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AN ABSORBING BOUNDARY CONDITION FORMULATION FOR FAST FREQUENCY SWEEP OF UNBOUNDED ACOUSTIC FINITE ELEMENT PROBLEMS

Pierre Mariotti[1*] **Huina Mao**[1] **Sebatian Schopper**[3]
Romain Rumpler[1,2]
[1] The Marcus Wallenberg Laboratory for Sound and Vibration Research (MWL),
[2] The Centre for ECO2 Vehicle Design and Digital Futures,
Department of Engineering Mechanics,
KTH Royal Institute of Technology, SE-100 44, Stockholm, Sweden
[3]Technical University of Munich, Chair of Structural Mechanics, Arcisstr. 21, 80333 Munich

ABSTRACT

In exterior acoustic simulations using the finite element method, accurate modelling of an infinite domain using a finite computational space is challenging due to reflections at the truncated boundaries. This study introduces a second-order operator for implementing absorbing boundary conditions with frequency-independent matrices. This new implementation is based on the coupling of the Helmholtz equation and the absorbing boundary condition operator equation with a common term. It involves a small expansion of the matrix, which is not costly compared to the advantage of the frequency-independent system matrix. In addition, the matrix formulation is well-suited for moment matching model order reduction due to the polynomial frequency combination of frequency independent matrices. This advancement presents a robust solution for large-scale acoustic problems, reducing computational time and resource requirements.

Keywords: *Unbounded domain, Absorbing Boundary Condition, Fast Frequency Sweep, Model Order Reduction*

**Corresponding author*: *pimariot@kth.se.*

1. INTRODUCTION

To model unbounded domain situations, Boundary Element Method (BEM) and Finite Element Method (FEM) are the two main methods in engineering cases. The BEM [1] reduces dimensionality by discretizing only the boundary of the domain, making it highly efficient for infinite or semi-infinite problems. In contrast, the FEM [2] discretizes the entire domain, providing greater flexibility and robustness for problems involving complex geometries, nonlinearities, or heterogeneous materials. For these reasons, the FEM is often preferred over the BEM in practical engineering applications. Commercial software that implement FEM, such as COMSOL, Ansys, and so on, make the construction of the FEM rather invisible for the user. But the mathematical development of the method is, from a general point of view, rather straightforward. In the development, the boundary conditions of the problem can be naturally introduced. Having a propagating wave in an open space, such as an acoustic wave produced by a car [3], a magnetic field [4], a seismic wave [5], are very classical situations. In order to avoid unwanted reflected waves along an artificial boundary, new assumptions must be considered. Among the different assumptions to model outgoing waves, perfect matched layers are also very popular [6]- [7]. They usually suffer from the expansion of the volume domain, it highly increases the size of the problem. In this work, the artificial boundary of the model is treated with an Absorbing Boundary Condition [8]- [9]. ABCs are directly applied along the artificial surface, therefore, the problem size remain the same. In addition, even if there is a lot of great work done for it [10], PML implementation suffers also from its large number of parameters that needs to be tuned, and can be frequency dependent. From an implementation point of view, ABCs suffer from the presence of normal derivatives that are known for being numerically instable [11]. Moreover, in exterior acoustic situations, very large scale modeling can be required. For instance, the study of the noise created by a plane requires to model a large domain size, and solving the problem for multiple different frequencies can become very expansive, even impossible. To solve this issue, it is possible to reduce the model by a Model Order Reduction (MOR) technique. In the case of frequency sweeps, it is possible to use a Moment Matching method. The term Moment, stands for the knowledge of the problem around a certain frequency. In this work, the MOR used is the Well-Condition Asymptotic Waveform Equation (WCAWE), a moment matching method creates a basis vectors from an iterative schema [12]- [13]. To create this basis, the matrix problem known at the interpolation frequency is used, but also the derivatives of the matrix problem w.r.t frequency, at this exact same frequency.

It is clear that having a system matrix expanded in the form of a (linear) combination of frequency-independent matrices has a double impact. Not only are the matrices assembled once for all, but the computation of derivatives of the system matrix with respect to frequency are also straightforward. This work proposes a new way to implement ABCs that leads to frequency-independent matrices. The system matrix is then only a polynomial frequency system.Therefore, it is possible to consider a larger amount of vector in the basis for the projection-based MOR technique chosen here.

