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REDUCED-ORDER MODELING ANALYSIS OF RANDOM GEOMETRIC CHANGES ON THE VIBROACOUSTIC BEHAVIOR OF METAMATERIALS

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ABSTRACT

As metamaterials are coming closer and closer to becoming a mature industrial solution, the question of manufacturing has started to gain importance. Indeed, variability related to industrial processes can cause geometric variability which might degrade the attenuation performance of manufactured samples. Hence, there is a strong need to quantify the impact of uncertainty on the behavior of vibroacoustic metamaterials. However, full-scale metamaterial models can require a relatively high computational time to solve. This makes uncertainty quantification methods like Monte-Carlo simulations either computationally intensive or infeasible due to the high number of model solves required. Hence, this paper proposes a parametric reduced-order model based on the Krylov subspace to predict the vibroacoustic response of a metamaterial structure due to geometric changes without requiring another solve of the full system. Derivation of the response statistics is then achieved by applying a Monte-Carlo simulation on the obtained reduced-order model.

Keywords: *Geometric Variability, Model-Order Reduction, Vibroacoustic Metamaterials.*

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1. INTRODUCTION

When it comes to lightweight solutions for sound and vibration attenuation, metamaterials are showing more and more potential for noise and vibration harnessing in different areas of the industry [1, 2]. The locally resonant nature of metamaterials allows them to create stopbands in which the vibration response is attenuated. However, the problem of variability in metamaterials and the prediction of its effects on their performance is currently a topic of interest. As metamaterials often rely on either in tune resonators or periodicity, imperfect manufacturing of metamaterial structures could lead to underperforming solutions. This is why, the impact of such uncertainty needs to be assessed.

Currently, few design procedures for metamaterials take into account the impact of uncertainties. At present, metamaterials are typically designed by Unit-Cell (UC) modeling [3]. This method takes advantage of the metamaterial translational symmetry to reduce the size of the problem, as only the smallest repeating period of the material is modeled. Some authors used this method to characterize the impact of uncertainty at the UC level [4, 5]. However, as this representation assumes perfect periodicity of the material between the UCs, it is not able to describe the random imperfections that occur when the material is actually manufactured. Still, to take into account these imperfections, different approaches were already proposed in current literature. Some authors suggest to take into account the impact of variability in a finite metamaterial model. Some references [6, 7] present studies of metamaterial structures through uncertainty quantification techniques such as Gaussian process modeling to characterize e.g. the uncertainty bounds on its response metrics. In [8], a model-order reduction strategy is presented to improve the solving time of nearly periodic structure, based on periodic structure theory, which allows to partially overcome the computational limitations of statistical analysis for large metamaterial structures.

However, the study of uncertainty in metamaterials has been mostly limited to point scatterers or idealized resonators [9], and the study of geometric deviations in such structures has received less attention. Indeed, geometric perturbations of a structure leads to nonlinear changes in the finite element matrices, making model-order reduction more complex. Additionally, parametric model-order reduction schemes need to be developed for efficient sampling and uncertainty quantification of the system as the number of solves needed for convergence of the statistics under consideration might be high in some cases.

In this paper, a parametric model-order reduction scheme based on [9] is developed for finite-size metamaterials. The method is applied to a phononic crystal beam to obtain the statistics of its vibration response to a point excitation. The accuracy of the method is assessed by comparison with the full-order model for individual simulations, but also for the statistics of the system's response.

The rest of the paper is structured as follows: section 2 provides a description of the problem under consideration. Then, section 3 gives an overview of the parametric model-order reduction approach. A statistical analysis of the structure as well as a comparison with the full-order model are given in section 4. An extension of the proposed method by projection basis enrichment is proposed in 5. Finally, the results are summarized in section 6.

2. PROBLEM DEFINITION

In this paper, the phononic Euler-Bernoulli beam of [10] is considered as case study. A one-dimensional structure is considered, for which geometric changes are straightforwardly defined and have a direct effect on the stopband behavior of the metamaterial. The system also has the advantage of being practically manufacturable, allowing for possible experimental validation of the numerical study in future work.

