}{\Delta t} - \frac{\log(|\gamma_m|)}{\Delta t} . \qquad (6)$$

### 2.2 Spurious eigenvalues

While the MPM provides good estimates of the system eigenvalues, the estimates are often polluted with spurious eigenvalues, see, e.g., Refs. [18, 19].

As an example of this behavior, a simulated time-dependent signal is used in this section to demonstrate the spurious estimates obtained. The signal, shown in Fig. 1, is given by

$$h(t) = \mathrm{Re}\left(\sum_{m=1}^{M} a_m \mathrm{e}^{\lambda_m t}\right), \tag{7}$$

where $a_m = 1, \forall m$, $f_m = (30, 70, 110, ..., 990)$, $\sigma_m = 2\pi, \forall m$, $f_s = 3$ kHz, and the signal has a duration of 1 s. Pink noise has been added to the signal, with a signal-to-noise ratio of 100 dB.

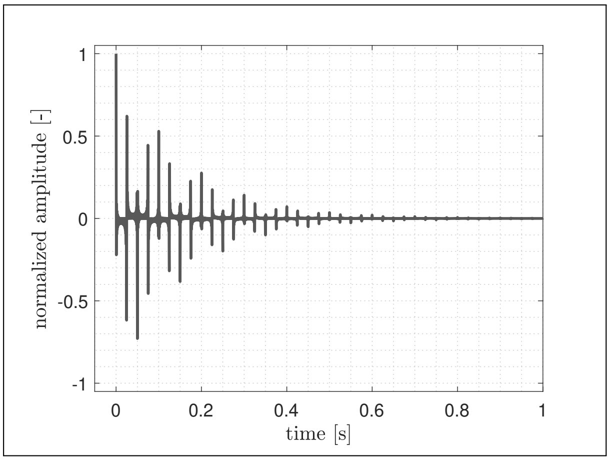


**Figure 1**. The time-dependent signal used to demonstrate the spurious eigenvalues. Note that the signal has been normalized by its maximum absolute value.

This signal is analyzed using the MPM, and a selected range of estimates are compared to reference eigenvalues in Fig. 2. We observe that the MPM provides valid estimates of the reference eigenvalues. However, due to the presence of noise, spurious eigenvalues are also estimated. The spurious eigenvalues pollute the estimates, making it difficult to identify the valid estimates. Note that there are many more MPM solutions than those shown here; The complete solution set ranges from $f = -f_s/2$ to $f = f_s/2$, and from $\sigma = -2846$ to $\sigma = 620.1$.

Various attempts to remove the spurious eigenvalues can be devised. For example, we could choose the eigenvalues which are closest to the frequencies at which the

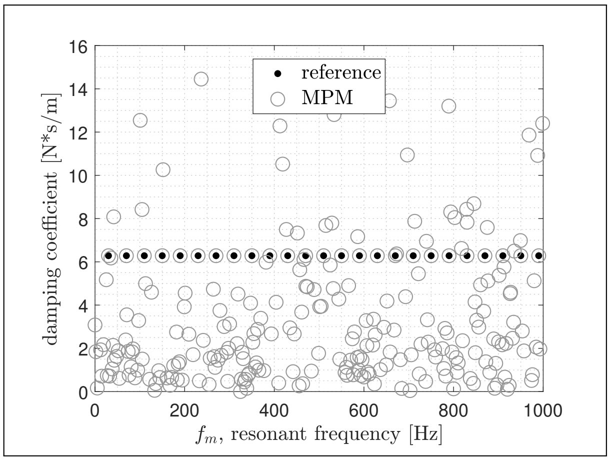


**Figure 2**. Comparison of reference eigenvalues and eigenvalue estimates obtained through use of the MPM.

peaks lie in the transfer function. This approach would have the disadvantage of omitting eigenvalues that are close in frequency and are, therefore, indistinguishable from a transfer function peak analysis. An alternative approach would be to perturb the eigenvalue system, given in Eq. (3), in the hope that the estimates whose values do not significantly change are valid estimates. However, this approach is not robust in the presence of noise.

