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A MULTI-MODAL ANALYSIS OF ANISOTROPY IN POROUS MEDIA

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ABSTRACT

This paper introduces a modal decomposition method to retrieve the effective fluid parameters of a sample of anisotropic porous material from multi-modal reflection and transmission coefficients measured in a squared impedance tube. The approach extends on traditional impedance tube methods to include higher-order modes, allowing to characterize the test specimen for plane-wave and multi-modal reflection and transmission coefficients. The proposed formulation uses a modal expansion of the sound pressure and normal particle velocity in the tube, reducing the problem to a system of two differential equations for the components of the pressure and particle velocity projected over the normal modes. The system is solved analytically, which leads to an exact algebraic formulation of the reflection and transmission matrices. An inverse problem is further formulated to infer the bulk modulus and density tensor coefficients of the anisotropic specimen. The method is evaluated numerically on a synthetic anisotropic sample of known porous properties.

Keywords: *Anisotropic porous media, multi-modal analysis, impedance tube*

1. INTRODUCTION

Acoustic wave propagation and viscothermal dissipation of acoustic energy in porous media can be described macroscopically by an equivalent fluid model, which requires the knowledge of several pore parameters. Although some of the pore parameters can be measured directly, these methods require specialized equipment and are often difficult to perform. In this context, inverse characterization methods that are based on the inversion of the scattering matrix [1] are particularly interesting, as they allow to recover the dynamic density and bulk modulus of the equivalent fluid from simple measurements of the normal incidence reflection and transmission coefficients [2]. If the material is homogeneous and isotropic, these two frequency-dependent fluid parameters are also sufficient to determine the six Johnson-Champoux-Allard-Lafarge (JCAL) parameters [3] that control the dissipation of acoustic energy in the medium.

However, most porous materials are anisotropic and exhibit different properties along orthogonal directions known as the *principal directions*. Specifically, the influence of anisotropy translates into a full symmetric density tensor (in place of a scalar), drastically increasing the number of effective fluid parameters required to describe the medium.

Several studies extended the inversion procedure presented in [2] to characterize anisotropic materials. However, most of these studies are based on simplified assumptions on the nature of anisotropy; e.g., anisotropic materials with principal directions belonging to the layer plane interface [4, 5], or two-dimensional anisotropic materials with principal directions arbitrarily tilted with respect to the reference coordinate system [6–8]. To the authors' knowledge, the only study that characterizes fully anisotropic porous materials in three dimensions (i.e., anisotropic porous materials having principal axes tilted in *a priori* unknown directions) is due to Terroir et al.

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[9], who described a general method to recover the effective fluid parameters and the orientation of the material's principal directions from knowledge of the reflection and transmission coefficients at six angles of incidence. Experimental validation was recently achieved by Cavalieri et al. [10], who measured the normal incidence reflection and transmission coefficients of six samples of glass wool cut in six different orientations in an impedance tube.

Following a different approach, the present study exploits the concept of modal decomposition [11–14] to elicit higher order modes in the tube and characterize the test specimen for plane-wave and multimodal reflection and transmission coefficients. Such approach is significant in that it allows to retrieve the effective fluid parameters and the principal directions without changing the orientation of the sample in the tube.

2. THEORY

Consider a layer of homogeneous anisotropic porous material with thickness L, bulk modulus B and density tensor $\boldsymbol{\rho}$ mounted in a rectangular impedance tube with cross section $\mathcal{A} = [0, w_1] \times [0, w_2]$. In the reference coordinate system $(\mathrm{O}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ with position coordinates (x_1, x_2, x_3), its boundaries are defined by the equations $x_3 = 0$ and $x_3 = L$. We further assume that the density tensor $\boldsymbol{\rho}$ is symmetric; that is $\boldsymbol{\rho}^\top = \boldsymbol{\rho}$, where $^\top$ denotes non-conjugate transposition. In particular, the orthonormal coordinate system $(\mathbf{e}_{\mathrm{I}}, \mathbf{e}_{\mathrm{II}}, \mathbf{e}_{\mathrm{III}})$ of the principal directions of the layer can be defined so that the density tensor is diagonal in this system; that is $\boldsymbol{\rho} = \boldsymbol{\rho}^* = \mathbf{diag}(\rho_{\mathrm{I}}, \rho_{\mathrm{II}}, \rho_{\mathrm{III}})$, where ρ_{I}, ρ_{II} and ρ_{III} are the principal densities and the superscript * designates the diagonal tensor. In the reference coordinate system, the density tensor reads $\boldsymbol{\rho} = \mathbf{R}_{rot}\boldsymbol{\rho}^*\mathbf{R}_{rot}^\top$, where $\mathbf{R}_{rot} = \mathbf{R}_3(\theta_{\mathrm{III}})\mathbf{R}_2(\theta_{\mathrm{II}})\mathbf{R}_1(\theta_{\mathrm{I}})$ is the rotation matrix between the two coordinate systems, with $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ being elementary rotation matrices and θ_{I}, θ_{II} and θ_{III} the roll, pitch, and yaw angles, respectively.