The beam is composed of 9 UCs with subsequent thickness variations (Fig. 1). The UC has a length $L_{UC} = 40$ mm and is composed of two equal-sized parts with different heights $h_1 = 10$ mm and $h_2 = 3$ mm respectively. The material properties of the beam are described in table Tab. 1. Free-free boundary conditions are considered and a unit transverse point force is applied on the first node on the left side of the beam.

Table 1. Properties of the phononic beam.

Property	Symbol	Value
Width	b	10 mm
Mass density	ρ	$1135\ \mathrm{kg/m^3}$
Young's modulus	E	1.9 GPa
Structural damping	η	5 %

Due to the presence of variability, the position of each UC center is assumed to randomly deviate from its nominal value and is described by the parameters l_n. This choice is motivated by the fact that this phononic beam relies on periodicity to produce a stopband. Therefore, modifying l_n directly impacts the periodicity but also preserves its total length as well as the location of the forcing and response nodes. The length deviation of l_n is assumed to follow a uniform distribution with limits at $\pm 3\%$ of the nominal length. There is currently no justification for this choice as data in the metamaterial literature on such variation is rather limited. However, the statistical distribution of the l_n could be determined by production of a large enough series and statistical analysis of the dimensions of the system.

To conduct the numerical study, the beam is discretized using the Finite Element Method (FEM). Euler-Bernoulli beam elements are used to obtain a small, one-dimensional problem instead of a full elastodynamic model. 8 elements per UC are used, ensuring a minimum of 10 elements per wavelength in either of the sections of the beam. The FE-discretized, l_n-dependent equations of motion then write as

$$\mathrm{D}(l_n)\mathbf{u}(l_n) = \mathbf{F}, \tag{1}$$

where $\mathrm{D}(l_n)$ is the dynamic stiffness matrix, $\mathbf{u}(l_n)$ is the displacement vector and $\mathbf{F}$ is the force vector. The dependency with respect to the l_n is accounted for by moving the nodes of the structure in space such that the elements of the beam are either contracted or dilated by a small percentage compared to the case $l_n = 0$ to match the desired geometric variation. In general, metamaterial UCs can become quite intricate, and UC meshes rather large, making full structure models of the UC assembly even

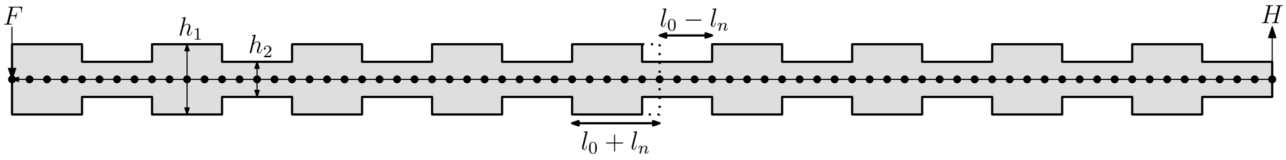

Figure 1. Phononic beam structure under consideration. F represents the forcing point, while H represent the velocity response point.

larger. Solving equation Eqn. (1) might then be too computationally expensive for full-order uncertainty propagation simulations requiring high amount of samples as the computational cost for one solve is already high. In the following section, a parametric model-order reduction scheme is developed to take into account the impact of geometrical uncertainties. The model under consideration is rather small and does not need to be reduced to perform uncertainty quantification. In principle, the method proposed is however generally valid, and in this paper it will be applied to this simple case as a first validation. The goal of the method is to obtain a parametric reduced-order model to quickly evaluate the response of a structure subject to geometric variations.