The first section of this work presents the finite element formulation leading to the resulting frequency-dependent system involved by the implementation of ABC. In the same section, the proposed implementation resulting in a combination of frequency-independent matrices is introduced. The section will then recall the WCAWE method, and highlights the advantage of having frequency affine development of the system matrix. Then the results section presents the exact similarity between the old and the new implementation before presenting the results concerning the ROM, and the need to consider a larger basis to entirely fit the range of frequency. The paper will then be conclude with a discussion section.

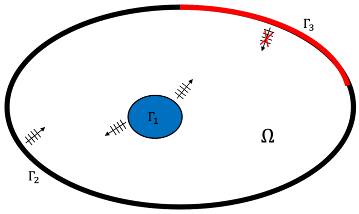


Figure 1: Description of the acoustic domain and boundaries.

2. FROM FULL-ORDER MODEL TO REDUCED-ORDER MODEL

This section first details the development of the Finite Element Method in order to introduce an Absorbing Boundary Condition, which, in its original implementation, leads to a frequency-dependent system matrix. It is followed by the proposed implementation which allows to express the problem in a polynomial frequency expression. Secondly, this section recalls the WCAWE moment matching Model Order Reduction method, which benefits from the implementation improvements proposed.

2.1 Finite Element formulation

A classical vibro-acoustic generic problem is described in Fig. 1. The acoustic domain is denoted by Ω, the boundaries of the domain are referred to as Γ_1 for a vibrating structure, and Γ_2 for a rigid wall where the acoustic wave is fully reflected. Finally, Γ_3 refers to an artificial boundary where the so-called Absorbing Boundary Condition will be applied.

The governing equation associated with the propagation of a harmonic wave in a homogeneous fluid domain, i.e. the Helmholtz equation, is given by

$$\forall M \in \Omega, \quad \nabla^2 p + k^2 p = 0, \tag{1}$$

where p correspond to the acoustic pressure fluctuation in Ω and k corresponds to the wavenumber.

The Eq. (1) is called the strong formulation of the Helmholtz equation, and does not present the boundary conditions of the problem. To do so, the weak formulation needs to be expressed. The weak formulation of the Helmholtz equation is

$$\forall v \in V, \int_\Omega \nabla p \cdot \nabla v d\Omega - k^2 \int_\Omega p \cdot v d\Omega - \int_{\partial\Omega} \frac{\partial p}{\partial n} \cdot v d\partial\Omega = 0, \tag{2}$$

where the function v corresponds to the test function, associated with the function space V. The boundary of the fluid domain, $\partial\Omega$ can be expressed as the reunion of boundaries such as $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Along the first two boundaries, a Neumann boundary condition is applied, such that

$$\begin{cases} \Gamma_1 : \frac{\partial p}{\partial n} = 1, \\ \Gamma_2 : \frac{\partial p}{\partial n} = 0. \end{cases} \tag{3}$$

The third term in Eq. (2) may be developed into three terms associated with each boundary. The boundary term associated with Γ_2 leads to a null term, the boundary term associated with Γ_1 leads to the so-called external excitation term, or right-hand side term, and the resulting term from the artificial boundary Γ_3, referred to as the "absorbing term" in the following, is further discussed in the next two sections.

2.1.1 Absorbing Boundary Condition: Frequency dependent matrix

The operator considered in this work is the Bayliss-Gunzburger-Turkel (BGT) operator [14]. In view of a focussed discussion on the scope of the present contribution, only the second order of the BGT is developed, but all results presented stand for the first order, and are currently being extended to higher orders. The second order BGT operator [11] is given by

$$\forall M \in \Gamma_2, \quad \frac{\partial^2 p(M)}{\partial r^2} + P_1(r,k)\frac{\partial p(M)}{\partial r} + P_2(r,k)p(M) = 0, \tag{4}$$

with r the distance from the point source, and the function $P_1(r,k) = 2jk + 4/r, P_2(r,k) = 2/r^2 - k^2 + 4jk/r$. Note here that P_n functions are polynomial expansions of the wavenumber. In its classical implementation, the ABC operator may be used to express the radial derivative of the pressure field, leading to a Neumann boundary condition. Using Eq. (4) to obtain the normal derivative of p, then inserting it in Eq. (2), results in

$$\forall v \in V, \quad \int_\Omega \nabla p \cdot \nabla v \, d\Omega - k^2 \int_\Omega p \cdot v \, d\Omega + \int_{\Gamma_3} \frac{1}{P_1(r,k)} \frac{\partial^2 p}{\partial n^2} \cdot v + \frac{P_2(r,k)}{P_1(r,k)} p \cdot v d\Gamma_3 = \int_{\Gamma_1} 1 \cdot v d\Gamma_1. \tag{5}$$

