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# NUMERICAL SIMULATIONS OF ELASTIC WAVE PROPAGATION IN SOLID MATERIALS CONTAINING A RANDOM DISTRIBUTION OF SPHERICAL SCATTERERS

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## ABSTRACT

A numerical approach of multiple scattering based on the resolution of multiple scattering equations for elastic waves inside solid heteregenous media is presented. It is validated by extracting the effective velocity and attenuation of longitudinal and transverse coherent inside a dispersion of spherical scatterers embedded in a duralumin matrix.

**Keywords:** *multiple scattering, elastic waves, coherent wave, effective parameters, numerical approach.*

## 1. INTRODUCTION

The propagation of elastic waves in multiply scattering solid media is an old problem that has been less studied than the case of acoustic waves (in fluids) or electromagnetic waves. One of the reasons lies in the existence of conversions between longitudinal and transverse waves at the surface of each scatterer. This work presents a new method for simulating elastic wave propagation in heterogeneous media composed of spherical inclusions embedded in a solid matrix. The MuScat code, initially developed to simulate acoustic wave propagation in heterogeneous fluids [1], is adapted to take into account the vector nature of elastic waves in three dimensions. It is based on the analytical solution of the multiple scattering equations using the spherical harmonic expansion of the incident and scattered fields, which allows to consider very large dispersions of particles. In a first section, we describe this general principle behind the code. In a second section, we show the validation of the numerical approach on dispersions of spherical cavities filled with water embedded in a duralumin solid matrix. Note that a similar study has been performed in two dimensions (cylindrical cavities filled with water) in ref. [2].

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## 2. GENERAL PRINCIPLE OF THE MODELING

The modeling behind the MuScat code for elastic waves will be presented in details in a future work. In the following, we indicate the main principles behind it.

We consider an isotropic and homogeneous visco-elastic medium of Lamé parameters $(\lambda, \mu)$ and density $\rho$ (we denote $k_L$ the longitudinal wavenumber and $k_T$ the transverse one) inside which are embedded $N_p$ spheres of respective radii $a_p$, Lamé parameters $(\lambda_p, \mu_p)$ and densities $\rho_p$ (fig. 2). There is no limitation on the polydisperisty either in size or elasticity.

We follow the T-matrix approach already used in electromagnetism [3, 4], adding the mode conversations between longitudinal and transverse waves that exist in the elastic case (details about this can be found in ref. [5]). Within this framework, the incident displacement field upon the distribution of scatterers is expanded upon the usual Vector Spherical Wavefunctions (VSWF) $\mathbf{L}_{nm}$, $\mathbf{M}_{nm}$ and $\mathbf{N}_{nm}$, that are solutions of the elastic wave equations in an homogeneous medium (see ref. [6] for their definition and properties), as

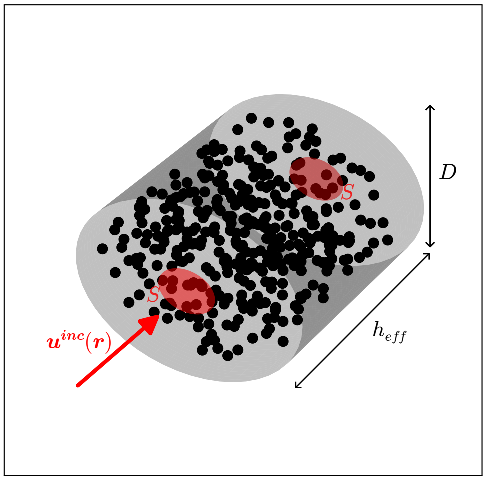


**Figure 1**. An incident displacement field (plane wave) is sent on a distribution of spherical scatterers placed in a cylindrical slab inside the solid matrix.

$$\mathbf{u}^{inc}(\mathbf{r}_0) = \sum_{n,m} (a^L_{nm}\mathbf{L}^{(1)}_{nm}(\mathbf{r}_0) + a^T_{nm}\mathbf{N}^{(1)}_{nm}(\mathbf{r}_0) + a^S_{nm}\mathbf{M}^{(1)}_{nm}(\mathbf{r}_0)). \tag{1}$$