2.1 Formulation

Outside the layer (i.e., for $x_3 < 0$ and $x_3 > L$), the sound pressure and particle velocity fields $(p, \mathbf{v})$ are governed by the equations of mass and momentum conservation:

$$\mathrm{div}(\mathbf{v}) = \frac{\mathrm{i}\omega}{B_0}p, \tag{1a}$$

$$\mathrm{i}\omega\mathbf{v} = \frac{1}{\rho_0}\nabla p, \tag{1b}$$

where B_0 and ρ_0 are the bulk modulus and scalar density of air, respectively, $\omega = 2\pi f$ is the angular frequency and the time dependency $\mathrm{e}^{-\mathrm{i}\omega t}$ is omitted. Similarly, the sound pressure and particle velocity fields in the layer (i.e., for $0 < x_3 < L$) verify

$$\mathrm{div}(\mathbf{v}) = \frac{\mathrm{i}\omega}{B}p, \tag{2a}$$

$$\mathrm{i}\omega\mathbf{v} = \mathbf{h}\nabla p, \tag{2b}$$

where $\mathbf{h}$ is the symmetric inverse density tensor and reads

$$\mathbf{h} = \begin{pmatrix} h_{11} & h_{12} & h_{13} \\ h_{12} & h_{22} & h_{23} \\ h_{13} & h_{23} & h_{33} \end{pmatrix} = \begin{pmatrix} \mathbf{h}_\perp & \mathbf{h}_3 \\ \mathbf{h}_3^\top & h_{33} \end{pmatrix}, \tag{3}$$

where $(h_{11}, h_{12}, h_{13}, h_{22}, h_{23}, h_{33})$ are the individual inverse density tensor components, $\mathbf{h}_\perp$ is a $2{\times}2$ matrix, and the subscript $\perp$ designates the transverse components. This tensor is diagonal in the orthonormal coordinate system of the principal directions of the layer.

As the tube is of constant cross-section, the boundary condition at its walls read:

$$\mathbf{n}^\top\mathbf{v} = 0, \tag{4}$$

where $\mathbf{n} = [\mathbf{n}_\perp; 0]$ is the normal vector. The system of differential equations (2a) and (2b) can be rewritten as follows

$$\begin{cases} h_{33}\dfrac{\partial p}{\partial x_3} = -\mathbf{h}_3^\top\nabla_\perp p + \mathrm{i}\omega v_3, \\ \dfrac{\partial(\mathrm{i}\omega v_3)}{\partial x_3} = -\omega^2 B^{-1}p - \mathrm{div}_\perp(\mathbf{h}_\perp\nabla_\perp p + \mathbf{h}_3\dfrac{\partial p}{\partial x_3}), \end{cases} \tag{5}$$

with the boundary condition

$$\mathbf{n}_\perp^\top(\mathbf{h}_\perp\nabla_\perp p + \mathbf{h}_3\frac{\partial p}{\partial x_3}) = 0. \tag{6}$$

Outside the layer, the problem reads

$$\begin{cases} (\Delta + \omega^2\rho_0 B_0^{-1})p = 0 \\ \mathbf{n}_\perp^\top\nabla_\perp p = 0 \end{cases} \tag{7}$$

2.2 Modal decomposition

The pressure p and the axial particle velocity v_3 are now expressed as the superposition of an infinite number of transverse modes $\psi_b(x_1, x_2)$, $b \in \mathbb{N}$:

$$p(x_1, x_2, x_3) = \sum_{b\in\mathbb{N}} \psi_b(x_1, x_2)p_b(x_3), \tag{8a}$$

$$\mathrm{i}\omega v_3(x_1, x_2, x_3) = \sum_{b\in\mathbb{N}} \psi_b(x_1, x_2)U_b(x_3), \tag{8b}$$

