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TRANSFER MATRIX MODELLING OF FOLDED QUARTER-WAVELENGTH RESONATORS WITH 90° BENDS

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ABSTRACT

Acoustic metamaterials that consist of multiple folded quarter-wavelength resonators in parallel have the potential to realise high and broadband low-frequency sound absorption while maintaining a compact volume. In a previous study, it was shown that folding can alter the acoustic behaviour of a quarter-wavelength resonator significantly and a length correction factor was proposed to account for the effect of a single 90° bend in analytical models. In this paper, a transfer matrix approach to model quarter-wavelength resonators with multiple 90° bends is proposed. Firstly, the different steps of the transfer matrix method are discussed and the two-port matrix of a 90° bend is predicted as a function of the resonator width by applying a passive two-port characterisation technique to the result of numerical simulations. Then, the transfer matrix method is applied to determine the resonance frequency of a set of single-bend and two-bend quarter-wavelength resonators. Lastly, these results are verified against other modelling approaches and finite element simulations in COMSOL Multiphysics.

Keywords: *acoustic metamaterial, quarter-wavelength resonator, transfer matrix method, 90° bend*

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1. INTRODUCTION

Sound absorbing panels made of porous material (e.g. mineral wool) offer a broadband solution to indoor noise problems, but since the needed layer thickness scales with the wavelength, issues might arise when targeting low-frequency sound in confined spaces. In such case, resonant acoustic metamaterials can be used instead, offering a compact and broadband solution to low-frequency sound problems [1–3].

This work focusses on quarter-wavelength (λ/4) acoustic metamaterials, which consist of multiple folded or coiled λ/4 resonators in parallel and/or series. In a previous study, it was shown that folding can affect the resonance frequency of a λ/4 resonator significantly [4]. Hence, an empirical length correction factor was derived to account for the effect of a 90° bend [5]. Although the proposed correction factor was successfully verified and validated [6], its use is limited to λ/4 resonators with a single 90° bend, whereas real-life resonators are often folded multiple times.

Hence, this paper investigates the use of a transfer matrix approach to determine the absorption coefficient and resonance frequency of a λ/4 resonator with multiple 90° bends. The remainder of this paper is structured as follows. In Section 2, the different steps of the proposed transfer matrix method are discussed and transfer matrices for straight and 90° corner elements are defined. Section 3 covers the numerical verification of the transfer matrix method against various single-bend and two-bend resonator configurations. Finally, the main conclusions are given in Section 4.

2. TRANSFER MATRIX METHOD

The transfer matrix method (TMM) is a well-established approach to determine the acoustic properties (e.g. the absorption coefficient) of a complex, multi-element structure [7]. In this work, the structure is a folded $\lambda/4$ resonator, which can be seen as a series arrangement of straight and 90° corner elements, embedded in a baffle. The different steps of the TMM are discussed in the following subsections.

2.1 Transfer matrix of an acoustic element

In order to apply the TMM, the transfer matrix $\underline{T}$ of each element must be known. This matrix relates the total pressure p and axial particle velocity v at the inlet and outlet of the element, as given by Eqn. (1) [7]:

$$\begin{Bmatrix} p \\ v \end{Bmatrix}_{out} = \underline{T} \begin{Bmatrix} p \\ v \end{Bmatrix}_{in}. \quad (1)$$

Depending on the type of element, $\underline{T}$ is calculated directly or derived from the scatter matrix $\underline{S}$, which relates the complex amplitudes of the left and right propagating pressure wave at the inlet and outlet of the element, as given by Eqn. (2) [8]:

$$\begin{Bmatrix} p^+_{out} \\ p^-_{in} \end{Bmatrix} = \underline{S} \begin{Bmatrix} p^+_{in} \\ p^-_{out} \end{Bmatrix}. \quad (2)$$

2.1.1 *Straight element*

For a straight element, $\underline{T}$ is given by Eqn. (3):

$$\underline{T} = \begin{bmatrix} \cos(kL) & -\mathrm{j}Z\sin(kL) \\ -\mathrm{j}\frac{1}{Z}\sin(kL) & \cos(kL) \end{bmatrix}, \quad (3)$$

with j the imaginary unit, L the length of the element, and k the wavenumber and Z the characteristic impedance of the equivalent fluid inside the element, respectively [7, 9].