One remedy that the authors have found quite useful in this respect is presented in the next section and forms the main contribution of this work.

## 3. RAYLEIGH QUOTIENT ITERATION

To avoid the spurious estimates, we choose to use Rayleigh quotient iteration (see, e.g., Saad [20, p. 90]) to solve Eq. (3). Because the solution of a large eigenvalue system is avoided, Rayleigh quotient iteration comes with the benefit of potentially reducing the computational complexity of the estimation process. This, of course, depends on the number of required eigenvalue estimates.

Rayleigh quotient iteration requires initial guesses of the eigenvalues and eigenvectors of the signal. The initial guesses used in this work are based on *a priori* knowledge of the system's geometry and damping coefficient estimates. Other approaches are available, and these are discussed in Sec. 3.3. For the moment, we re-derive Eq. (3) to identify the signal eigenvectors used as initial guesses for the Rayleigh quotient iteration algorithm.

### 3.1 Eigenvalue problem derivation

Eq. (2) can be written as:

$$\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_N \end{bmatrix} = \underbrace{\begin{bmatrix} e^{\lambda_1 t_1} & e^{\lambda_2 t_1} & \dots & e^{\lambda_M t_1} \\ e^{\lambda_1 t_2} & e^{\lambda_2 t_2} & & e^{\lambda_M t_2} \\ \vdots & & \ddots & \vdots \\ e^{\lambda_1 t_N} & e^{\lambda_2 t_N} & \dots & e^{\lambda_M t_N} \end{bmatrix}}_{\mathbf{E}} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_M \end{bmatrix} . \tag{8}$$

Noting that $h(t_n + \Delta t) = \sum_{m=1}^{M} a_m e^{\lambda_m (t_n + \Delta t)}$, we can write:

$$\mathbf{H}_1 = \mathbf{E} \underbrace{\begin{bmatrix} a_1 & a_1 e^{\lambda_1 \Delta t} & \dots & a_1 e^{\lambda_1 (p-1)\Delta t} \\ a_2 & a_2 e^{\lambda_2 \Delta t} & & a_2 e^{\lambda_2 (p-1)\Delta t} \\ \vdots & & \ddots & \vdots \\ a_M & a_M e^{\lambda_M \Delta t} & \dots & a_M e^{\lambda_M (p-1)\Delta t} \end{bmatrix}}_{\mathbf{A}} , \tag{9}$$

and,

$$\mathbf{H}_2 = \mathbf{E} \underbrace{\begin{bmatrix} e^{\lambda_1 \Delta t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 \Delta t} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & e^{\lambda_M \Delta t} \end{bmatrix}}_{\mathbf{\Gamma}} \mathbf{A} . \tag{10}$$

Writing $\mathbf{H}_1$ and $\mathbf{H}_2$ in terms of $\mathbf{E}$, we obtain

$$\mathbf{H}\mathbf{\Psi} = \mathbf{\Psi}\mathbf{\Gamma} , \tag{11}$$

where $\mathbf{H} = (\mathbf{H}_1)^{\dagger}\mathbf{H}_2$, $\mathbf{\Psi} = \mathbf{A}^{\dagger}$, and $\mathbf{\Gamma}$ is a diagonal matrix populated with eigenvalues. By choosing the $m$th eigenvalue, this can be written as a linear eigenvalue problem

$$\mathbf{H}\boldsymbol{\psi}_m = \gamma_m \mathbf{I} \boldsymbol{\psi}_m , \tag{12}$$

which is solved to obtain a set of eigenvalues $\boldsymbol{\gamma}$. It should be clear that we have re-derived the MPM presented in Sec. 2.1. Since eigenvector $\boldsymbol{\psi}_m$ can have arbitrary amplitude, $\boldsymbol{\psi}_m$ can be constructed from knowledge of $\lambda_m$, $p$, and $\Delta t$ (cf. Eq. (9)).