3. METHODOLOGY

3.1 Parametric Model-Order Reduction Scheme

As a first approach, one might consider expanding $\mathbf{u}$ as a first order perturbation series around $\mathbf{u}_0$:

$$\mathbf{u} \simeq \mathbf{u}_0 + \sum_{n=1}^{N_p} l_n \mathbf{u}_1^{(n)}, \tag{2}$$

with N_p the number of random parameters, equal to the number of UCs in this study, $\mathbf{u}_0 = \mathbf{u}(l_n = 0)$ and $\mathbf{u}_1^{(n)} = \mathrm{D}^{-1}(0)\frac{\partial \mathrm{D}}{\partial l_n}|_{l_n=0}\mathbf{F}$. The matrices $\frac{\partial \mathrm{D}}{\partial l_n}$ can be straightforwardly evaluated using the analytical expression of the Euler-Bernouilli beam element matrices

$$K_e^p = \frac{EI_p}{l_e^3}\begin{pmatrix} 12 & 6l_e & -12 & 6l_e \\ 6l_e & 4l_e^2 & -6l_e & 2l_e^2 \\ -12 & -6l_e & 12 & -6l_e \\ 6l_e & 2l_e^2 & -6l_e & 4l_e^2 \end{pmatrix}, \tag{3}$$

$$M_e^p = \frac{\rho S_p l_e}{420}\begin{pmatrix} 156 & 22l_e & 54 & -13l_e \\ 22l_e & 4l_e^2 & 13l_e & -3l_e^2 \\ 54 & 13l_e & 156 & -22l_e \\ -13l_e & -3l_e & -22l_e & 4l_e^2 \end{pmatrix}, \tag{4}$$

where K_e^p and M_e^p are respectively the Euler-Bernoulli beam element stiffness and mass matrices of the p-th cross section of the beam with $p = 1, 2$, S_p and I_p are the surface and quadratic moment of inertia of the p-th cross section of the beam and l_e is the element length.

Eqn. (2) usually suffers from accuracy and instability, especially in the presence of resonances, making it unsuitable for accurate computation of the frequency response of a vibrating structure in which many resonances are present [11].

This is why an alternative moment-matching approach [9, 12] is used instead in this paper. The moments of the first-order perturbation series $\mathbf{u}_0$ and $\mathbf{u}_1^{(n)}$ are multiplied by a set of coefficients for the chosen set of parameters by multiplying them by a correction contribution amplitude w_n.

$$\mathbf{u} \simeq w_0\mathbf{u}_0 + \sum_{n=1}^{N_p} w_n l_n \mathbf{u}_1^{(n)}. \tag{5}$$

These additional degrees of freedom will later allow for an optimal correction of the displacement vector. This is equivalent to taking the assumption that the solution $\mathbf{u}$ lies in the subspace

$$\mathrm{Q} = orth\left(\left[\mathbf{u}_0, \mathbf{u}_1^{(1)}, ..., \mathbf{u}_1^{(N_p)}\right]\right). \tag{6}$$

Projecting Eqn. (1) onto the basis Q, the following reduced-order problem is obtained

$$\tilde{\mathrm{D}}(l_n)\mathbf{w}(l_n) = \tilde{\mathbf{F}}, \tag{7}$$

with $\tilde{\mathrm{D}}(l_n) = \mathrm{Q}^*\mathrm{D}(l_n)\mathrm{Q}$, and $\tilde{\mathbf{F}} = \mathrm{Q}^*\mathbf{F}$. Eqn. (7) can then be used to quickly evaluate the changes of $\mathbf{u}$ to parametric variations, without the need to solve for the full-order problem once again, and only requiring a reassembly of the system matrices with a transformed mesh according to the l_n.

3.2 Monte-Carlo simulations

To quantify the random changes in the structure's response, Monte-Carlo simulations are used to propagate the uncertainty. For a sufficiently large number of samples $N_{samples}$, the set of l_n is drawn from a uniform distribution using a random number generator. In this study, the Python function from the *Numpy* library `numpy.random.uniform` is used.

For each set of l_n, and at each frequency the matrix $\mathrm{D}(l_n)$ is assembled. Then the system Eqn. (7) is derived and solved. Equation Eqn. (5) is finally used to obtain the displacement vector and the response. Once the procedure is repeated for the full set of l_n, statistical analysis can be run on the obtained set of responses.

