DOI: 10.61782/fa.2025.0868

FORUM ACUSTICUM
EURONOISE 2025

The influence of nonlinear behaviour of reinforced concrete structures on numerical modelling for vibration

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ABSTRACT

In the context of vibration predictions, numerical models typically form an integral part of the process. This is particularly important for newly planned structures near railway lines, where such calculations are considered as "state of the art" and fundamental for reliable predictive computations. Numerical calculations generally assume linear-elastic behavior of concrete in its uncracked state (State I) . However, the design of structures aims to allow a certain degree of cracking in the final state (State II) to activate the reinforcement. This, however, leads to a reduction in the stiffness of the structure, resulting in diminished natural frequencies. Natural frequencies, in particular, represent a critical parameter in the dynamic assessment of buildings [1].

This work aims to investigate, through nonlinear numerical calculations, the influence of cracking and the transition to State II on the dynamic properties of structural constructions.

Keywords: *cracked concrete, bilinear dynamics, vibration prognosis.*

1. INTRODUCTION

Vibration forecasts based on numerical calculation models are widely used due to the further development of computing power and the general and largely cost-effective availability of the corresponding computing programs. The basis for the calculations is usually the design drawings for the building structure. The material parameters of the calculations are chosen according to the building materials and cross-sections used. In most cases, these programs have stored the corresponding calculation parameters for concrete of different strength classes, for example.

Linear-elastic approaches are usually chosen as material models. More specifically, it is assumed that the rigidity of the structure remains constant with deformation. To put it simply, in Hook's law:[2]

$$F_s = kx \tag{1}$$

This means that above all the ceiling structures relevant for the vibration prediction are assumed to have a linear-elastic behavior. However, this is precisely not the approach used as a basis for the static design of the structures. In the static design, it is assumed that due to the low tensile strength of the concrete, cracks will form in the tensile area if the tensile strength of the concrete is exceeded, and the reinforcing steel will be activated [3].

However, the crack formation leads to a change in cross-section, which has a direct influence on the bending stiffness via the connection in Eq. 2 and Eq. 3 [4].

$$S = EI \tag{2}$$

with

$$I = \frac{bh^3}{12} \tag{3}$$

Cracking in the tensile area of the beam causes the cross-sectional area to change, which influences the bending stiffness by means of the moment of inertia I Ultimately, increasing cracking leads to a decrease in bending stiffness and thus also to a change in the natural frequency of the system.

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In the present work, a simple rectangular cross-section is to be investigated, which is subsequently approximated by a single-mass oscillator with a bilinear spring. This should serve to estimate the dynamic properties of cracked systems.

2. BASICS

2.1 The linear single-mass oscillator

For a first introduction to the topic, the single-mass oscillation is a good starting point. A harmonic excitation of a damped single-mass oscillator with forced harmonic excitation [5], [6], [7] can be represented by the following well-known equation of motion.

$$m\ddot{y} + c\dot{y} + ky = F_0 \cos\left(\omega_f t\right) \tag{4}$$

By introducing various simplifications and terminologies, Eq. 4 can be rewritten into the following form [5]

$$\ddot{y} + 2D\omega_o\dot{y} + \omega_0^2 y = \omega_0^2 F_0 \cos\left(\omega_f t\right) \tag{5}$$

The following terms were introduced.

$$\omega_0^2 = \frac{k}{m}, 2D\omega_0 = \frac{d}{m}, \omega_D = \omega_0\sqrt{1 - D^2}, \quad \eta = \frac{\omega_f}{\omega_0} \tag{6}$$

In the course of this work, we are primarily interested in stationary solution with continuous harmonic excitation, i.e. in the forced oscillation that we obtain as a solution of the inhomogeneous differential equation (Eq. 5). By means of the following two-part solution approach (Eq. 7), a stationary solution of differential euqtion can be found.

$$\mathrm{y(t)} = C_1 sin\left(\omega_f t\right) + C_2 cos\left(\omega_f t\right) \tag{7}$$

The stationary solution of the differential equation then results in [5], [8]:

$$\mathrm{y(t)} = \frac{y_0}{\sqrt{N(\eta, D)}} sin\left(\omega_f t + \varphi\right) \tag{8}$$

Based on this solution from Eq. 8, the deflection of the system according to harmonic force excitation can be calculated for each time *t*. By deriving the formulation from Eq. 8 accordingly, the course of the velocity or acceleration can also be determined.