2.1.2 *90° corner element*

For a 90° corner element, a direct representation of $\underline{T}$ like Eqn. (3) is not available in literature. Hence, $\underline{T}$ is calculated from $\underline{S}$ using Eqn. (4):

$$\underline{T} = \begin{bmatrix} \frac{1-S_{12}}{2} & \frac{Z(1+S_{12})}{2} \\ \frac{-S_{22}}{2} & \frac{ZS_{22}}{2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{S_{11}}{2} & \frac{ZS_{11}}{2} \\ \frac{S_{21}-1}{2} & \frac{Z(1+S_{21})}{2} \end{bmatrix}. \quad (4)$$

The scatter matrix of a 90° corner element is derived by applying a passive two-port characterisation technique to the results of 2D finite element simulations in COMSOL Multiphysics [10]. In this work, only 90° corners with a square cross-section with side length b ranging from 1 to 6 mm are considered. It is found that, for each corner, $\underline{S}$ is symmetric (i.e. $S_{11} = S_{22}$ and $S_{12} = S_{21}$) and can be obtained using Eqn. (5)-(8):

$$\Re(S_{11}) = -0.287(k_0b)^2 - 0.013(k_0b) + 0.999, \quad (5)$$

$$\Im(S_{11}) = -0.783(k_0b) - 0.001, \quad (6)$$

$$\Re(S_{12}) = -0.161(k_0b)^2 - 0.004(k_0b) - 0.0002, \quad (7)$$

$$\Im(S_{12}) = -0.221(k_0b) - 0.0003. \quad (8)$$

The coefficients in the above equations are rounded to enhance readability. Eqn. (5)-(8) are valid for $k_0b \leq 0.16$, with k_0 the wavenumber of the lossless medium. The R^2-value of each fit is greater than 0.98.

2.2 Transfer matrix of the entire structure

Once the transfer matrix of each element is defined, the total transfer matrix $\underline{T}'$ of the $\lambda/4$ resonator can be calculated using Eqn. (9):

$$\underline{T}' = \left(\prod_{i=m}^{i=1} \underline{T}_i\right) \underline{T}_{EC}\underline{T}_{\phi}, \quad (9)$$

with $\underline{T}_i$ the transfer matrix of the i-th element, $\underline{T}_{EC}$ the end correction matrix and $\underline{T}_{\phi}$ the porosity matrix [11].

2.3 Absorption coefficient and resonance frequency

From $\underline{T}'$, the absorption coefficient α of the $\lambda/4$ resonator can be calculated using Eqn. (10):

$$\alpha = 1 - \left| \frac{T'_{21} + \frac{T'_{22}}{Z_0}}{\frac{T'_{22}}{Z_0} - T'_{21}} \right|^2, \quad (10)$$

with Z_0 the characteristic impedance of the lossless medium [7]. The resonance frequency f_R of the resonator is the frequency at which α reaches its maximum value.

3. NUMERICAL VERIFICATION

The use of Eqn. (3)-(10) is verified by comparing the results of the TMM to the results of 3D finite element simulations for a wide variety of folded $\lambda/4$ resonators.

3.1 Approach

To verify the formulas presented in Section 2, various single-bend and two-bend $\lambda/4$ resonators are modelled in 3D in COMSOL Multiphysics. Each resonator has a square cross-section and is embedded in a square baffle, as shown in Fig. 1.

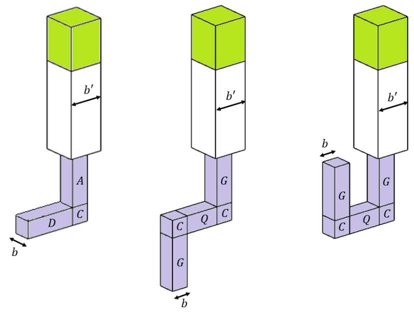

Figure 1. 3D model of a single-bend (left), Z-shaped two-bend (middle) and U-shaped two-bend (right) $\lambda/4$ resonator. Each capital letter indicates an acoustic element.

An overview of the modelled resonators is given in Tab. 1. In total, 54 unique single-bend and 192 unique two-bend resonators are modelled.

Table 1. Overview of the modelled $\lambda/4$ resonators. The porosity ϕ equals $(b/b')^2$.

