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REAL-TIME DEMONSTRATORS TO EXHIBIT, INVESTIGATE AND TEACH TRANSIENT BEHAVIORS IN NONLINEAR DYNAMICS, FOCUSED ON MUSICAL INSTRUMENTS

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ABSTRACT

Nonlinear dynamical systems exhibit various transient behaviors, whose understanding is required to fully apprehend how the system responds to sollicitations. This work focuses on multistable systems, where several stable solutions coexist for a given value of the system's parameters. In these settings, transients condition which solution actually appears during time integration. Then, a comprehensive description of the system's behavior entails mapping out the infinite possibilities of variations of the system's parameters with respect to time. This is impractical for models whose phase space is of high dimension, or in limited timescales such as practical activities for students. This work proposes real-time demonstrators of dynamical systems as a way to still investigate these phenomena. The construction of such demonstrators is detailed, including a display outlining the zones of stability of the solutions, and control inputs mimicking a musical instrument (such as a MIDI controller). We show how these elements coupled with audio and visual feedback enables users to quickly gauge a large variety of transient behaviors, for academic proofs of concept or pedagogical activities. A demonstrator for a Van der Pol toy model and one for a woodwind model used for a lab in a master's degree are used as examples.

Keywords: *musical acoustics, nonlinear dynamics, real-time synthesis, bifurcations*

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1 INTRODUCTION

It is very rarely possible to exhaustively predict the behavior of a nonlinear dynamical systems. Therefore, a lot of work in the dynamical systems community is descriptive: one observes, documents and explains the various behaviors that the system displays in its different configurations [1]. This leads to graphical representations such as bifurcation diagrams that represent behavior as a function of one parameter, or cartographies when two or three parameters are taken into account. This work presents two real-time construction of bifurcation diagrams. In a pedagogical context, the real-time audio and graphical representation tool allows impactful and flexible demonstrations, in which the students and the professor can decide to investigate a specific point and do so immediately. The audio feedback is especially interesting for music-related curriculums, where students want to interpreted dynamical system behaviors in terms relevant to music as often as possible.

2 A TOY OSCILLATOR ILLUSTRATES DIRECT, INVERSE AND DYNAMICAL HOPF BIFURCATIONS

A fifth order Van der Pol model [2], close to the normal form of a Bautin bifurcation [3], displays a Hopf bifurcation [4] that can be either direct or inverse. Its toy model nature makes it easy to write with a single equation

$$\ddot{x} + (\mu + \sigma(\dot{x}^2 + x^2) + \nu(\dot{x}^2 + x^2)^2)\dot{x} + x = 0 \quad (1)$$

Its solutions as well as the locus of its bifurcations are both completely described analytically. This good *a priori* knowledge of the system's steady states means the demonstration can focus on transient aspects such as hysteresis

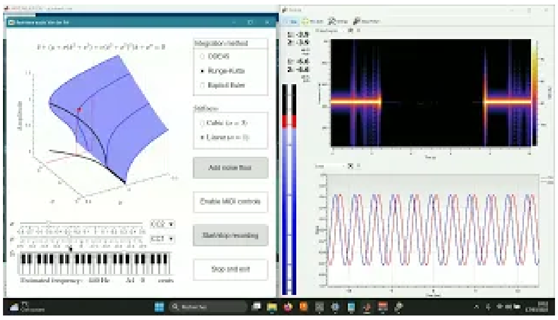

Figure 1. Snapshot from the demonstration video linked at https://youtu.be/a0S7l7Hjl1s.

and dynamical bifurcations. Figure 1 links to a video capture of such a demonstration. The full demonstrator is available online [5].

Negative values of σ lead to an inverse Hopf bifurcation at $\mu = 0$, which is followed by a saddle-node bifurcation [6]. In the region between these two bifurcations, the system is multistable. Therefore, the system displays a hysteresis cycle (0:00 timestamp) wherein the existing stable solution (constant or periodic) is maintained until the parameter leave the multistability region. When demonstrating this phenomenon, another feature is immediately apparent : the oscillations start and stop brutally. For music-oriented students, this observation is very interesting as it means that self-oscillating instruments that have an inverse Hopf bifurcation will likely be difficult to play at a *pianissimo* nuance. To the contrary, the Hopf bifurcation is direct for positive σ values. The user can experience that a fine adjustment of parameter μ enables arbitrarily low amplitude oscillations (0:23 timestamp).

