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LOCATING AND DYNAMICALLY ENCIRCLING AN EXCEPTIONAL POINT IN AN EXPERIMENTAL MECHANICAL SYSTEM

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ABSTRACT

Eigenmodes remain a recurring concept in several branches of physics for the analysis of dynamical systems. As long as they are Hermitian, they satisfy standard orthogonal properties. This is no longer true when gains and/or losses are taken into account. For specific values of some parameters, eigenvalues as well as their associated eigenvectors can coalesce because of the existence of an exceptional point (EP). These EPs have gained much interest in recent years because of the counter-intuitive concepts associated with them, like strong attenuation or mode switching phenomenon.

In the fields of acoustics and vibration, EP control may lead to a better understanding of energy exchanges and dissipation between modes. These aspects are also integral elements of metamaterials, since their design is based on resonators and their coupling.

This work aims at exploring the key concepts related to EPs by revisiting the well-known coupled pendulums problem in the presence of damping. First, the free response of the experimental system is investigated after tuning it on an EP. Then an encircling is performed by varying the parameters through time. Experimental results allow us to observe nearly-optimal dissipation, energy exchange as well as chirality effects which have already been studied in physics.

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1. PROBLEM STATEMENT

The experimental setup consists of two simple pendulums of mass m_i and length L_i coupled by a spring of stiffness k located at a distance d from the pivots, as shown in Fig. 1. As losses are needed to make this system non-Hermitian, a magnetic damping c_2 is created by means of a conductive plate fixed at the end of the second pendulum oscillating in a controllable magnetic field. In order to achieve an encirclement in section 3, the length L_2 is also controllable by an electric linear actuator. The angles of rotation θ_i of each pendulum are recorded by a Hall effect sensor.

The equation of motion in terms of the generalized coordinates $\mathbf{q} = \left[\begin{smallmatrix}\theta_1\\ \theta_2\end{smallmatrix}\right]$ reads

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}, \tag{1}$$

with

$$\begin{aligned}\mathbf{M} &= \begin{bmatrix} m_1L_1^2 & 0 \\ 0 & m_2L_2^2 + I_2 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix}, \\ \mathbf{K} &= \begin{bmatrix} m_1gL_1 + kd^2 & -kd^2 \\ -kd^2 & m_2gL_2 + \Omega_2^2 I_2 + kd^2 \end{bmatrix},\end{aligned} \tag{2}$$

where we take into account the moment of inertia I_2 and natural angular frequency Ω_2 added to the second pendulum due to the plate.

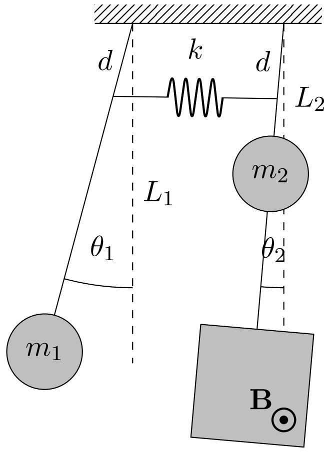
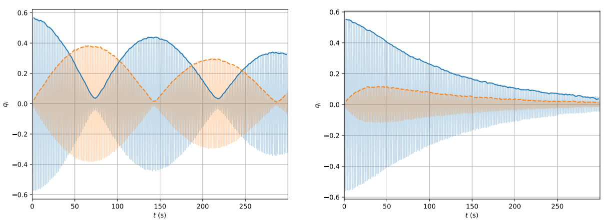


Figure 1: Sketch of the coupled pendulums.

The first objective of this paper is to find out how to achieve optimal energy dissipation of this system. We start by taking a look at how it behaves after moving the first pendulum from its equilibrium for different damping values. If the damping is very low, there are periodic exchanges of energy between the two pendulums, as shown by the beat pattern in Fig. 2a. This is because both pendulums transfer energy to each other at a faster rate than the second pendulum dissipates it. On the contrary, if the damping is sufficient, the second pendulum will dissipate it faster than they exchange it, thus ending the periodic exchanges, as shown in Fig. 2b.



(a) (b)

Figure 2: Free response envelopes of the first (solid blue line) and second pendulum (orange dashed line) after moving the first one from its equilibrium position for (a) strong and (b) weak coupling.