2.2 The bilinear single-mass oscillator

The bilinear single-mass oscillator is characterized by the fact that the spring stiffness *k* has different values in the tensile and compression ranges. Figure 1 shows an example of a working line of a bilinear spring.

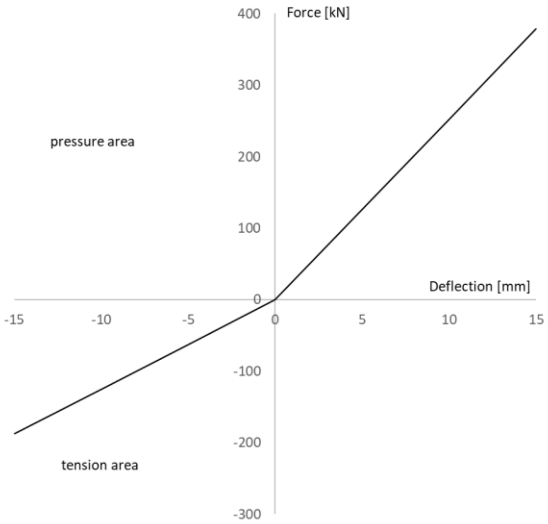

Figure 1. Working law for a bilinear spring

This is a property that can also be found, for example, in beams under nonlinear consideration. This will be discussed in more detail in the following section. In this section, the analytical solution of the bilinear single-mass oscillator will be discussed.
In contrast to the linear single-mass system, the two stiffnesses k_1 and k_2 in the compressive and tensile ranges also mean that two natural frequencies or period durations are to be expected. The mean duration T and the mean stiffness of the system can be determined according to Eq. 9 and Eq. 10 [9].

$$T = \frac{1}{2}(T_1 + T_2) \tag{9}$$

$$K = \frac{4k_1k_2}{\left(\sqrt{k_1} + \sqrt{k_2}\right)^2} \tag{10}$$

The eigen-circular frequency of this system is then obtained according to Eq. 11 [5].

$$\omega = \frac{2\pi}{T} = \sqrt{\left(\frac{K}{m}\right)} \tag{11}$$

Taking into account the definitions in Eq. 12, the differential equation from Eq. 4 can be converted into the dimensionless form, shown in Eq. 13.

$$\eta = \frac{\omega_f}{\omega}, Y = \frac{y}{\frac{F_0}{K}}, \tau = \omega_f t, \beta = \frac{k_2}{k_1}, \xi = \frac{c}{2m\omega} \quad (12)$$

$$\ddot{Y} + 2\frac{\xi}{\eta}\dot{Y} + K_i Y = \frac{1}{\eta^2}\sin(\tau) \qquad i = 1,2 \quad (13)$$

The two dimensionless stiffnesses, expressed by the bilinearity factor β can be seen in Eq. 14 and Eq. 15.

$$K_1 = \frac{\left(1+\sqrt{\beta}\right)^2}{4\beta\eta^2} \quad (14)$$

$$K_2 = \frac{\left(1+\sqrt{\beta}\right)^2}{4\eta^2} \quad (15)$$

The bilinearity factor β is one of the decisive parameters in connection with the calculations cited. In the following section, its relevance in relation to a simple bar will be discussed in detail.
By superposition of the particulate solution and the homogeneous solution of the differential equation from Eq. 13, the general solution of Eq. 13 can be determined.

$$Y = A_1 sin(\tau) + A_2 \cos(\tau) + e^{-\frac{\xi\tau}{\eta}}(A_3 \sin(\omega_D \tau) + A_4 \cos(\omega_D \tau)) \quad (16)$$