Out of interest, an alternative derivation that uses the first order gradient, $\mathrm{d}h/\mathrm{d}t$, to generate $\mathbf{H}_2$ results in an eigenvalue system of the form: $\mathbf{H}\boldsymbol{\psi}_m = \lambda_m \mathbf{I}\boldsymbol{\psi}_m$. However, the accuracy of this alternative approach is limited by the accuracy of the method used to obtain $\mathrm{d}h/\mathrm{d}t$. Acceptable results might be obtained by computing the gradient in the frequency domain as long as the decay of the signal is adequately captured, but this alternative approach has been found to be less accurate than the approach derived above.

### 3.2 Rayleigh quotient algorithm

To find only valid (*i.e.*, non-spurious) estimates of the eigenvalues, Rayleigh quotient iteration can be used to solve Eq. (12). An initial guess for eigenvector $\boldsymbol{\psi}_m$ can be iteratively refined, using the formula:

$$\boldsymbol{\psi}_m^{(q+1)} = \frac{\left(\mathbf{H} - \gamma_m^{(q)}\mathbf{I}\right)^{-1} \boldsymbol{\psi}_m^{(q)}}{\left\| \left(\mathbf{H} - \gamma_m^{(q)}\mathbf{I}\right)^{-1} \boldsymbol{\psi}_m^{(q)} \right\|} , \tag{13}$$

and the updated eigenvalues can be estimated by using

$$\gamma_m^{(q+1)} = \frac{\left(\boldsymbol{\psi}_m^{(q+1)}\right)^T \mathbf{H}\boldsymbol{\psi}_m^{(q+1)}}{\left(\boldsymbol{\psi}_m^{(q+1)}\right)^T \boldsymbol{\psi}_m^{(q+1)}} , \tag{14}$$

where $q = 1, 2, \dots, Q$ indicates the number of iterations. The initial guess for the eigenvector $\boldsymbol{\psi}_m^{(1)}$ is given by the $m$th column of $\mathbf{\Psi}$, and $\gamma_m^{(1)} = e^{\tilde{\lambda}_m \Delta t}$ is the initial guess for the eigenvalue. Note that $\tilde{\lambda}_m$ comprises initial guesses of the $m$th resonant frequency and $m$th damping coefficient. It has been observed that the magnitude of the difference between the $q$th and $q+1$th eigenvalue can be used as a measure of convergence. Thus, the iteration may be stopped before $Q$ is reached, if $|\gamma_m^{(q+1)} - \gamma_m^{(q)}| \leq \tau$, where $\tau$ is a small, predefined value.

We must now provide initial guesses for the unknown eigenvalues.

### 3.3 Initial eigenvalue guesses

Using well-reasoned initial guesses can aid in the identification of valid eigenvalues, while also reducing the computational effort by reducing the number of iterations. Additionally, good initial guesses that reduce the number of required iterations can help to keep the conditioning under control.

One might obtain initial guesses in many ways. For example, Mäkivirta *et al.* [7] estimate the resonant frequencies from the resonant peaks in the short-term Fourier transform of the transfer function, and estimate the damping coefficients by fitting an exponential decay model to the decaying signal at each estimated resonant frequency, as proposed by Karjalainen *et al.* [21]. This approach might be used when one is interested in estimating the resonant frequencies and damping coefficients of specific resonant peaks. Although, with the caveat that this approach might not find modes that, due to the coalescence of closely spaced resonant peaks, are buried in the data.

Alternatively, with knowledge of the room's geometry, one might determine the resonant frequencies of the undamped system. For example, when considering a shoebox-shaped room there exists an analytical formula for computing the undamped resonant frequencies [22]. More generally, if the geometry of the room is complex, performing an eigenvalue analysis of the undamped room can provide initial guesses.

For lightly damped rooms, the undamped and damped resonant frequencies might be quite similar, in which case good results might be obtained by using the undamped resonant frequencies as initial guesses. However, it cannot always be expected that the undamped and damped resonant frequencies will lie close to each other in the complex plane. In such cases, initial guesses of the damping coefficients can also be utilized to improve estimates. The method provided by Karjalainen *et al.* [21] can be used to obtain good estimates.

In this work, eigenvalue analysis of the undamped room, and damping coefficient estimation from a room impulse response [21], are used to generate initial guesses.