In the case of coupled oscillators, these two distinct regimes are known as weak and strong coupling. Moreover, it is known that the transition between these two regimes is linked to the presence of an EP [1, 2].

This kind of damping-dependent behavior also appears in the well-known damped harmonic oscillator, which can either be underdamped or overdamped depending on its damping ratio. The transition between these two regimes corresponds to the critical damping, for which the system returns to equilibrium as quickly as possible without oscillating. *The critical damping actually corresponds to an EP*, as it is the moment where the complex conjugate eigenvalues will merge and become purely real [3].

Similarly, can optimal attenuation of the system be achieved by tuning L_2 and c_2 on an EP? To do so, a modal analysis is necessary in order to gain a better insight.

2. MODAL ANALYSIS

The modal analysis of Eq. Eqn. (1) starts by considering the exponential ansatz $\mathbf{q} = \mathbf{v}e^{\lambda t}$ which yields the quadratic eigenvalue problem (QEP)

$$\left(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}\right)\mathbf{v} = \mathbf{0}, \tag{3}$$

By construction, eigenvalues are the roots of the characteristic polynomial

$$p(\lambda) = \det\left(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K}\right). \tag{4}$$

Because matrices $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$ are real, eigenvalues and eigenvectors are real or come in complex conjugate pairs [4]. Using the Laplace transform notation, we may write the complex eigenvalue as $\lambda = -\sigma \pm \mathrm{i}\omega$, where ω corresponds to the natural angular frequency and σ to the exponential decay constant, both being real. In this paper, we are interested in situations where an eigenvalue, call it λ_0, becomes a double root, which signifies:

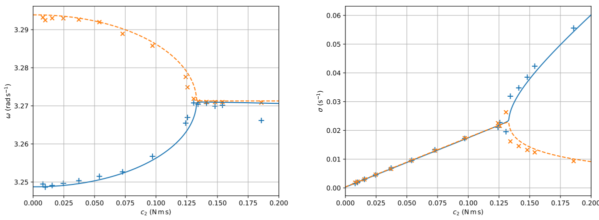
$$p(\lambda_0) = \partial_\lambda p(\lambda_0) = 0. \tag{5}$$

In theory, to obtain an EP, one needs either a complex parameter or two real parameters. While the first case is easy to solve with a numerical solver, the second requires more effort. An algorithm has been developed, not detailed here, allowing us to find a pair (L_2^0, c_2^0) corresponding to an EP.

In order to observe experimentally the impact of the EP on the eigenvalues, let us fix $L_2 = L_2^0$ and observe their evolution as a function of c_2. The experimental eigenvalues are obtained by curve fitting algorithm assuming sum of exponential solution

$$\mathbf{q}(t) = 2\mathrm{Re}\left(\mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t}\right), \tag{6}$$

and shown in Fig. 3. First is the hermitian case, as there is nearly no damping. The eigenvalues are purely imaginary, $\lambda = \pm i\omega$, meaning that each mode does not decay through time. Next, as the damping increase, so does the exponential decay constants while the natural angular frequencies are getting closer and closer. Then, both the real and imaginary parts of the eigenvalues coalesce, this is the EP. Finally, from here, increasing the damping will continue to increase the exponential decay constant of one mode, but will have the opposite effect on the other.



(a) (b)

Figure 3: (a) Imaginary and (b) real parts of the eigenvalues as a function of c_2 for $L_2 = L_2^0$.

Thus, the EP corresponds to an optimal damping of the free response of the system for an arbitrary initial condition, in the sense that it corresponds to the configuration where the least damped mode is the most damped.

3. MODE SWITCHING

The Fig. 3 shows separately each parts of the eigenvalues as a function of c_2. To get the full picture, we want to display the complex eigenvalues as a function of L_2 and c_2. For this purpose, it is common to use a Riemann surface, as shown in Fig. 4, computed numerically, where the curves in Fig. 3a have also been plotted. This surface provides a *static* picture made of a collection of independent coupled pendulum. The natural angular frequencies are plotted on the z-axis and the exponential decay constants are indicated by the colour of the surface, which is either blue if less than the EP decay ($\sigma_0 = 0.023\,\mathrm{s}^{-1}$) or red otherwise ($\max \sigma = 0.067\,\mathrm{s}^{-1}$). It is therefore obvious that the EP corresponds to the best compromise, as one cannot have two red surfaces for the same set of parameters.

