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ON THE USAGE OF ACOUSTIC RADIATION MODES FOR KRYLOV SUBSPACE REDUCTION OF ACOUSTIC FINITE ELEMENT MODELS IN THE TIME DOMAIN

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ABSTRACT

Recent works have shown that Krylov subspace reduction of vibro-acoustic finite element models can lead to accurate reduced order models that are only a fraction of the size of the full system. However, one of the disadvantages of this reduction method is that the resulting reduced order model size is dependent on the number of uncorrelated inputs. This can be an issue for vibro-acoustic models in which the structural vibration is not explicitly modelled. For example, this could occur when the structural vibration is measured, such as with high-speed cameras. In this case we might not know the structural properties, but we can measure full-field vibration patterns and apply them as acoustic boundary conditions. Thus, effectively implementing the boundary condition as a high number of uncorrelated inputs. It is known that at low frequencies a few acoustic radiation modes are responsible for most of the far-field sound radiation. Therefore, in this work we are investigating the creation of reduced order models through a decomposition of the input into radiation modes. It is shown how the input can be approximated by a few radiation modes and how to use these modes to create a reduced order model.

Keywords: *Acoustic radiation modes, model order reduction, Finite Element Method* s

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1. INTRODUCTION

The usage of Krylov subspace Model Order Reduction (MOR) for vibro-acoustic system generally leads to small Reduced Order Models (ROMs) with a high accuracy [1]. Since the spanned subspace is implicitly matching moments of the frequency response, the accuracy is higher than with classical MOR using normal modes. Recent research has shown that this method also can be applied for time-domain problems with frequency dependent damping [2]. One issue of the moment matching procedure is that the size of the ROM is highly dependent on the number of inputs. For certain applications, such as predicting the sound field directly from many structural measurements, this leads to ROMs that are potentially only marginally smaller than the full order model (FOM).

In this paper we investigate a potential way to circumvent this issue, namely through the usage of acoustic radiation modes. Acoustic radiation modes are orthonormal vibration patterns of a structure that radiate sound power independently [3]. They are only dependent on the geometry of the radiator and the acoustical properties. Thus, in contrast to normal vibration modes, no knowledge is required of the structural boundary conditions, excitation location/amplitude and material properties. Additionally, it is known that at low frequencies only a few radiation modes are sufficient to calculate the majority of the active sound power. Therefore, in this paper we aim to investigate if we can get an accurate ROM by building the reduction basis using only a limited number of acoustic radiation modes. After explaining the theory in Section 2, we will show the performance of such a ROM in Section 3.

2. METHODS

Consider a free-field acoustic domain with a radiator placed in a sound hard baffle. The domain is meshed using finite elements and infinite elements, to approximate the Sommerfeld radiation condition, leading to n Degrees Of Freedom (DOF). The discretized radiator with d DOF is described by $\boldsymbol{p}(\omega) = \mathbf{Z}_{ac}(\omega)\boldsymbol{a}(\omega)$ in which $\boldsymbol{p} \in \mathbb{C}^{d\times d}$ is the pressure vector and $\boldsymbol{a} \in \mathbb{C}^{d\times d}$ is the acceleration, with angular frequency ω. For ease of notation we assume throughout the paper that each element of the discretized radiator has the same surface area. As shown in [4], the matrix $\mathbf{Z}_{ac}$ can be calculated as follows from the infinite/finite element model:

$$\begin{aligned}\mathbf{Z}_{ac} = &-\omega^2\rho_f(\boldsymbol{\Theta}_{\Gamma\Gamma} - \mathbf{A}_{\Gamma o}\mathbf{A}_{oo}^{-1}\boldsymbol{\Theta}_{o\Gamma})^{\mathsf{T}} \\ &(\mathbf{A}_{\Gamma\Gamma} - \mathbf{A}_{\Gamma o}\mathbf{A}_{\Gamma\Gamma}^{-1}\mathbf{A}_{o\Gamma})^{-\mathsf{T}}\boldsymbol{\Theta}_{\Gamma\Gamma}.\end{aligned} \tag{1}$$