This field is expressed in an arbitrary coordinates system $R_0$ ($\mathbf{r}_0$ denotes the position in this system), generally placed at the center of the distribution of scatterers. For convenience we have used the notation $\sum\limits_{n,m} \equiv \sum\limits_{n=0}^{+\infty}\sum\limits_{m=-n}^{+n}$. The superscript $(1)$ denotes the use of the spherical Bessel function of the first kind $j_n$ in the definition of the VSWF. The coefficients $a^M_{nm}$ are the amplitudes of the incident field and are known. The $L$ part refers to longitudinal contributions to the displacement field wheareas the $T$ and $S$ parts represent the transverse ones. The total field $\mathbf{u}$, which we seek to express, is

$$\mathbf{u} = \mathbf{u}^{inc}(\mathbf{r}_0) + \sum_{p=1}^{N_p} \mathbf{u}^{sc}_p(\mathbf{r}_p), \tag{2}$$

where $\mathbf{u}^{sc}_p(\mathbf{r}_p)$ is the scattered field of the $p$-th particule expressed in its local coordinates system $R_p$. Those fields are also expanded on the VSWF and are thus written

$$\mathbf{u}^{sc}(\mathbf{r}_p) = \sum_{n,m} (b^{p,L}_{nm}\mathbf{L}^{(3)}_{nm}(\mathbf{r}_p) + b^{p,T}_{nm}\mathbf{N}^{(3)}_{nm}(\mathbf{r}_p) + b^{p,S}_{nm}\mathbf{M}^{(3)}_{nm}(\mathbf{r}_p)). \tag{3}$$

The superscript $(3)$ indicates here the use of the spherical Hankel function of the first kind $h^{(1)}_n$ in the definition of the VSWF. The coefficients $b^{p,M}_{nm}$ are the amplitudes of the mode $M = L, T, S$ of the scattered field of the $p$-th particle and are thus the unknowns of the problem. Using the addition theorems for VSWF [4,7] we can express the excited field on the $p$-th particle $\mathbf{u}^{ex}_p = \mathbf{u}^{inc} + \sum\limits_{q \neq p} \mathbf{u}^{sc}_q$ in its local coordinates system $R_p$. Knowing the T-matrix of the particle, we can eventually write a linear system that allows to find the coefficients $b^{p,M}_{nm}$ and then determine the total field. This system is solved numerically by introducing a modal truncature and using iterative methods.

## 3. EXTRACTION OF EFFECTIVE PARAMETERS OF A MULTIPLE SCATTERING MEDIUM

### 3.1 The coherent field

The validation of the numerical approach is here realized by comparison with a vector multiple scattering model (denoted as the LCVB model) giving the effective parameters of the coherent longitudinal and transverse waves [5]. Indeed, the displacement field for a realization of the disorder can be written $\mathbf{u} = \langle \mathbf{u} \rangle + \delta\mathbf{u}$, where $\langle \cdot \rangle$ denotes the configurational average. $\langle \mathbf{u} \rangle$ is thus the coherent field. It can be written

$$\langle \mathbf{u} \rangle = u_0 \mathrm{e}^{\mathrm{i}k_{eff}z}\hat{\mathbf{p}}, \tag{4}$$

where $\hat{\mathbf{p}}$ is the polarization of the incident wave and $k^{eff}_M$ is the so-called effective wave number ($M = L$ for a longitudinal wave, $M = T$ for a transverse one). We have $k^{eff}_M = \omega / c^{eff}_M + \mathrm{i}\, \alpha^{eff}_M$, where $c^{eff}_M$ is the effective velocity of the coherent wave and $\alpha^{eff}_M$ its effective attenuation. The propagation of the coherent wave is thus modeled as the propagation of an elastic wave of polarization $\hat{\mathbf{p}}$ inside an effective homogeneous medium. The two parameters $c^{eff}_M$ and $\alpha^{eff}_M$ characterize the multiple scattering medium.

### 3.2 Results and discussion

We consider the propagation of an elastic wave inside a dispersion of spherical cavities (all with the same radius $a = 250$ µm) filled with water embedded inside a duralumin matrix ($\rho = 2800$ kg/m$^3$,$\lambda = 57.8$ GPa, $\mu = 28$ GPa). The spheres are put in a virtual volume of transverse diameter $D$ and thickness $h_{eff}$ (see fig. 2) and illuminated by a plane wave [1]. We measure the transmissions coefficients for $N_{avg}$ dispersions of spheres. We denote $\mathcal{T}$ the configurationally averaged transmission coefficient [2]. The effective parameters are then extracted from the following expression

$$\mathcal{T} = \frac{e^{ik_{eff}h_{eff}}}{e^{ikh_{eff}}}, \tag{5}$$

with $k = k_L$ (resp. $k = k_T$) if the incident wave is longitudinal (resp. transverse).