Following the Galerkin approach, after the discretization of the space, and the appropriate selection of the polynomial function space, each of the integral terms in Eq. (5) may be assembled in a matrix. Each component corresponds to the integral term where the unknowns are approximated with N_i, a polynomial function of the chosen nodal basis. The resulting matrix formulation of the problem has the form

$$\left[K + C(k) - k^2 M\right] P = F, \tag{6}$$

where P is the vector of nodal unknowns and F the vector of external excitations. The matrices K and M are the stiffness and mass matrices, whereas $C(k)$, is the frequency-dependent matrix associated with absorbing boundaries, which needs to be computed and reassembled at each frequency iteration.

2.1.2 Proposed implementation of the operator

The previous section explained how the implementation of ABC classically leads to frequency dependent matrices. It is well known that assembling such matrices for each step of a frequency sweep, is memory and CPU consuming. This subsection, proposes a strategy to implement ABC in a way that the matrices are frequency independent. Moreover, the matrix formulation of the problem has another advantage to the use of the moment-matching method WCAWE where the matrix derivatives w.r.t frequency are needed.

The strategy is rather simple, the only difficulty can be in the finite element programming process. The strategy is to transform a common term of Eq. (5) and Eq. (4) in a new variable. After obtaining the weak formulation of the BGT Eq. (4)), a system of two equations and two unknowns is then possible to write, such as

Find $p \in V$ and $q \in U$ such as:

$$\begin{cases} \forall v \in V, & \int_\Omega \nabla p \cdot \nabla v \, d\Omega - k^2 \int_\Omega p \cdot v \, d\Omega - \int_{\Gamma_3} q \cdot v \, d\Gamma_3 \\ & = \int_{\Gamma_1} 1 \cdot v \, d\Gamma_1 \\ \forall u \in U, & \int_{\Gamma_3} \frac{\partial^2 p}{\partial n^2} \cdot u \, d\Gamma_3 + \int_{\Gamma_3} P_1 q \cdot u \, d\Gamma_3 \\ & + \int_{\Gamma_3} P_2 p \cdot u \, d\Gamma_3 = 0 \end{cases} \tag{7}$$

In this paper, the change of variable is done on the normal derivative, $\frac{\partial p}{\partial n} = q$. It is now possible to take advantage of the polynomial development w.r.t frequency of P_n. The two terms have been previously developed, but can be written as

$$P_n = \sum_{i=0}^{n} \frac{\alpha_i}{r^{n-i}} (jk)^i, \quad n = 1, 2 \tag{8}$$

with $alpha_i$ numerical values. Using this, the two last integral terms of the second equation in the system Eq. (7), can be expressed with the sum of frequency independent integrals with frequency dependent coefficients, *e.g*

$$\int_{\Gamma_3} P_1 q \cdot u d\Gamma_3 = \int_{\Gamma_3} \frac{4}{r} q \cdot u d\Gamma_3 + jk \int_{\Gamma_3} 2q \cdot u d\Gamma_3. \tag{9}$$

This development can similarly be done for the other integral term that concerns p.