The subscript Γ denotes the boundary DOF and subscript o denotes all the other DOF. The matrix $\mathbf{A}$ is the dynamic stiffness matrix $\mathbf{A}(\omega) = \mathbf{K}+i\omega\mathbf{C}-\omega^2\mathbf{M}$, with the mass matrix $\mathbf{M} \in \mathbb{R}^{n\times n}$, the damping matrix $\mathbf{C} \in \mathbb{R}^{n\times n}$ and the stiffness matrix $\mathbf{K} \in \mathbb{R}^{n\times n}$. $\boldsymbol{\Theta} \in \mathbb{R}^{n\times n}$ is the boundary mass matrix defined on the radiator:

$$\boldsymbol{\Theta} = \int_{\Gamma} \boldsymbol{N}^{\mathsf{T}}\boldsymbol{N}d\Gamma, \tag{2}$$

were $\boldsymbol{N}$ are shape function vectors following from the FE discretisation. The proposed method to perform input reduction starts by recognizing that we can write the acceleration as:

$$\begin{aligned}\boldsymbol{a}(\omega) &= \mathbf{Z}_{ac}^{-1}(\omega)\mathbf{Z}_{ac}(\omega)\boldsymbol{a}(\omega) \\ &= \mathbf{V}\boldsymbol{\Sigma}^{-1}\mathbf{U}^{\mathsf{H}}\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}}\boldsymbol{a}(\omega),\end{aligned} \tag{3}$$

where we used the Singular Value Decomposition (SVD) of $\mathbf{Z}_{ac} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^{\mathsf{H}}$. Note that, by taking the real part of $\mathbf{Z}_{ac}$, the columns in $\mathbf{U}$, $\mathbf{V}$ are known as the far-field radiation modes [3]. In the current paper we are interested in both near-field and far-field radiation, so instead we are taking the SVD on the complex-valued matrix. As mentioned in [3], it is common that the active (far-field) sound power at low frequencies can be described by only a few radiation modes, where the singular values of the radiation modes are related to the radiation efficiency of that particular mode. Therefore, the main idea of the current paper is to take a low rank approximation of $\mathbf{Z}_{ac}$, thus

$$\begin{aligned}\boldsymbol{a}(\omega) \approx \boldsymbol{a}_r(\omega) &= \mathbf{V}_r\boldsymbol{\Sigma}_r^{-1}\mathbf{U}_r^{\mathsf{H}}\mathbf{U}_r\boldsymbol{\Sigma}_r\mathbf{V}_r^{\mathsf{H}}\boldsymbol{a}(\omega) \\ &= \mathbf{V}_r\mathbf{V}_r^{\mathsf{H}}\boldsymbol{a}(\omega).\end{aligned} \tag{4}$$

This expression shows how the input vector $\boldsymbol{a}(\omega)$ can be approximated using a few radiation modes with the highest singular values. While the current expression is written in the frequency domain, instead we are interested in time domain analysis. Since the matrix $\mathbf{Z}_{ac}$ is frequency dependent, conversion to the time domain of Eq. (4) would lead to a convolution integral, which is undesired. Instead, we only calculate $\mathbf{Z}_{ac}$ at the highest frequency of interest. The rationale is that the radiation modes follow the nesting property, which means that the radiation modes at lower frequencies form a subset of higher frequency radiation modes [5]. Under this assumption, the following time domain expression results

$$\boldsymbol{a}(t) \approx \boldsymbol{a}_r(t) = \mathbf{V}_r\mathbf{V}_r^{\mathsf{H}}\boldsymbol{a}(t). \tag{5}$$

Since $\boldsymbol{a}(t)$ is typically a real-valued signal, a conversion from the complex-valued orthonormal basis $\mathbf{V}_r$ to a real-valued basis is performed

$$\mathbf{V}_{r,real} = \mathrm{orth}(\begin{bmatrix} \mathrm{Re}(\mathbf{V}_r) & \mathrm{Im}(\mathbf{V}_r) \end{bmatrix}). \tag{6}$$