### 3.4 Signal reconstruction

The eigenvalue estimates are found using Eq. (6) with the solutions of Eq. (14). Eq. (2) can then be rewritten to estimate the unknown amplitudes, i.e.,

$$\tilde{\mathbf{a}} = \mathbf{E}^{\dagger}\mathbf{h}\,. \tag{15}$$

The signal can be reconstructed from the real part of Eq. (2) with $a_m = \tilde{a}_m$. In scenarios for which there is no reference data, reconstructing the signal and comparing it to the original signal can serve as a sanity check that the estimates are reasonable. Since the reconstruction will have a limited number of eigenvalues, comparison of the transfer functions rather than the time-domain signals can sometimes be more informative.

## 4. EIGENVALUE ESTIMATION

We now demonstrate the performance of the RQMPM, using both simulated and measured data. Using simulated data gives us access to reference eigenvalues, with which we can verify the approach. However, when using measured data, we do not have access to any reference eigenvalues. Instead, we infer the validity of the approach by reconstructing the measured signal.

### 4.1 Test case and setup

Both the simulated and measured data are obtained from the same room. A model of the room is shown in Fig. 3. It can be seen that the room's geometry is complex. A loudspeaker is placed in one corner of the room, and simulated and measured impulse responses are obtained at the same randomly chosen position within the room.

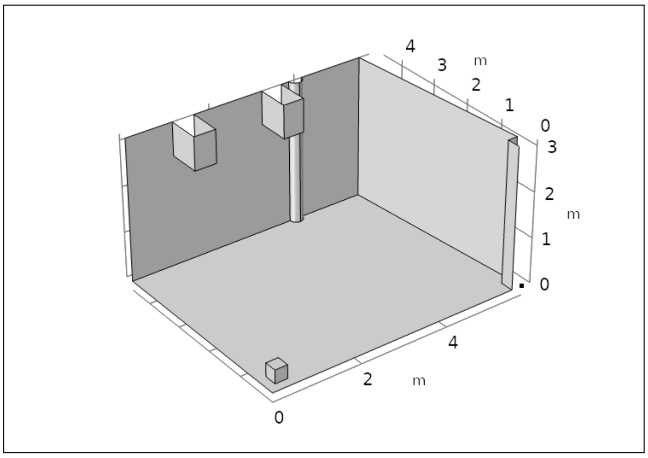


**Figure 3**. Geometry of room under test. The source is located close to the coordinate system origin.

For the construction of the matrix pencil from the simulated and measured impulse responses, we use $p = \mathrm{round}(N/2)$. To reduce computational effort, for the generation of the matrix pencil from the measured impulse response downsampling is used. Assuming the geometry of the room under test is known, the initial guesses for the resonant frequencies are taken from a finite element-based eigenvalue analysis of the hard-walled room. Initial guesses of the damping coefficients are generated using the method given by Karjalainen *et al.* [21].

### 4.2 Simulation

A uniform, frequency-independent normalized impedance of 60 is imposed on all bounding surfaces of the room. We choose a constant impedance because obtaining accurate reference solutions for a frequency-dependent impedance cannot be guaranteed. A simulated room impulse response has been generated using a Gaussian pulse with a bandwidth of 300 Hz, a delay of 0.05 s, a sampling frequency of 2 kHz, and a duration of 1 s (following the approach described by Prinn [23]). The simulated impulse response is shown in Fig. 4.