Dynamical bifurcations are a phenomenon whereby the system's parameter cross a bifurcation point, but the effect of that bifurcation is only felt some time after the crossing [7]. For a Hopf bifurcation, this can mean that the oscillations only appear after a certain delay after μ crosses 0. This delay can be rather long, especially when the system is allowed to converge very close to the equilibrium. When the system is near the equilibrium and it becomes unstable, the distance between it and the position of the system is amplified exponentially. If this distance is very small, the time needed for it to become big enough for oscillations to be heard can be long. In the case of our real-time Van der Pol oscillator, 0:52 timestamp showcases a long bifurcation delay (of about 1 s). In a musical context, this duration is not negligible at all

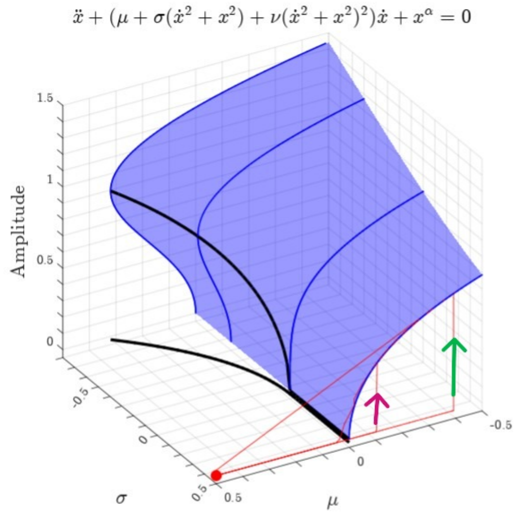

Figure 2. Bifurcation diagram: theoretical static solution amplitudes in blue, and time-integration trajectories in red for two increases in parameter μ from -0.5 to 0.4: first without added noise (green arrow) and then with added white noise (pink arrow).

(half a bar at 120 bpm). Then, a small white noise of standard deviation 10^{-5} is added to the variables of the system at the 1:04 timestamp. By preventing the system to approach the equilibrium too much, this modification greatly shortens the bifurcation delay. Figure 2 compares the two trajectories : compared to a short bifurcation delay, the long bifurcation delay also entails that oscillations appear when μ is larger and thus with a greater amplitude – which translates to a more sudden attack.

This demonstration shows the importance of the nature of the bifurcation in a musical context, as well as the more subtle phenomenon that is bifurcation delay.

3 A CLARINET MODEL SHOWS SIMPLE AND COMPLEX MULTISTABLE BEHAVIOR, AS WELL AS CHAOTIC REGIMES

A more applicative example, used in a lab for a master's degree, puts the student with a real-time integration of a woodwind model, whose modal parameters can be fixed by the user. The two variable control parameters are the

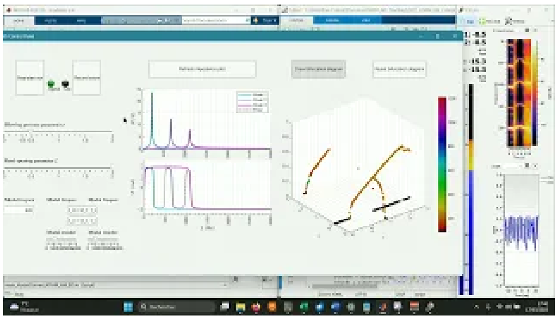

Figure 3. Snapshot from the demonstration video linked at https://youtu.be/D8AUCYr-XXg.

pressure in the mouth and the opening of the reed at rest. Although it is more involved than the previous toy model, this woodwind model writes rather simply as

$$\begin{cases} \dot{p_1} &= s_1 p_1 + C_1 u \\ \dot{p_2} &= s_2 p_2 + C_2 u \\ \dot{p_3} &= s_3 p_3 + C_3 u \\ p &= 2\Re(p_1 + p_2 + p_3) \\ u &= \zeta \max(p - \gamma + 1, 0) \mathrm{sign}(\gamma - p)\sqrt{|\gamma - p|} \end{cases} . \tag{2}$$

where p_1, p_2 and p_3 are complex modal pressures associated with the three modes whose complex residues and poles are C_i and s_i, and ζ and γ are reed opening and blowing pressure dimensionless parameters [8].

Figure 3 shows the time integration results for this model, when the user manipulates sliders to control the parameters of the system.

The video first showcases the apparition of oscillations through direct Hopf bifurcations when increasing the blowing pressure γ (0:00 timestamp). This is explained as the system only producing sound if the musician blows into it sufficiently hard. Another Hopf bifurcation with respect to the reed opening parameter ζ (1:25 timestamp) is interpreted as the clarinet only producing sound if the reed is open enough at rest. A large multistability zone is explored through a hysteresis cycle at timestamp 0:35 timestamp. Then, the system's ability to produce chaos for a large reed opening is showcased though intermittent regimes (1:55 and 2:35 timestamps). A real-time sound synthesis model enables direct comments of the produced sounds. This parameter region also displays subharmonic regimes (2:21 timestamp). This subharmonic regime produces a note that is unexpected when looking at the resonances of the instrument, as it is lower than the lowest resonance frequency. Then (3:20 timestamp), it is shown that through chaos the system can transition from the first register (regime associated with the first resonance) to the second register (regime associated with the second resonance). These two registers are actually multistable for a large portion of the parameter space : when the second register is obtained it can then be easily maintained.

4 ACKNOWLEDGMENTS

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