The damping coefficients for the compression and tension range of the spring are thus [9].

$$\xi_1 = \frac{2\xi\sqrt{\beta}}{1+\sqrt{\beta}}, \xi_2 = \frac{2\xi}{1+\sqrt{\beta}} \quad (15)$$

2.3 Derivation of bilinearity from beam theory

If a simple rectangular cross-section is considered, the change in the moment of inertia can be determined as a function of the geometric factors or the degree of reinforcement of the upper and lower layers according to Eq. 4 and Eq. 5 [10].

$$I = \frac{bx_2^3}{3} + (r-1)P_1 b d_2 (x_2 - c_1)^2 + rP_2 b d_2 (d_2 - x_2)^2 \quad (16)$$

$$r = \frac{280}{3\sigma_{cbc}} \quad (17)$$

Figure 1 shows a rectangular cross-section with the corresponding geometric parameters used in Eq. 4 and Eq. 5.

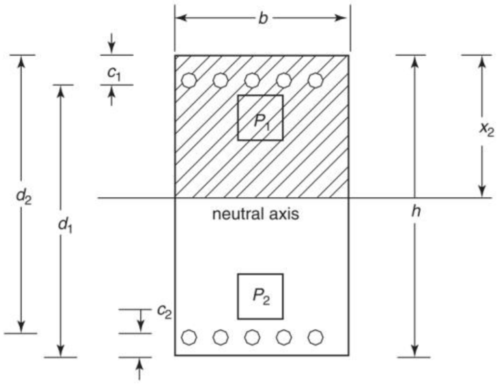

Figure 2. Reinforced rectangular cross-section [10]

If one considers the rectangular cross-section in Fig. 1 under the assumption of different reinforcement degrees in the upper and lower reinforcement layers or an almost non-existent tensile strength of the concrete, an asymmetry, different stiffness of the upper and lower parts occurs in the course of a bending stress. In the lower part of the cross-section, cracking will occur and thus a significant decrease in bending stiffness. In practice, the reduction in bending stiffness due to the reduction in cross-section due to cracking will be less than theoretically expected. This is due to "*tension stiffening*", i.e. those areas in the tensile area that are not affected by cracking and still contribute to the bending stiffness.
In the upper part, the bending stiffness will not change under stress. From this asymmetry, β can be considered as a bilinearity factor for a simplified view.

3. CALCULATION EXAMPLE

3.1 Model assumptions

To illustrate the analytical considerations shown above, an FE model of a single-mass oscillator was created with the program package Sofistik. Table 1 shows an overview of the calculation parameters.
A representation of the working line of the spring used can be found in Figure 1.

The calculations were carried out nonlinearly using the time step method (Hilbert-Hughes-Taylor [11]). The time step size was chosen as 0.001s.

Table 1. FE-model assumptions

Parameters	Value
k_2	25,250 kN/m
k_1	12,500 kN/m
measure	2.500kg
D	2%

The linear natural frequency of the system shown above is about 16Hz. The natural frequency of the reduced stiffness of *12,500kN/m* is about *11.3Hz.*

Figure 3. Single degree of freedom System (SDOF) calculation model

This means, according to Eq. 9, an average natural frequency of the bilinear system of *13.6Hz* at a $\beta=2$.

3.2 Simulation

Simulations were carried out on the basis of the model parameters listed in the previous section. In the following, for the sake of comparability with the analytical solution, a harmonic excitation (sine) with an excitation frequency of *16Hz* was chosen. The excitation was carried out over a period of 3s. The calculation duration itself was chosen as *5s*, so that the oscillation of the system was still recorded.