In the following section, the previously derived reduction scheme and the Monte-Carlo simulation approach are used for statistical analysis of the phononic beam structure.

4. STATISTICAL ANALYSIS OF THE PHONONIC BEAM

In this section, the method of section 3 is validated on the structure from section 2. Then an error analysis is performed to verify the accuracy of the method.

4.1 Response analysis

In (Fig. 2), the endpoint velocity response of the beam is displayed in dashed lines for the nominal case, where $l_n = 0$. The nominal periodic phononic beam shows a frequency range of strong vibration reduction around 1050 Hz. As shown in [10], this is due to the presence of a Bragg stopband around this frequency. Destructive interference occurs, which prevents waves from propagating to the probing location at the other side of the beam. As this effect is linked to the periodicity of the structure, including disorder in the UC geometry through the l_n, and therefore breaking the periodicity of the systems is expected to have an impact on the stopband effect.

The median of the response (blue) as well as the confidence interval between its 5-th and the 95-th percentiles (light blue region) are also displayed. A total of 1000 Monte-Carlo sample are computed using the reduced-order model from Eqn. (7) method and used to obtain the statistics of the response. After 1000 samples, it was observed that the relative variation in the statistical indicators under consideration is less than 3% over the frequency range $50 - 2500$ Hz when increasing the number of iterations.

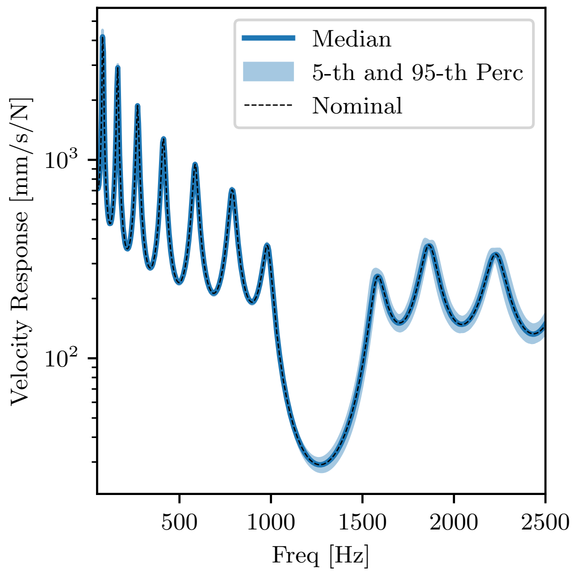

Figure 2. Endpoint velocity response of the phononic response in the nominal case, median, 5-th and the 95-th percentiles of the response.

Thanks to these response statistics, the effect of geometric variations on the vibration attenuation performance of the beam can be assessed. In the higher frequency part of the stopband region, as well as above it, the difference between the 95-th and 5-th percentile responses is seen to be bigger than in the lower part of the frequency range, meaning that the magnitude of the response is more uncertain in this frequency region. For the geometrical changes considered, small variations affect the response in the band gap. However, a clear attenuation region still persists for the disorder strength considered. This was also observed for locally resonant disordered beam metamaterial structures in the literature [6, 13, 14].

4.2 Error Analysis

Following computation of the response statistics, an analysis is conducted to compare how representative the reduced-order model is of the full-order model. After performing statistical computations with the reduced-order model, a Monte-Carlo simulation is run for the full-order model Eqn. (1) for the same set of l_n to conduct an error analysis. This is feasible since the model considered here has a small number of degrees of freedom. To verify the accuracy of the reduced-order model, the maximum of the relative error between the reduced and full-order models across all realizations of the structure is evaluated. This error metric can serve as an estimate of the minimum degree of accuracy attained by the method in the desired frequency range.

