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CONDITIONS FOR WEAK SCATTERING AND THE BORN APPROXIMATION IN BEAMS WITH MULTIPLE RESONATORS

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ABSTRACT

We report the necessary conditions for weak scattering of longitudinal and flexural waves in a beam loaded by multiple resonators. The equations of motion of a one-dimensional elastic waveguide with several point resonators are derived by a Green's matrix approach, for any resonator with any number of natural frequencies and morphology modeled by the transfer matrix method. The multiple scattering theory is used to express the response os the system by an infinite series whose convergence is closely linked to the scattering intensity provided by the resonators. The spectral radius of the scattering matrix is used to study of the convergence. Furthermore, we show that the Born approximation is provided by the leading order of the multiple scattering expansion. The work offers approximate expressions for the spectral radius, providing a preliminary step to a clear physical interpretation of weak scattering.

Keywords: *Multiple Scattering, Locally resonant materials, Weak scattering, Born approximation*

Metamaterials are structured composites with engineered wave propagation properties. In elasticity and acoutstics, they offer advanced control over flexural waves, with applications in vibration isolation, energy harvesting, and cloaking [1–5]. The wave response in such systems is influenced by multiple scattering interactions between discrete resonators. When these interactions are weak, simplified models, such as the Born approximation, provide accurate solutions. However, a rigorous criterion for weak scattering in flexural wave systems is necessary.

We consider longitudinal and flexural waves propagating through a straight beam-rod waveguide of infinite length, and also assume that the reference axis x coincides with the centroid of the cross-section, so no coupling between both types of waves occurs if the medium is homogeneous [6]. In this paper, an arrangement of N scatterers with internal resonances distributed along the beam at locations $x_1 < x_2 < \cdots < x_N$, will also be considered. We do not make any restriction except that the motion of both beam and resonators is restricted to the xz–plane. This approximation, known as plane-strain, usually provides good representations of real-life problems.

Assuming time harmonic motion, the state vector can be written as $\mathbf{U}(x,t) = \mathbf{u}(x)\, e^{i\omega t}$ and hence the time dependence is understood and omitted hereafter, with

$$\mathbf{u}(x) = \{u(x),\ w(x),\ \theta(x),\ N(x),\ V(x),\ M(x)\}^T. \tag{1}$$

being $u(x)$ and $w(x)$ the displacements in x and z direction of the reference axis and $\theta(x)$ the cross section rotation; the functions $N_x(x), V_z(x), M_y(x)$ represent the generalized forces associated to the previous kinematics variables, i.e. the normal and shear forces, and bending moment at position x. The system of equations (**??**) is frequency–dependent and can be written in matrix form as

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{A}\,\mathbf{u} + \mathbf{q}(x), \tag{2}$$

where

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$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 0 & 1/EA & 0 & 0 \\ 0 & 0 & 1 & 0 & 1/GA_z & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/EI_y \\ -\rho A\omega^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\rho A\omega^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho I_y\omega^2 & 0 & -1 & 0 \end{bmatrix}, \qquad \mathbf{q}(x) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -q_x \\ -q_z \\ -m_y \end{Bmatrix},$$

represents the constitutive matrix and the external forces, respectively.

The closed-form expression found for the matrix $\mathbf{K}_\alpha$, representing the response of the α-scatterer and that can be obtained by the Transfer Matrix Method [7], enables us to write Eq. (2) as follows

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}x} = \mathbf{A}\,\mathbf{u} + \sum_{\alpha=1}^{N} \mathbf{K}_\alpha \mathbf{u}(x_\alpha)\,\delta(x - x_\alpha) + \mathbf{q}_e(x)\,. \qquad (3)$$

The total solution is written as

$$\mathbf{u}(x) = \boldsymbol{\psi}_0(x) + \mathbf{u}_s(x), \qquad (4)$$

where $\boldsymbol{\psi}_0(x)$ denotes the state vector corresponding to the incident wave and is the solution of

$$\frac{\mathrm{d}\boldsymbol{\psi}_0}{\mathrm{d}x} = \mathbf{A}\,\boldsymbol{\psi}_0 + \mathbf{q}_e(x). \qquad (5)$$

Moreover, $\mathbf{u}_s(x)$ represents the state vector of the scattered wave, which is solution of

$$\frac{\mathrm{d}\mathbf{u}_s}{\mathrm{d}x} = \mathbf{A}\,\mathbf{u}_s + \sum_{\alpha=1}^{N} \mathbf{K}_\alpha \mathbf{u}(x_\alpha)\,\delta(x - x_\alpha). \qquad (6)$$

The solution will be presented in terms of the Green's matrix of the system, $\mathbf{G}(x)$, defined as the solution of

$$\left(\frac{\mathrm{d}}{\mathrm{d}x} - \mathbf{A}\right)\mathbf{G} = \mathbf{I}\,\delta(x). \qquad (7)$$

The general solution of Eq. (3) can then be written as

$$\mathbf{u}(x) = \boldsymbol{\psi}_0(x) + \mathbf{u}_s(x) = \boldsymbol{\psi}_0(x) + \sum_{\beta=1}^{N} \mathbf{G}(x - x_\beta)\,\mathbf{K}_\beta\,\mathbf{u}(x_\beta). \qquad (8)$$