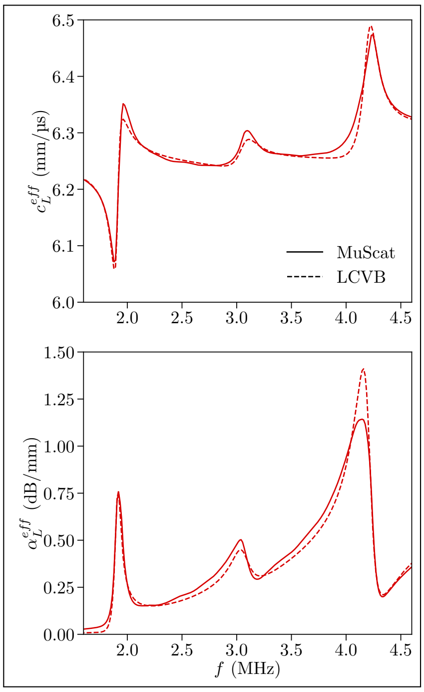


**Figure 2**. Frequency dependant effective velocity (top) and attenuation (bottom) of coherent longitudinal waves in a dispersion of cavities (filled with water) embedded inside a duralumin matrix ($\phi_v = 2\%$) around several resonances. Comparison between the LCVB model and numerical results obtained with MuScat.

[1] The number of modes to take into account for the truncature is determined from the scattering cross section of the scatterers. For an incident longitudinal (resp. transverse) wave, a maximum of 5 (resp. 6) modes has been taken.

[2] For each configuration of the disorder, we also perform spatial average (over a surface $S$ as seen on fig. 2), as it is often do practically in experiments, to achieve a better averaging process.

Fig. 2 shows the effective velocity and attenuation of the coherent longitudinal wave in the dispersion of spherical cavities for a volume fraction $\phi_v = 2\%$. The spheres are placed inside the heteregenous medium with just a condition of non-overlaping, i.e. the distance between the centers of two scatterers shall be superior to $2a$. The figure shows a good agreement between the LCVB model and the numerical results obtained with MuScat. Several resonances can be observed, they are characterized by a strong dispersion on the effective velocity of the coherent wave and thus also by an effective attenuation peak.

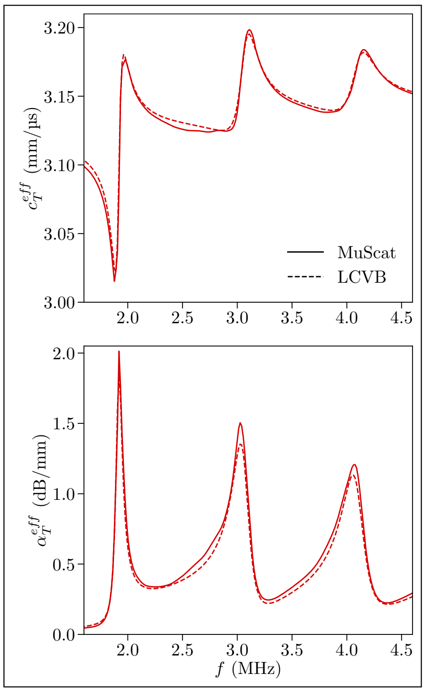


**Figure 3**. Frequency dependant effective velocity (top) and attenuation (bottom) of coherent shear waves in a dispersion of cavities (filled with water) embedded inside a duralumin matrix ($\phi_v = 2\%$) around several resonances. Comparison between the LCVB model and numerical results obtained with MuScat.

Fig. 3 shows the effective velocity and attenuation for the coherent shear waves in those dispersions. A good agreement can be seen between analytical and numerical results. Several resonances are also observed over the frequency band. In this kind of system, for both longitudinal and transverse waves, those resonances are linked to the existence of whispering gallery waves and Rayleigh waves modes at the interface of the cavities [2].

## 4. CONCLUSION

We have presented a numerical approach, MuScat, aiming to study the propagation of elastic waves inside heterogeneous media. It has been validated here by comparison with a vectorial model of elastic wave multiple scattering. This numerical approach opens new perspectives for the study of elastic waves propagation inside dispersions of scatterers both on the coherent part of the field and on its fluctuations. Several types of incident waves can be used (point source, plane wave, Gaussian beam) and the distribution of scatterers can be polydisperse both in size and elasticity.

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