Furthermore, by defining $\boldsymbol{\nu}(t) = \mathbf{V}_{r,real}^{\mathsf{T}}\boldsymbol{a}(t)$ the equations of motion with the derived low-rank input approximation becomes

$$\mathbf{M}\ddot{\boldsymbol{p}}(t) + \mathbf{C}\dot{\boldsymbol{p}}(t) + \mathbf{K}\boldsymbol{p}(t) = \mathbf{V}_f\boldsymbol{\nu}_f(t), \tag{7}$$

in which $\mathbf{V}_{r,real}^{\mathsf{T}}$ and $\boldsymbol{\nu}(t)$ are extended to the full dimensions of the FE model by augmenting them with zeros. Model order reduction is then performed on Eq. (7) through Krylov subspace reduction, using the methods shown in [1], leading to a reduced basis $\mathbb{V} \in \mathbb{R}^{n\times k}$ that can be used to calculate the reduced order matrices, as follows

$$\mathbf{M}_k = \mathbb{V}^{\mathsf{T}}\mathbf{M}\mathbb{V}, \mathbf{C}_k = \mathbb{V}^{\mathsf{T}}\mathbf{C}\mathbb{V}, \tag{8}$$

$$\mathbf{K}_k = \mathbb{V}^{\mathsf{T}}\mathbf{K}\mathbb{V}, \mathbf{F}_k = \mathbb{V}^{\mathsf{T}}\mathbf{V_f}, \tag{9}$$

$$\boldsymbol{p}(t) \approx \hat{\boldsymbol{p}} = \mathbb{V}\boldsymbol{p}_k(t), \tag{10}$$

where $\boldsymbol{p}_k \in \mathbb{R}^{k\times 1}$ is the reduced order pressure vector.

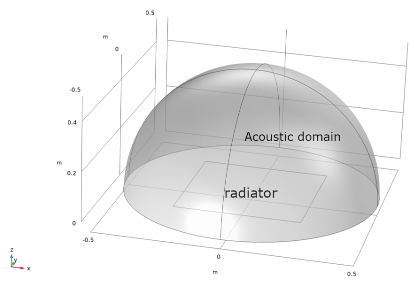

Figure 1. Geometry of the finite element domain.

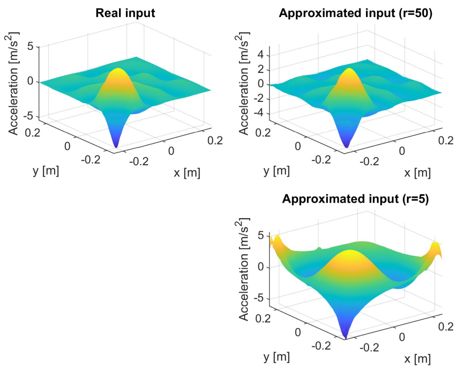

Figure 2. Real input compared to low-rank approximations of the input using radiation modes at t=0.01s.

3. NUMERICAL VALIDATION OF THE REDUCED ORDER MODEL

Consider the acceleration response in normal direction resulting from an impulsive force excitation of a clamped plate of 0.5 by 0.5 m, given as $\boldsymbol{a}(t)$. By looking at the acceleration approximation $\boldsymbol{a}_r(t)$ at a certain time step and comparing it the exact acceleration $\boldsymbol{a}(t)$ we can assess the approximation quality using the mode shapes from Eq. (6). We used a finite/infinite element model to calculate the acoustic radiation modes, see Fig. 1, using linear shape functions in the acoustic domain and a sound hard boundary condition at $z = 0$ m. The infinite elements were placed on the outer surface of the hemisphere and use (Jacobi) radial shape functions of order 6.

Given that we recorded the acceleration response at $d = 441$ nodes, it can be seen from Fig. 2 that choosing $r = 50$ radiation modes at $t = 0.01s$ gives only small discrepancies in the response. If only a very low rank input approximation is chosen ($r = 5$) it is clear that the acceleration response is approximated poorly. By looking at these results, it would be tempting to conclude that many modes are necessary to solve the acoustic problem accurately.

However, since our main goal is to use the reduced input basis to calculate the acoustic pressure, it is more informative to compare the acoustic pressure obtained with a low number of radiation modes. By looking at a cross-section of the acoustic domain through the $x = 0$ m symmetry axis, a very different result can be observed. In Figure 3, the variance accounted for (VAF) value of the time domain response of the full order input vs the reduced input is shown across the domain, were a value of 1 means a perfect match and lower values mean a discrepancy.