After evaluating Eq. (8) at $x_1, \ldots, x_N$, the following system of linear equations arises

$$\mathbf{u}(x_\alpha) - \sum_{\beta=1}^{N} \mathbf{G}(x_\alpha - x_\beta)\,\mathbf{K}_\beta\,\mathbf{u}(x_\beta) = \boldsymbol{\psi}_0(x_\alpha) \quad , \quad 1 \le \alpha \le N. \qquad (9)$$

Let us now introduce the following auxiliary variables associated with each resonator.

$$\boldsymbol{\phi}(x_\alpha) = \left[\mathbf{I} - \mathbf{G}(0^+)\mathbf{K}_\alpha\right]\,\mathbf{u}(x_\alpha). \qquad (10)$$

After some rearrangements, the system of equations (9) is written in terms of the new variables defined above as

$$\begin{bmatrix} \mathbf{I} & -\mathbf{G}(x_1 - x_2)\mathbf{t}_2 & \cdots & -\mathbf{G}(x_1 - x_N)\mathbf{t}_N \\ -\mathbf{G}(x_2 - x_1)\mathbf{t}_1 & \mathbf{I} & \cdots & -\mathbf{G}(x_2 - x_N)\mathbf{t}_N \\ \vdots & \vdots & \ddots & \vdots \\ -\mathbf{G}(x_N - x_1)\mathbf{t}_1 & -\mathbf{G}(x_N - x_2)\mathbf{t}_2 & \cdots & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\phi}(x_1) \\ \boldsymbol{\phi}(x_2) \\ \vdots \\ \boldsymbol{\phi}(x_N) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{\psi}_0(x_1) \\ \boldsymbol{\psi}_0(x_2) \\ \vdots \\ \boldsymbol{\psi}_0(x_N) \end{Bmatrix}, \qquad (11)$$

where the new matrices $\mathbf{t}_\alpha$ are defined as

$$\mathbf{t}_\alpha = \mathbf{K}_\alpha\,\left[\mathbf{I} - \mathbf{G}(0^+)\mathbf{K}_\alpha\right]^{-1}. \qquad (12)$$

This 6×6 matrix represents the dynamical impedance of each scatterer with respect to homogeneous parameters of the host beam. It contains the mechanical and inertia properties throughout the components of $\mathbf{K}_\alpha$ matrix, so fully described the resonant behavior of the scatterers. Eq. (11)

can be written in compact form as

$$[\mathbf{I}-\mathbf{g}]\ \mathbf{\Phi}=\mathbf{\Psi}_0, \tag{13}$$

where the matrix $\mathbf{g}=\hat{\mathbf{G}}\,\hat{\mathbf{T}}$ and

$$\hat{\mathbf{G}}=\begin{bmatrix} \mathbf{0} & \mathbf{G}(x_1-x_2) & \cdots & \mathbf{G}(x_1-x_N) \\ \mathbf{G}(x_2-x_1) & \mathbf{0} & \cdots & \mathbf{G}(x_2-x_N) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{G}(x_N-x_1) & \mathbf{G}(x_N-x_2) & \cdots & \mathbf{0} \end{bmatrix}, \quad \hat{\mathbf{T}}=\begin{bmatrix} \mathbf{t}_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{t}_N \end{bmatrix}. \tag{14}$$

The scattering coefficients $\mathbf{\Phi}$ are simply deduced from Eq. (13) as

$$\mathbf{\Phi}=[\mathbf{I}-\mathbf{g}]^{-1}\,\mathbf{\Psi}_0. \tag{15}$$

The state vector is obtained from

$$\mathbf{u}(x)=\boldsymbol{\psi}_0(x)+\sum_{\alpha=1}^{N}\mathbf{G}(x-x_\alpha)\,\mathbf{t}_\alpha\,\boldsymbol{\phi}(x_\alpha). \tag{16}$$

The solution of Eq. (15) and the state vector of Eq. (16) provide the exact approach to the problem for any frequency.

As shown above, the scattering coefficients are found from the solution of Eq. (15). If the elements of the *square* matrix $\mathbf{g}$ are sufficiently small, the inverse is efficiently evaluated by means of the Neumann series expansion, namely

$$[\mathbf{I}-\mathbf{g}]^{-1}=\mathbf{I}+\mathbf{g}+\mathbf{g}^2+\cdots=\sum_{n=0}^{\infty}\mathbf{g}^n. \tag{17}$$

The necessary and sufficient condition to guarantee that the series converges is that the spectral radius $\rho(\mathbf{g})$, defined as the maximum of the absolute values of the eigenvalues of $\mathbf{g}$, is less than unity. When $\rho(\mathbf{g})<1$, the scattering induced by the N resonators can be considered as weak, meaning that a few terms of the series are necessary to accurately reproduce the scattering behavior of the system. In other situations for which $\rho(\mathbf{g})>1$ holds, the inverse matrix cannot be approximated by the expansion of Eq. (17).

By employing a Green's matrix approach, this work reports analytical derivations and approximations to effectively model the interaction between waves and resonators en beams [7]. Equations show that weak scattering conditions and the Born approximation depends on the spectral radius is sensitive to both local resonator properties and their spatial configuration, which influence scattering intensity. Future work could explore more complex arrangements and resonator types, potentially extending these findings to applications in structural health monitoring and wave-based material design.

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