Doing so, it is possible to formulate the problem using only integral terms with spatial terms, multiplied by $(jk)^i$ powers. *Nota bene*, in the equation system Eq. (7), the variable p (*resp.* v) and q (*resp.* u) do not have the same number of degrees of freedom (dofs). The variable p is defined in an acoustic volumic domain, whereas the variable q is defined only on a surface. This results in rectangular matrices for coupled terms. The development of the integral terms as inEq. (9), the gathering according to their $(jk)^i$ powers, leads to the following matrix expression

$$\begin{bmatrix} K & -C \\ G_1 + G_2 & E_1 \end{bmatrix} + jk \begin{bmatrix} 0 & 0 \\ G_3 & E_2 \end{bmatrix} - k^2 \begin{bmatrix} M & 0 \\ G_4 & 0 \end{bmatrix} \tag{10}$$

The submatrices assembled with their corresponding integral terms are listed here

$$K = \int_{\Omega} \nabla p \cdot \nabla v \, d\Omega$$
$$M = \int_{\Omega} p \cdot v \, d\Omega$$
$$C = \int_{\Gamma_3} q \cdot v \, d\Gamma_3$$
$$G_1 = \int_{\Gamma_3} \frac{\partial^2 p}{\partial n^2} \cdot u \, d\Gamma_3$$
$$G_2 = \int_{\Gamma_3} \frac{2}{r^2} p \cdot u \, d\Gamma_3$$
$$G_3 = \int_{\Gamma_3} \frac{4}{r} p \cdot u \, d\Gamma_3$$
$$G_4 = \int_{\Gamma_3} p \cdot u \, d\Gamma_3$$
$$E_1 = \int_{\Gamma_3} \frac{4}{r} q \cdot u \, d\Gamma_3$$
$$E_2 = \int_{\Gamma_3} 2q \cdot u \, d\Gamma_3$$

The resulting matrix problem is expended by few degrees of freedom. If Ω has n_Ω degrees of freedom, Γ_3 has n_{Γ_3}, as Γ_3 is a subsurface of Ω, the expansion is very small. Finally, the matrix problem is written such as

$$\left[D_1 + jkD_2 - k^2 D_3\right] P_q = F_q \tag{11}$$

The vectors P_q and F_q are respectively the expanded unknowns vector and the expanded force vector, and can be expressed such as

$$P_q = \begin{bmatrix} p \\ q \end{bmatrix} \quad F_q = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{12}$$

2.2 Model Order Reduction: The new formulation for WCAWE

Among the MOR techniques, the projection based techniques is based on the built basis V, where the problem is projected, solved, and projected back with

$$P = V\alpha \tag{13}$$

By multiplying the matrix problem (6) by V^H, the Hermitian transpose of V, the problem is then reduced in a $N \times N$ problem with N the number of vector in V. And this reduced problem is much faster to solve for each frequency iteration as $N << n_\Omega$. The reduced matrix problem is

$$\left[K_r + C_r(k) - k^2 M_r\right] \alpha = F_r \tag{14}$$

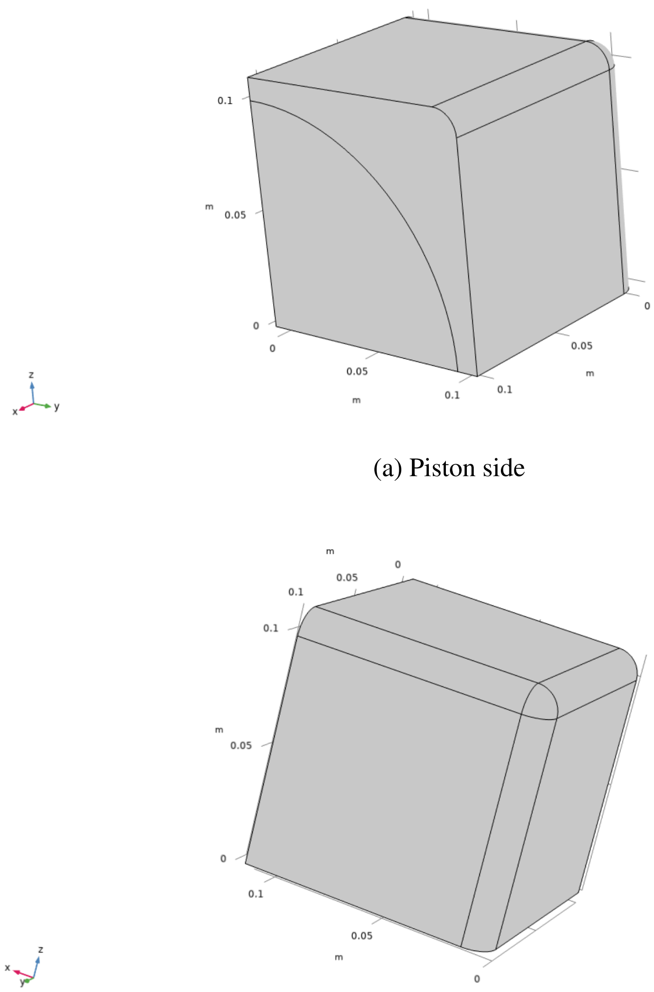
This development is valid for any matrix problem, it has been detailed with the matrix formulation (6), but it is the same with (11). The core of projection-based MOR, is the method used to build V. In this work, the WCAWE is considered, and the schema to do so is the following