where

$$\psi_b(x_1, x_2) = g_m^{(w_1)}(x_1) g_n^{(w_2)}(x_2), \tag{9}$$

with $g_m^{(w)}(x) = \sqrt{\varepsilon_m/w}\cos(m\pi x/w)$, $\varepsilon_0 = 1$ and $\varepsilon_{n>0} = 2$. The modal functions $\psi_b(x_1, x_2)$, $b \in \mathbb{N}$, are the eigenfunctions obeying the transverse eigenproblem $\mathrm{div}_\perp(\nabla_\perp \psi_b) = -\gamma_b^2 \psi_b$, with the boundary condition $\mathbf{n}_\perp^\top \nabla_\perp \psi_b = 0$. They form a complete orthonormal basis for the inner product $(f|g) = \int_{\mathcal{A}} \overline{f} g \mathrm{d}S$; i.e., $(\psi_a|\psi_b) = \delta_{ab}$. The modal decompositions in Eqs (8a) and (8b) can also be written as [15]

$$p = \boldsymbol{\psi}^\top \mathbf{p}, \tag{10a}$$

$$\mathrm{i}\omega v_3 = \boldsymbol{\psi}^\top \mathbf{u}, \tag{10b}$$

where $\mathbf{p}$, $\mathbf{u}$ and $\boldsymbol{\psi}$ are column vectors.

Equation (5) is now projected on the orthonormal basis $(\psi_a)_{a\in\mathbb{N}}$

$$\begin{cases} h_{33} p_a' = -\sum_{b\in\mathbb{N}} C_{ab} p_b + U_a, \\ U_a' = -\omega^2 B^{-1} p_a - \sum_{b\in\mathbb{N}} (D_{ab} p_b + \tilde{C}_{ab} p_b'), \end{cases} \tag{11}$$

where $C_{ab} = (\psi_a|\mathbf{h}_3^\top \nabla_\perp \psi_b)$, $\tilde{C}_{ab} = (\nabla_\perp \psi_a|\mathbf{h}_3 \psi_b)$ and $D_{ab} = (\nabla_\perp \psi_a|\mathbf{h}_\perp \nabla_\perp \psi_b)$. This yields a system of differential equations in $\mathbf{p}$ and $\mathbf{u}$

$$\begin{pmatrix} h_{33}\mathbb{1} & \mathbb{0} \\ -\tilde{\mathbf{C}} & \mathbb{1} \end{pmatrix} \begin{pmatrix} \mathbf{p}' \\ \mathbf{u}' \end{pmatrix} = \begin{pmatrix} -\mathbf{C} & \mathbb{1} \\ -\omega^2 B^{-1}\mathbb{1} + \mathbf{D} & \mathbb{0} \end{pmatrix} \begin{pmatrix} \mathbf{p} \\ \mathbf{u} \end{pmatrix}, \tag{12}$$

which can be solved using several numerical strategies [14]. Equation (12) can be rewritten as

$$\mathbf{p}'' + \frac{1}{h_{33}}(\mathbf{C} - \tilde{\mathbf{C}})\mathbf{p}' + \left(\frac{\omega^2}{h_{33}B}\mathbb{1} - \frac{1}{h_{33}}\mathbf{D}\right)\mathbf{p} = 0. \tag{13}$$

3. ANALYTICAL SOLUTION

In this study, we consider the particular case in which the principal directions are rotated only around the longitudinal axis $\mathbf{e}_3$ of the tube (i.e., in the transverse plane). In this case, $\mathbf{h}_3 = \mathbf{0}$ so that $\mathbf{C} = \tilde{\mathbf{C}} = \mathbb{0}$ and Eq. (13) reads

$$\mathbf{p}'' = \mathbf{M}\mathbf{p}, \tag{14}$$

where $\mathbf{M} = -h_{33}^{-1}(\omega^2 B^{-1}\mathbb{1} - \mathbf{D})$. The matrix $\mathbf{M}$ is diagonalized according to

$$\mathbf{M} = \mathbf{Q}\boldsymbol{\Gamma}^2\mathbf{Q}^{-1}, \tag{15}$$

where $\boldsymbol{\Gamma}^2$ is the diagonal matrix of eigenvalues and $\mathbf{Q}$ is the associated matrix of eigenvectors. This yields the general solution:

$$\mathbf{p}(x_3) = \mathbf{Q}\mathrm{e}^{\boldsymbol{\Gamma}x_3}\mathbf{Q}^{-1}\mathbf{c}^+ + \mathbf{Q}\mathrm{e}^{\boldsymbol{\Gamma}(L-x_3)}\mathbf{Q}^{-1}\mathbf{c}^-. \tag{16}$$