Resonators with one 90° bend	
Porosity ϕ [-]	0.072
Centreline length L_{CL} [m]	0.06; 0.15; 0.35
Side length b [mm]	1; 2; 3; 4; 5; 6
L_A/L_{CL} [-]	0.1; 0.5; 0.75
L_D [m]	L_{CL} - b - L_A
Resonators with two 90° bends	
Folding shape [-]	U; Z
Porosity ϕ [-]	0.072
Centreline length L_{CL} [m]	0.06; 0.1; 0.25; 0.4
Side length b [mm]	1; 2; 3; 4; 5; 6
L_Q/b [-]	0.2; 0.6; 1; 5
L_G [m]	(L_{CL} - 2b - L_Q) / 2

For each resonator, the absolute relative error in predicted resonance frequency $|\Delta f_R|$ is calculated using Eqn. (11):

$$|\Delta f_R| = \left| \frac{f_R(T) - f_R(C)}{f_R(C)} \right| .100\%, \qquad (11)$$

with $f_R(C)$ the resonance frequency extracted from the finite element simulations and $f_R(T)$ the resonance frequency obtained using the TMM. For the single-bend resonators, the accuracy of the TMM is verified against the accuracy of two other modelling approaches: the length correction factor (LCF) and the model used to model unfolded resonators, which is from now on referred to as the basic analytical model (BAM) [4,6]. For the two-bend resonators, the accuracy of the TMM is only verified against the accuracy of the BAM since the LCF is not applicable.

3.2 Single-bend resonators

The results for the single-bend resonators are visualised in Fig. 2. In this figure, it is seen that the TMM yields similar accuracy values as the LCF. Since the latter was successfully validated in earlier work [6], the use of the formulas presented in Section 2 is justified. It can also be seen that both methods yield significantly higher accuracies than the BAM. The maximum value of $|\Delta f_R|$ when applying the TMM is 0.18%.

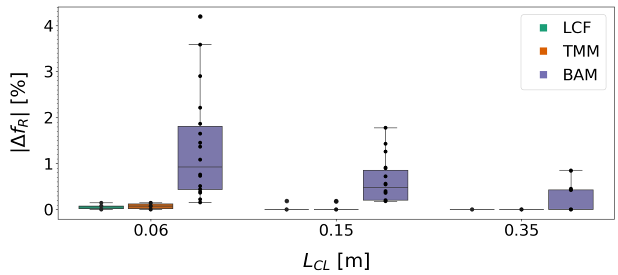

Figure 2. Results for the 54 single-bend resonators.

3.3 Two-bend resonators

The results for the two-bend resonators are visualised in Fig. 3. In this figure, it is seen that (i) the TMM yields significantly higher accuracies than the BAM and (ii) similar accuracy values are achieved for U- and Z-shaped resonators. The maximum value of $|\Delta f_R|$ when applying the TMM is 0.56%.

When looking into the values of $|\Delta f_R|$ in more detail, it is found that, for fixed values of b and L_{CL}, $|\Delta f_R|$ increases when the distance between the bends decreases. This holds for both U- and Z-shaped resonators and implies that interaction effects occur between closely spaced bends. However, since the TMM still yields accurate results, no additional correction is included to account for these effects.

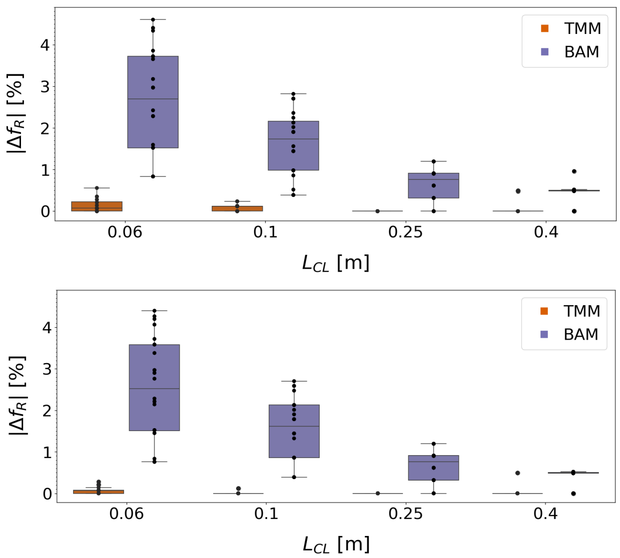

Figure 3. Results for the U-shaped (top) and Z-shaped (bottom) resonators.

4. CONCLUSIONS AND OUTLOOK

In this work, the TMM is applied to determine the resonance frequency of various square $\lambda/4$ resonators with one or two 90° bends and the results are verified against other modelling approaches and 3D finite element simulations in COMSOL Multiphysics. It is found that the TMM yields accurate results: the maximum absolute relative error in predicted resonance frequency is 0.18% and 0.56% for single-bend and two-bend resonators, respectively. In a next step, the use of the TMM will be verified for square $\lambda/4$ resonators with more than two 90° bends and experimental validation will be performed on 3D-printed samples.

5. ACKNOWLEDGMENTS

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