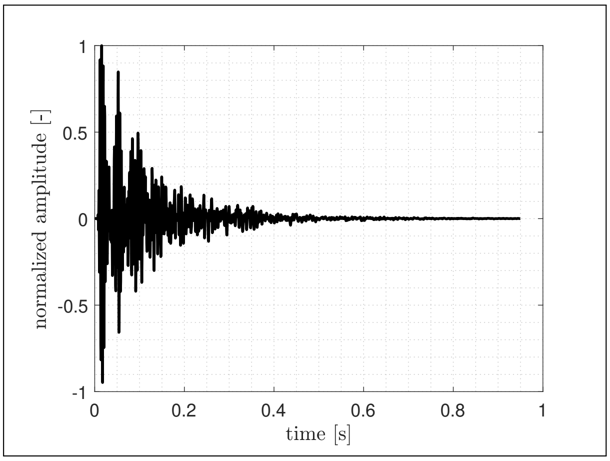


**Figure 4**. The simulated impulse response.

We estimate the eigenvalues of this impulse response using Eqs. (13) and (14). Upon consultation of Fig. 5, it can be seen that the eigenvalues are well estimated. However, the matrix pencil has failed to accurately capture all of the eigenvalues. Note that the RQMPM estimates can only be as accurate as the MPM estimates. Thus, the discrepancy seen here is an inherent limitation of the matrix pencil estimation. The RQMPM has avoided many of the spurious MPM estimates.

Shown in Fig. 6 is a reconstruction of the simulated transfer function. Only eigenvalues below 120 Hz have been estimated, which explains the missing peak above 120 Hz in the reconstructed transfer function. Upon comparison with the reference transfer function, good agreement is found.

### 4.3 Measurement

An impulse response is measured in the room with a sampling frequency of 48 kHz and a duration of 2 s. The signal-to-noise ratio is approximately 56 dB. For the matrix pencil construction, the impulse response is downsampled by a factor of 35, as this reduces the computational effort while still providing good estimates.

For the experimentally obtained data, we do not have access to reference values. Instead, the RQMPM estimates are compared to the initial guesses and the MPM estimates in Fig. 7. The RQMPM estimates are different from the initial guesses, and it would appear that the RQMPM approach has avoided many of the spurious MPM estimates. In the absence of reference values, we infer the validity of the estimates from a reconstruction of the measured signal. A comparison of the reference and reconstructed transfer functions, given in Fig. 8, indicates that reasonable estimates have been found. Due to the noise present in the measured impulse response, this is a more challenging test of the method.

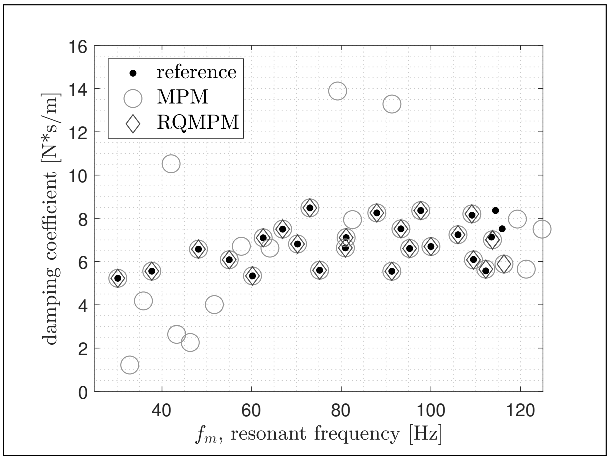


**Figure 5**. Comparison of the reference and estimated eigenvalues of the simulated room. Note that only a subset of MPM solutions is shown in this graph.