The required responses were calculated using a Newmark-beta time integration scheme and were obtained

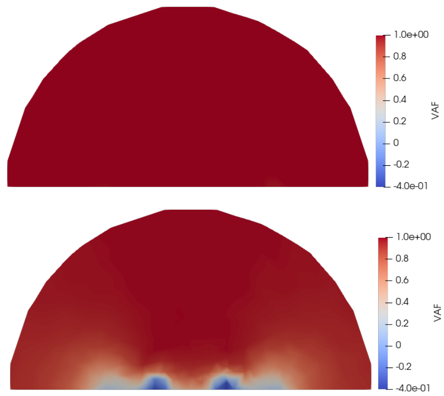

Figure 3. VAF using input reduction with 50 radiation modes (top) and 5 radiation modes (bottom).

with the full order model to just see the effect of the input reduction. The VAF at microphone location i is defined as

$$\mathrm{VAF}_i = \left(1 - \frac{\mathrm{var}(p_i - \hat{p}_i)}{\mathrm{var}(p_i)}\right). \quad (11)$$

As expected, $r = 50$ gives a good approximation of the acoustic pressure. But perhaps surprisingly also $r = 5$ gives a good approximation of the acoustic pressure, with the exception of some nodes near the radiator. Additional comparisons were done by looking at the time domain response at specific field points, namely at $(0.0637\mathrm{m}, 0.0448\mathrm{m}, 0.2497\mathrm{m})$ and $(-0.0750\mathrm{m}, -0.225\mathrm{m}, 0.0\mathrm{m})$ m, using $r = 5$ modes, see

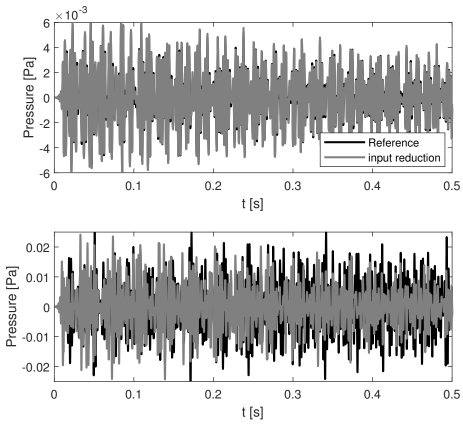

Figure 4. Comparison between pressure in the time domain from reference model (black) and using input reduction (grey) at $(0.0637\mathrm{m}, 0.0448\mathrm{m}, 0.2497\mathrm{m})$ (top) and $(-0.0750\mathrm{m}, -0.225\mathrm{m}, 0.0\mathrm{m})$ (bottom).

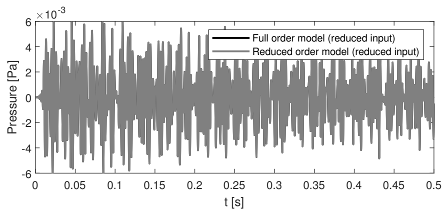

Figure 5. Comparison between pressure in the time domain from full order model with reduced inputs (black) and the reduced order model at $(0.0637\mathrm{m}, 0.0448\mathrm{m}, 0.2497\mathrm{m})$.

Fig. 4. It can be seen that acoustic pressure is approximated well for points that are a bit further from the radiator, while the pressure is poorly approximated close to the radiator. As a final comparison the ROM is calculated using $r = 5$. By using Krylov subspace reduction with an error tolerance of 0.1% between 0-500 Hz, a ROM of 407 DOFs is obtained, while the full order model is 6915 DOFs. Even larger reductions are expected for FE models with more DOFs. As can be seen in Fig. 5, both time domain responses are virtually indistinguishable and stability is preserved in the MOR process.

4. CONCLUSIONS

A method for input order reduction is presented based on acoustic radiation modes. It works by using the SVD to get a low-rank approximation of the derivative of the impedance matrix, which are complex-valued versions of the acoustic radiation modes. They are then used to derive an approximation of the acceleration of the radiator. To make the modes suitable for time domain simulation, we make use of the nesting property and convert the modes from complex-valued to real-valued ones.

While a low-rank approximation of the input does not necessarily lead to a good approximation of the acceleration, it does give an acceptable approximation of the acoustic pressure in the far-field, which shows that acoustic radiation modes can be used to construct a low-rank input for accurate acoustic analysis. The low-rank input approximation allows for efficient Krylov-subspace reduction, which potentially makes these models suitable for rapid acoustic radiation prediction of full-field structural measurements.

5. REFERENCES

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