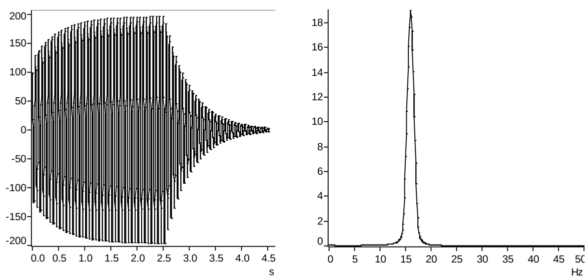

Figure 4. Calculation result of linear SDOF

Figure 4 shows the result of this simulation for a linear single-mass oscillator. The time course shows the transient part, the stationary part and the oscillation of the system. The frequency analysis shows very clearly that the natural frequency of *16Hz* clearly corresponds to the excitation frequency of *16Hz.*

In contrast, fig 5 shows the response of the bilinear system to the same excitation as in fig 4. Already in the course of time it is clearly recognizable that there is no transient in the resonance range.

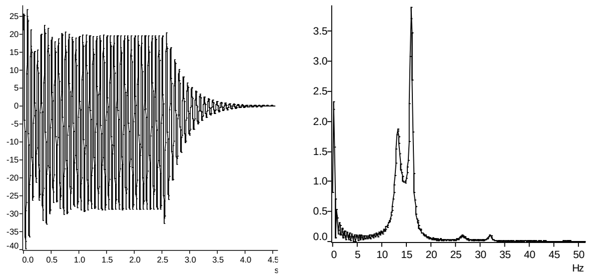

Figure 5. Calculation result of bilinear SDOF

The frequency analysis clearly shows that several peaks can be seen in the frequency spectrum. On the one hand, the *16Hz* from the excitation is decisive, as well as another significant peak in the range of about *13.4Hz*. This additional peak is due to the mean natural frequency of the bilinear system (Eq. 9). This becomes even clearer when the pure swinging out of the bilinear system is evaluated, as can be seen in Figure 6.

After the forced frequency of the system has disappeared, the natural frequency of the bilinear system becomes significant and agrees very well with the theoretical prediction from Eq. 9.

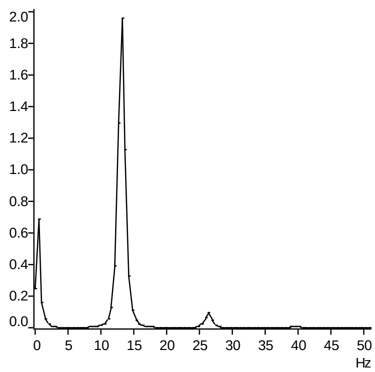

Figure 6. Frequency analysis of the decay process of a bilinear single-mass system

The very low frequency component at about 1Hz is due to the asymmetry of the vibration behavior due to the softer behavior in the tensile range. This leads to an offset in the signal, which is reflected in the low-frequency range of the spectrum.
Regarding the question of the relevance of these relationships for vibration investigations, the following can be said. The results shown above on a very simplified bilinear system show that resonance phenomena are significantly weaker, since these systems have a different natural vibration behavior according to the load. This could be one reason why often predicted strong resonance phenomena on e.g. floor ceilings cannot be observed in practice.

4. SUMMARY AND OUTLOOK

In the present work, an analytical method for the dynamic calculation of a bilinear single-mass oscillator was presented. This can be used as a simplified model for estimating bending natural frequencies of beams. Of particular interest is the dynamic behaviour of these systems when the tensile strength of the concrete is exceeded, i.e. when the system transitions into the cracked state. This is accompanied by a reduction in the bending stiffness and thus also a reduction in the natural frequency of the system. The results have shown that such systems occur in addition to the expected basic natural frequency, which are due to the reduction of stiffness in the tensile area.
In the example shown above, a bilinearity factor of $\beta=2$ was chosen. This was chosen because in practice no higher factors are to be expected due to "tension stiffening".
In a further step, however, it is still necessary to investigate the extent to which higher factors affect behavior. It [9], [10] has already been shown that higher factors for β lead to greater amplification but also to more significant minor frequencies resulting from the pull range of the system.

5. ACKNOWLEDGMENTS

In the course of numerous interesting discussions regarding the development of damage due to construction vibrations, the topic of vibration loading and non-linear issues was addressed for me for the first time. The company iC consulenten then gave me the opportunity to use the FE programs and also released funds to enable research in this area.

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