$$\left\{\begin{aligned}
&Z^{(0)}\bar{v}_1 = F^{(0)}\\
&\qquad\text{Normalization}\quad \bar{v}_1 \to v_1\\
&Z^{(0)}\bar{v}_2 = F^{(1)}e_1^T P_{Q_1}(2,1)e_1 - Z^{(1)}v_1\\
&\qquad\text{Orthonormalization}\quad \bar{v}_2 \to v_2\\
&\qquad\vdots\\
&Z^{(0)}\bar{v}_k = \sum_{j=1}^{k-1}\left(F^{(j)}e_1^T P_{Q_2}(k,j)e_{k-j}\right) - Z^{(1)}v_{k-1}\\
&\qquad - \sum_{j=2}^{k-1}\left(Z^{(j)}v_{k-j}P_{Q_2}(k,j)e_{k-j}\right)\\
&\qquad\text{Orthonormalization}\quad \bar{v}_k \to v_k\\
&\qquad\vdots\\
&Z^{(0)}\bar{v}_N = \sum_{j=1}^{N-1}\left(F^{(j)}e_1^T P_{Q_2}(N,j)e_{N-j}\right) - Z^{(1)}v_{N-1}\\
&\qquad - \sum_{j=2}^{N-1}\left(Z^{(j)}v_{N-j}P_{Q_2}(N,j)e_{N-j}\right)\\
&\qquad\text{Orthonormalization}\quad \bar{v}_N \to v_N
\end{aligned}\right. \tag{15}$$

For brevity, the terms P_{Q_ω} will not be detailed here, but is well explained in [13]. The point of this method, is to use the derivatives w.r.t frequency of the global matrix Z, that is the assembled matrix of Eq. (6) or Eq. (11). Here comes the contribution of having the new implementation of ABC. It is clear that computing the frequency derivatives of the system matrix of the old implementation (6) becomes quickly impossible to compute, even with symbolic programming language, because of the quotient of frequency polynomials functions. On the other hand, the canonical expansion w.r.t frequency of the global matrix, helps at constructing V with any number of vectors that is needed.

3. RESULTS

The studied system is quite simple and can be expanded to more complex situations. In order to test the presence of unwanted reflected waves, the system is the classical baffle system. It consists in a piston clamped in a rigid wall, vibrating in the air. The air domain is truncated by an artificial surface where the ABC is applied. In this case, there is a metric that does not depend on the truncated domain geometry : the radiation factor. The radiation factor will be considered as the ground truth. The Fig. 2 represents the geometry of the considered case. The flat surface defined by the normal $-y$, and $-z$ are symmetries, and the truncated surface is a cubic one where the edges have been curved in order to preserve continuity of the normal around the edges.



(a) Piston side

(b) Artificial boundaries side

Figure 2: Geometry of the cubic rounded truncated domain.