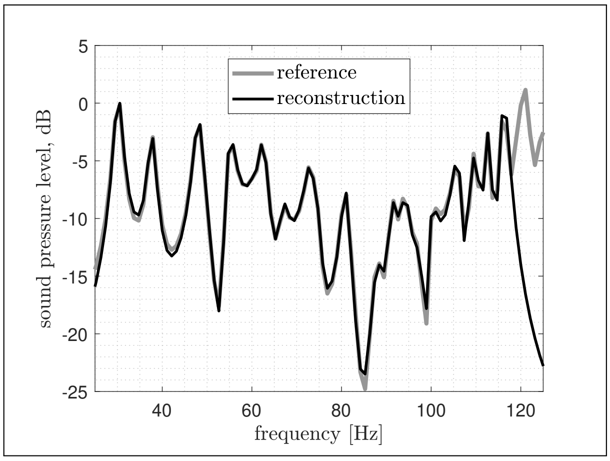


**Figure 6**. Comparison of the simulated (reference) and reconstructed room transfer functions.

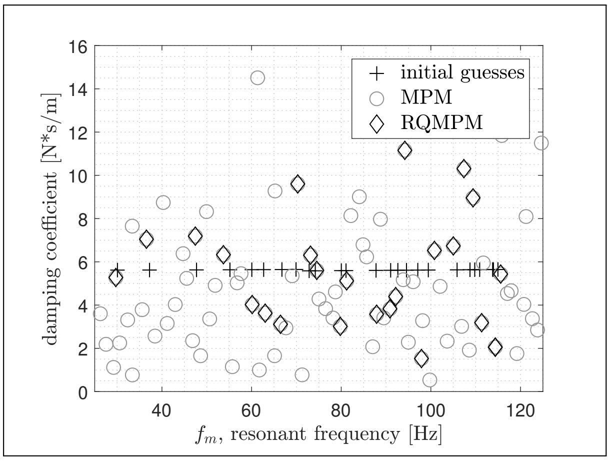


**Figure 7**. Comparison of the initial guesses, the MPM estimates, and the RQMPM estimates for the measured room. Note that only a subset of MPM solutions is shown in this graph.

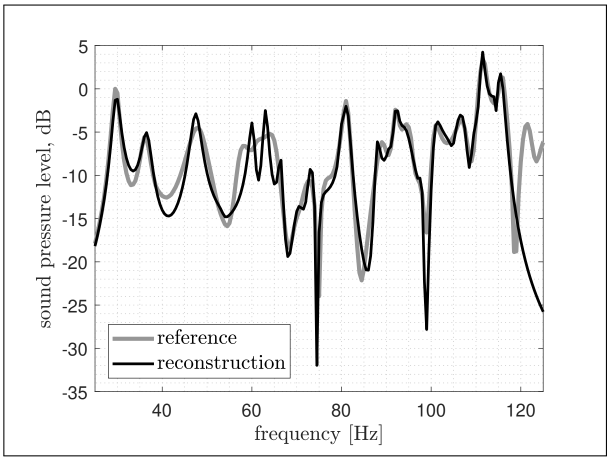


**Figure 8**. Comparison of the measured (reference) and reconstructed room transfer functions.

Note that it has been found that better agreement between the reference and reconstructed transfer functions can be obtained by analyzing impulse responses measured at multiple positions in the room. The RQMPM estimates can then be averaged to obtain improved eigenvalue estimates. However, averaging has not been used here, and the data presented is taken from one measured impulse response only.

### 4.4 Discussion

The RQMPM does not always provide reliable solutions. Depending on the quality of the MPM solutions and the quality of the initial guesses, the approach might still find a spurious eigenvalue. It is unclear how this might be avoided, and further investigation is required. However, if multiple measurements are used in a controlled setting, e.g., a reverberation chamber, in which good initial guesses are available from an *a priori* analysis, it is expected that good eigenvalue estimates can be obtained.

One further issue that might arise is the estimation of repeated eigenvalues. This is an indication that two eigenvalues are similar in value and, thus, that an eigenvalue has been missed. A simple remedy is to run the estimator again, twice. Once with an initial guess that is lower than the repeated estimate and once with a guess that is higher. If the repeated estimation does not find a new eigenvalue, then an eigenvalue may be missing from the solution set.

## 5. CONCLUSION

Of the many eigenvalue estimation methods available, the matrix pencil method provides good estimates. However, these good estimates are accompanied by spurious estimates. In practical settings, it is difficult to determine which solutions are valid estimates and which are spurious. In this work, Rayleigh quotient iteration has been used to solve the eigenvalue problem formulated by the matrix pencil approach. The matrix pencil has been rederived to arrive at initial guesses of the eigenvectors and eigenvalues required for the iterative procedure. The use of Rayleigh quotient iteration has been tested on simulated and measured data. In both cases, the estimates obtained have been used to reconstruct the transfer functions of the input signals. The agreement obtained between the reference and reconstructed room transfer functions implies that the proposed approach can provide reliable estimates of room mode frequencies and damping coefficients.

Future work should focus on advanced methods for identifying and removing spurious eigenvalues from the MPM solutions.

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