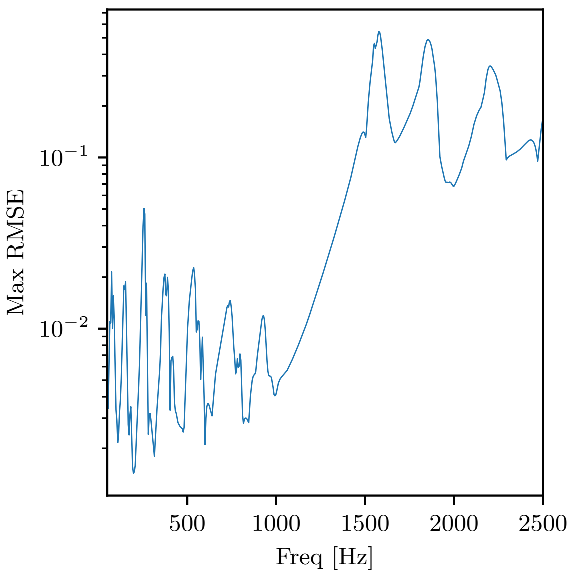

Figure 3. Maximum relative root mean square error between the reduced and full-order models observed across realizations of the system.

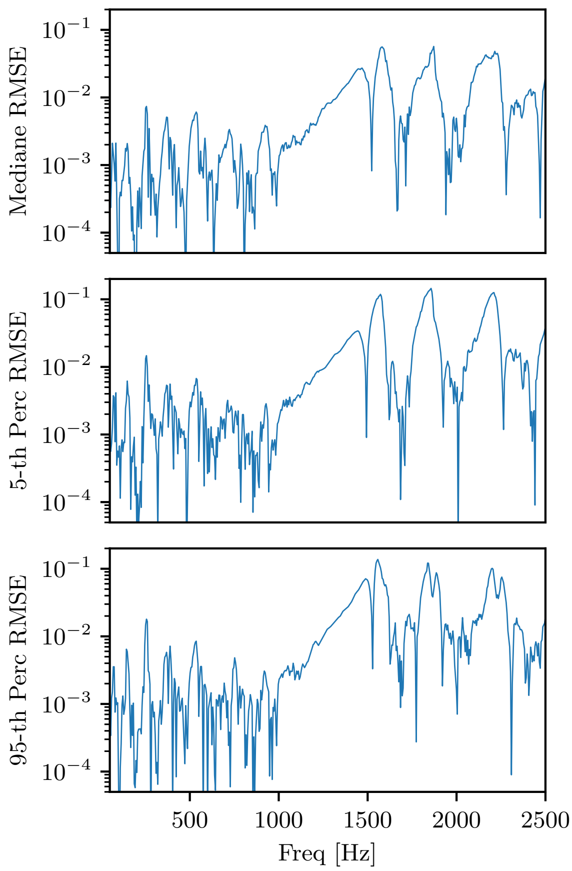

Figure 4. Relative root mean square error between the reduced and full-order models on the median, 5-th and 95-th percentiles of the endpoint velocity response.

The results of this first analysis are displayed in Fig. 3. From the error curve across the frequency range, we can observe that the reduced-order model performs best up to 1000Hz, before the stopband region, with under 5% relative error. At higher this frequencies, the maximum error between the two models is higher, and can get as high as 40% near some resonances. Indeed, past the stopband, the dynamics of the phononic beam are different and could explain the difference in accuracy. Additionally, error peaks occur near resonances of the system.

To verify the accuracy of the method, the error on the statistical indicators is computed for this system, namely, the median, 5th and 95th percentiles of the response is computed compared to the full-order system in (Fig. 4).

The error does not exceed 15% for either of the statistical quantities over the frequency range of interest. However, similarly to (Fig. 3), a much lower accuracy is obtained above the stopband region and error peaks are again notified near resonances of the system, confirming the inaccuracy problem notified earlier. The relative error on the statistics is seen to be much lower than the maximum error across realizations of the system. This is to be expected since the first error metric serves as a measure of the maximal error and single realizations of the system will have an error profile either lower or equal to the one of (Fig. 3).

If the desired accuracy of the reduced-order model is not acceptable for certain use cases. The origin for the error between the two models is that the subspace Q is not of sufficient rank to describe effect of geometric variations with sufficient accuracy, especially in the regions of high error. A higher accuracy could be attained by enriching the projection basis with higher order moments of $\mathbf{u}$ and is the object of the following section.