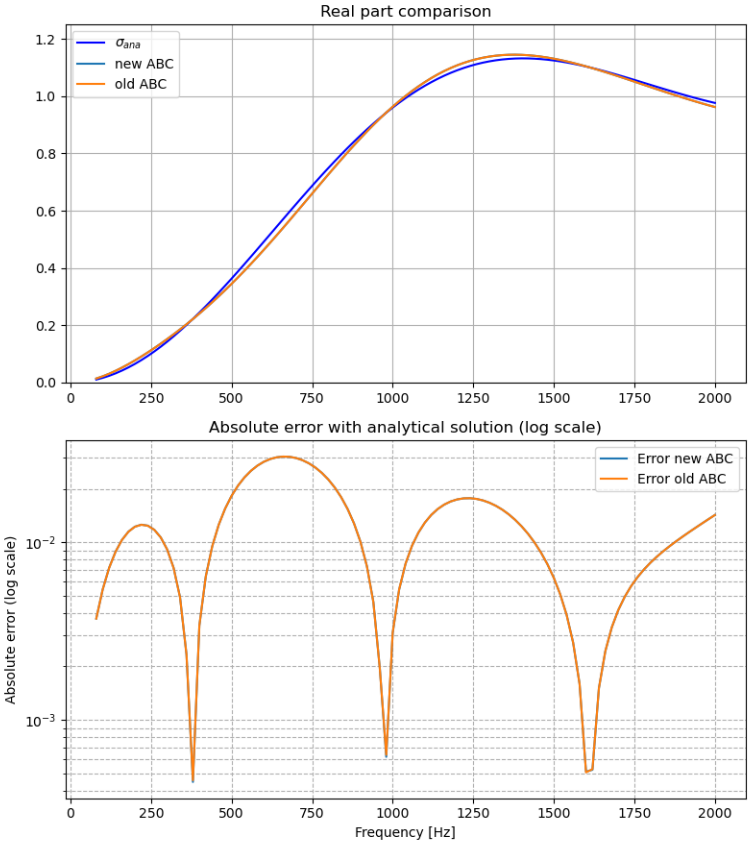


Figure 3: Comparison of old and new implementation of ABC

For the record, the operator in Eq. (4), has been developed for spherical truncated domain around a scattering point at the very center of the sphere. The operator was developed in a way that it is perfectly absorbing for a perfect sphere, in the present case, it is expected to have deviation from the ground truth. The authors are working on warping up the impact on different truncated geometries and the possibility to improve the operator to have a more general framework to model general structures.

The Fig. 3 shows the radiation factor of the considered case. As the curves are perfectly overlapping the both implementation provides the exact same results. It is case dependent, but in sake of comparison, in this case, the acoustic domain is composed by $n_\Omega = 20924$ dofs that is the size of the old implementation matrix, whereas the artificial surface add $n_{\Gamma_3} = 3070$ dofs.

It is clear that the both implementations lead to the exact same results. The expansion of the matrices involving a supposed longer computational time is actually compensated by the no-need of computing the $C(k)$ matrix for each frequency iteration. However this expansion, gave the idea of using MOR on the system. To present the results of the MOR, the ground truth is now the FOM. Fortunately the both FOM are exactly the same. Therefore, ROM of both implementations can be compared to the same ground truth curve.

It is expected that the MOR frequency sweep provides better and better results as the number of vector in V increases. Actually, the new implementation is very alike a coupled problem, and as it has been highlighted in [15], the basis V has to be split, such as

$$\widetilde{\mathbf{V}} = \begin{bmatrix} \mathbf{V}_p & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_q \end{bmatrix}. \tag{16}$$

It surely doubles the size of the reduced order model, but in comparison to the FOM it remains incredibly smaller. And moreover, there is still the advantage in the new implementation of the non-assembling process for each frequency iteration thanks to the frequency independent matrices. From now, when the results are presented, when V is said to be composed by N vector, it is for the old implementation but the size of the projection basis for the new implementation is $\widetilde{N} = 2N$. In Fig. 4 are presented the results of the ROM for both implementations with a number of vector $N = 5$. In some extent, the results are similar. The log-error plot shows that the interval of convergence is similar, and the first plot shows that there is still some improvement that can be achieved by increasing the number of vectors in the basis. However, the comparison has been done with $N = 5$ vectors because for the old implementation, it is the maximum number of derivatives that can be efficiently computed to construct V with the schema (15). Above the fifth derivative, the computational time to compute the derivative rapidly increases and looses the advantage of using MOR technique. Here comes the need to use the new implementation. Thanks to it, it is possible to populate the basis V with much more vectors and having a ROM that perfectly fits the FOM. An example with $N = 20$ is presented Fig. 5. Doing a computational time comparison can be tricky, but just for an overview, the FOM needs around 2 minutes to be performed whereas the ROM needs less than 3 seconds.