Let us now consider an incident plane wave propagating towards the layer from the left (i.e., in the region $x_3 < 0$). The multimodal solution to the differential equation (7) reads

$$\mathbf{p}(x_3 < 0) = \mathrm{e}^{\rho_0 \mathbf{Y}_c x_3}\mathbf{p}^{(i)} + \mathrm{e}^{-\rho_0 \mathbf{Y}_c x_3}\mathbf{p}^{(r)}, \tag{17}$$

where $\mathbf{p}^{(i)}$ is a vector containing the modal components of the incident wave at $x_3 = 0$, $\mathbf{p}^{(r)} = \mathbf{R}\mathbf{p}^{(i)}$ contains the modal components of the reflected wave with $\mathbf{R}$ the reflection matrix, and

$$\mathbf{Y}_{cab} = \frac{\mathrm{i}}{\rho_0}\sqrt{\omega^2\frac{\rho_0}{B_0} - \gamma_a^2}\,\delta_{ab} \tag{18}$$

is the characteristic admittance of the waveguide. In the region $x_3 > L$, the transmitted sound pressure reads

$$\mathbf{p}(x_3 > L) = \mathrm{e}^{\rho_0 \mathbf{Y}_c (x_3 - L)}\mathbf{p}^{(t)}, \tag{19}$$

where $\mathbf{p}^{(t)} = \mathbf{T}\mathbf{p}^{(i)}$ contains the modal components of the transmitted wave with $\mathbf{T}$ the transmission matrix.

The scattering matrix can be written as

$$\mathbf{S} = \begin{pmatrix} \mathbf{R} & \mathbf{T}' \\ \mathbf{T} & \mathbf{R}' \end{pmatrix}, \tag{20}$$

where $\mathbf{R}$ and $\mathbf{T}$ are the reflection and transmission matrices for a right-going incident wave (i.e., propagating towards the layer from the left), and $\mathbf{R}'$ and $\mathbf{T}'$ are the reflection and transmission matrices for a left-going incident wave (i.e., propagating towards the layer from the right). Since $\mathbf{h}_3 = \mathbf{0}$, we have $\mathbf{R} = \mathbf{R}'$ and $\mathbf{T} = \mathbf{T}'$. The matrices $\mathbf{R}$ and $\mathbf{T}$ are obtained from the continuity of the pressure and normal component of the particle velocity at $x_3 = 0$ and $x_3 = L$, and read:

$$\begin{cases} \mathbf{R} = (\mathbf{Y}_c + \mathbf{Y}_0)^{-1}(\mathbf{Y}_c - \mathbf{Y}_0), \\ \mathbf{T} = \mathbf{T}_r(\mathbb{1} - \mathbf{H}\mathbf{R}_l\mathbf{H}\mathbf{R}_r)^{-1}\mathbf{H}\mathbf{T}_l, \end{cases} \tag{21}$$

where $\mathbf{Y}_0 = \tilde{\mathbf{Y}}_c(\mathbb{1} - \mathbf{H}\mathbf{R}_r\mathbf{H})(\mathbb{1} + \mathbf{H}\mathbf{R}_r\mathbf{H})^{-1}$ is the admittance matrix at $x_3 = 0$, $\tilde{\mathbf{Y}}_c = h_{33}\mathbf{Q}\boldsymbol{\Gamma}\mathbf{Q}^{-1}$, $\mathbf{H} = \mathbf{Q}\mathrm{e}^{\boldsymbol{\Gamma}L}\mathbf{Q}^{-1}$, $\mathbf{R}_r = \mathbf{R}_l = (\tilde{\mathbf{Y}}_c + \mathbf{Y}_c)^{-1}(\tilde{\mathbf{Y}}_c - \mathbf{Y}_c)$, $\mathbf{T}_r = (\mathbf{Y}_c + \tilde{\mathbf{Y}}_c)2\tilde{\mathbf{Y}}_c$, and $\mathbf{T}_l = (\mathbf{Y}_c + \tilde{\mathbf{Y}}_c)2\mathbf{Y}_c$.

4. NUMERICAL RESULTS

The validity of the method is examined numerically on a synthetic anisotropic sample with known porous properties. Based on the reflection and transmission matrices derived in Sect. 3, the procedure described in [9] and [10] is applied to retrieve the bulk modulus B and the coefficients h_{11}, h_{12}, h_{22} and h_{33} of the inverse density tensor. The results were not available at the time of writing.

5. CONCLUSION

A multimodal method for the characterization of anisotropy in porous media has been formulated. The method is based on the inversion of the scattering matrix to retrieve the effective fluid parameters from multimodal reflection and transmission coefficients measured in an impedance tube.

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7. REFERENCES

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