5. SECOND ORDER PROJECTION SCHEME

To obtain additional accuracy, one may consider adding the second order terms of the perturbation series to the reduced-order basis. The subspace Q then becomes,

$$\mathrm{Q}_2 = orth\left(\left[\mathrm{Q}, \mathbf{u}_2^{(1,1)}, \mathbf{u}_2^{(1,2)}, ..., \mathbf{u}_2^{(N_p,N_p)}\right]\right), \quad (8)$$

where the $\mathbf{u}_2^{(m,n)}$ are defined as

$$\mathbf{u}_2^{(m,n)} = \mathrm{D}^{-1}(0)\left[2\frac{\partial \mathrm{D}}{\partial l_m}\mathbf{u}_1^{(n)} + \frac{\partial^2 \mathrm{D}}{\partial l_m \partial l_n}\mathbf{u}_0\right]_{l_m=0}. \quad (9)$$

The procedure from section 3 is then followed using Q_2 instead of Q. Monte-Carlo simulations are run on this enriched reduced-order model. The maximum error across realizations of the system is computed again and displayed in Fig. 5.

An increase of accuracy of about one to two orders of magnitude is observed compared to Fig. 3 for the whole frequency range. Error peaks are still observed but their magnitude is lower. However, this gain of accuracy comes at the cost of a higher need in computational resources as the size of the reduced-order model might increase significantly from $1 + N_p$ to $1 + N_p + N_p^2$ at most due to the number of correction vectors to be computed increasing.

6. CONCLUSION

In this paper, a parametric model-order reduction scheme was derived to analyze the effect of random geometric variations on the behavior of a phononic crystal beam. A comparison of the reduced-order model with the full-order model response and statistics was performed, verifying the accuracy of the method.

Statistical analysis of the structure was conducted through Monte-Carlo simulation of the reduced-order model. A statistical analysis of the metamaterial structure allowed for quantification of the effect of geometrical variations stopband behavior of the structure. The attenuation performance of the structure was observed to be impacted by the random length changes in some cases,

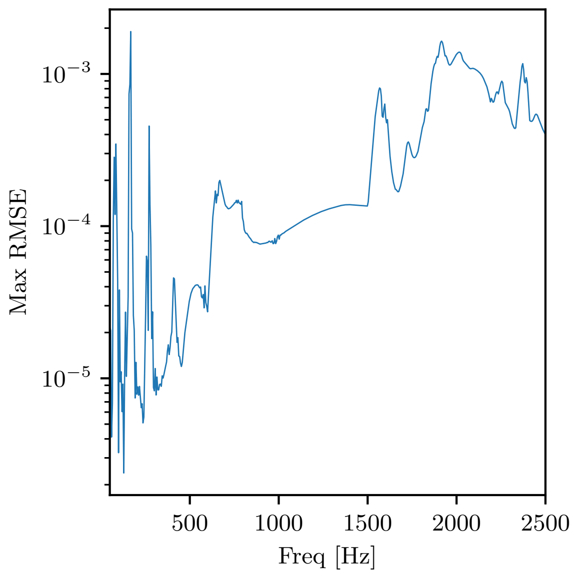

Figure 5. Maximum relative root mean square error between the reduced and second order full-order models observed across realizations of the system.

meaning that some samples of the beam might underperform compared to its target performance, which was observed in the literature for different structures. In this situation, the vibration attenuation region stays relatively close to the nominal case for the considered geometric perturbations.

In the specific case study of this paper, the full and reduced-order models take similar time to run. In the upcoming works, larger structures will to be analyzed as the computational cost reduction is expected and needs to be quantified. The impact of the number basis of correction vectors used in the model needs as well to be assessed in terms of accuracy gain and computational cost increase for a wider range metamaterial structures and problems, as it is important to verify that the accuracy is sufficient for the study under consideration. To do so in a different, specific case study, the analysis from (Fig. 3), using only a few realizations (say 10 to 50) of the system could serve as a benchmark.

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