Moreover, we can notice that this surface has the surprising property of being self-intersecting (not the case for Hermitian system). Because of this self-intersecting manifold, one can move from one sheet to another by turning

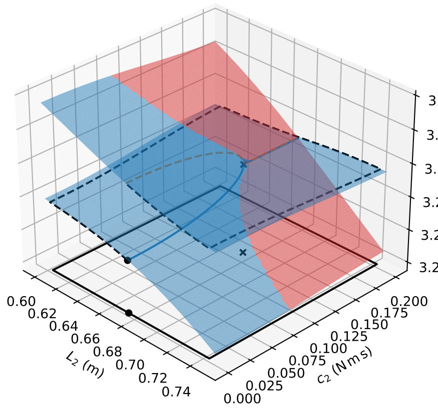
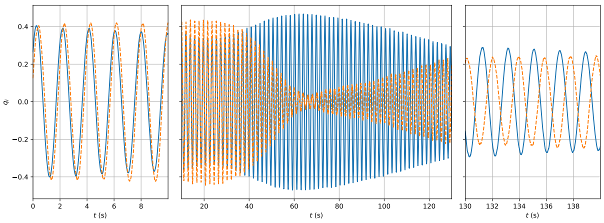


Figure 4: Riemann surface of the complex eigenvalues in the parameter space. The z-axis corresponds to ω while the color refers to σ. The path of the encirclement is also displayed in the parameter space (solid line) as well as its projection on the Riemann surface (dashed line) in the clockwise direction starting from the lower sheet, with the circle indicating the starting point and the cross the EP.

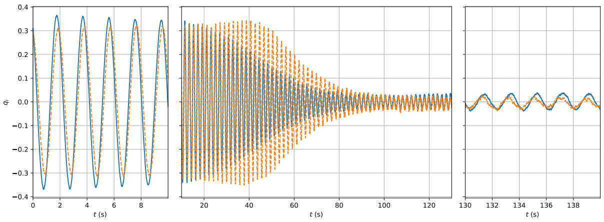
around the EP (see the dashed line) and switch from one mode to another. Does this mean that this *static* property is conserved when the EP is *dynamically* encircled in the parameter space ?

This was done experimentally by slowly varying the second pendulum length and the inductance current, starting at $(L_2, c_2) = (L_2^0, 0)$ where the initial conditions were nearly the in-phase mode, that is the lower sheet, and encircling the EP in both direction. For the clockwise encirclement, shown in Fig. 5a, the results match our intuition, as starting in-phase, we end up out-of-phase, as suggested by the dashed line in Fig. 4. However, this is not the case in the counter-clockwise direction, shown in Fig. 5b, where we remain in phase and with a very high attenuation.

To understand what happened, we have to go back to Fig. 4. As we move the parameters, the system is perturbed, and these perturbations will populate both modes. However, once we reach the high damping zone, the energy injected into the most attenuated mode (in red) will vanish faster than in the least attenuated one (in blue) and this latter will therefore predominate. This is why, as we turn around the EP in the clockwise direction, we stay



(a)



(b)

Figure 5: Response of the system for a 2-minute encirclement of the EP in the (a) clockwise and (b) anticlockwise directions. Responses are also displayed before and after the encirclement to highlight the mode switching phenomenon.

"close" to the blue surface. On the counterclockwise direction, however, we soon reach the red surface, and the perturbation will populate the other surface more, ending at the same point once the encirclement is done. This explains why on Fig. 5b the system ends up in the same mode and well attenuated. This yield to chiral behavior, as the final state is determined by the direction of encirclement [5, 6].

4. CONCLUSION

In this work, an experimental set-up is proposed to highlight and to encircle EP for a first time in a mechanical system. This system allows a direct visual observation of the main phenomena linked to EPs with a great accuracy.

During the encircling, chiral mode switching has been evidenced. These results are conform to those observed in other fields like in [7, 8]. Work is ongoing to investigate the influence of the contour shape [9, 10], the contour starting point [8, 11] and the connection with Floquet-EP [12] to provide a better understanding of this phenomenon.

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