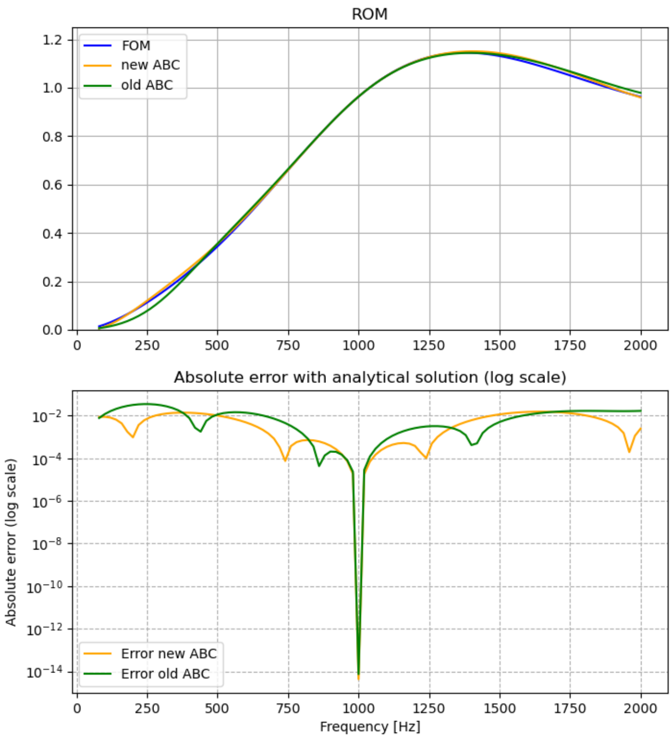


Figure 4: Comparison of old and new implementation of ABC

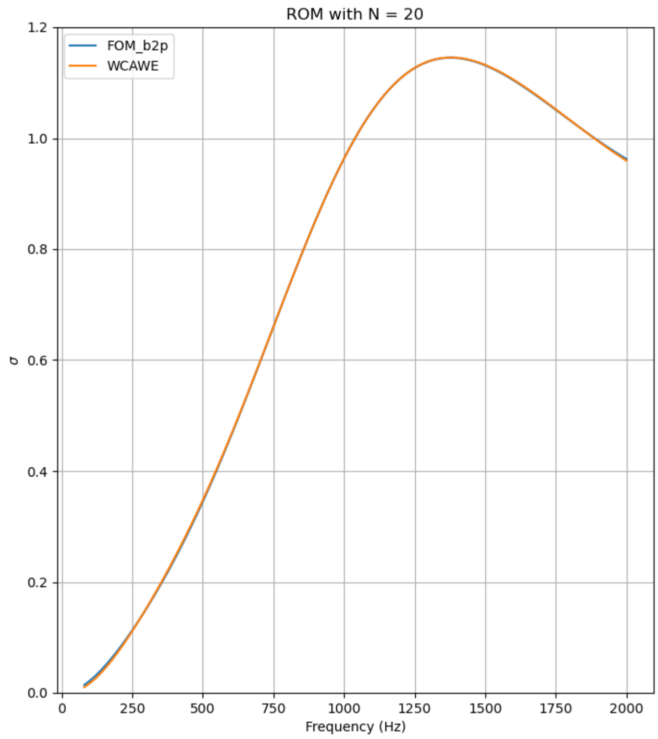


Figure 5: WCAWE technique with high number of vector to have a fitting ROM

4. CONCLUSION AND DISCUSSION

A new implementation of frequency independent Absorbing Boundary Condition (ABC) is presented. The motivation was to have frequency independent matrices in the problem and the application was a moment-matching method that requires frequency derivatives of the problem. By coupling the ABC equation to the Helmholtz equation and allowing a small expansion of the problem, it is possible to obtain a frequency independent problem classically expressed with the frequency affine expansion. It also allows the possibility to efficiently compute a projection based reduced model, with a high number of vectors, to perform fast frequency sweep.

The new implementation can be expand to higher order ABC, so far the authors are facing usual problems, especially numerical instabilities at low frequencies due to higher normal derivatives in the formulation. The main limitation of this work is the coupled system reduce order model, that implies to split the projection basis, hence, doubling the size of the ROM. A possible framework on that would be to use the basis computed with the new implementation, with larger number of vectors, keeping the vectors concerning the pressure degrees of freedom, and use it to reduce the problem formulated with the old implementation.

5. ACKNOWLEDGMENTS

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