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# TUNABLE ENVELOPE SOLITONS IN A PIEZOELECTRIC METAMATERIAL OBEYING THE NONLINEAR SCHRÖDINGER EQUATION

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### ABSTRACT

Nonlinear shunt circuits can substantially alter the dynamic response of piezoelectric structures, offering a great potential to use in metamaterials and metastructures for wave manipulation. These circuits, implemented as analog or digital shunts, offer tunable and programmable nonlinear characteristics. In this work, we investigate the solitary wave dynamics in a metamaterial beam with a periodic array of piezoelectric patches shunted to Duffing circuits. Our focus is placed on the tunability of the solitary wave propagation by varying the linear and nonlinear inductance values. The wave evolution in the proposed piezoelectric metamaterial beam follows the nonlinear Schrödinger equation (NLSE), where the dispersion and nonlinear coefficients are dependent on the inductance parameters. By adjusting the linear inductance, the dispersion curve can be tuned, enabling the generation of solitary waves with either softening nonlinearity or hardening nonlinearity. The amplitude and shape of the solitary wave can also be controlled through the variation of inductance values. These findings may open new avenues for programmable elastic/acoustic metamaterials, with potential applications in elastic wave control, targeted energy transfer, and physical intelligence.

**Keywords:** piezoelectric metamaterial, nonlinear dynamics, nonlinear Schrödinger equation, solitons

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### 1. INTRODUCTION

Shape-invariant solitary waves in nonlinear mechanical metamaterials attracted significant attention due to its ability to propagate without dispersion, making them valuable for energy and information transmission. While previous research on solitons in mechanical metamaterials have mainly focused on the structural design with geometric nonlinearity, recent advances in piezoelectric structures [1] highlight the potential for controlling structural responses by introducing the source of nonlinearity through cubic inductive shunts. Such nonlinear shunt circuits can be implemented using either analog circuits or digital signal processing-based synthetic impedance circuits. Compared to purely structural nonlinearity, these nonlinear circuits offer a great flexibility of wave manipulation by controlling mechanical responses through electric signals. In this work, we investigate the tunability of solitary wave motion in a piezoelectric metamaterial beam [2] with an array of piezoelectric patch pairs shunted to linear and cubic inductive circuits. We exclusively focus on how variations in linear and cubic nonlinear inductance influence soliton formation and propagation.

### 2. SYSTEM DESCRIPTION

We consider a piezoelectric metamaterial beam with unit cells made of two piezoelectric layers sandwiching a central substrate layer (i.e. bimorph setting) as shown in Fig. 1. The piezoelectric layers are oppositely poled in different directions and segmented periodically. The surface electrodes of each unit cell are connected to a nonlinear shunt circuit with a linear inductance and a cubic nonlinear inductance (Fig. 1). Based on the Euler-Bernoulli

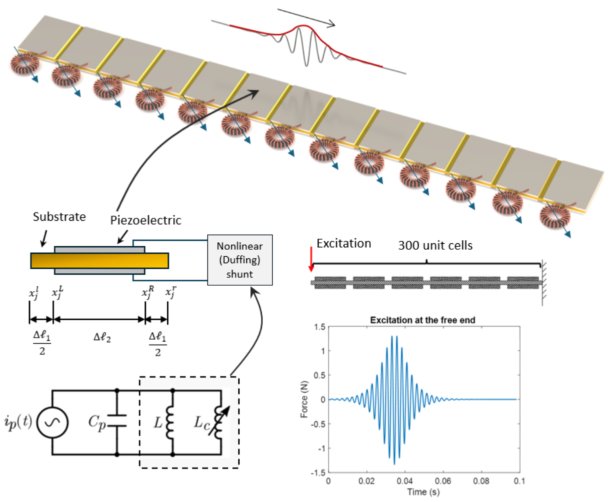


**Figure 1**. System overview depicting the metamaterial beam, unit cell detail, the excitation and its location on the beam with free-fixed boundary conditions.

beam theory, the undamped flexural wave equation of the beam with electromechanical coupling and circuit current balance equations are:

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}(EI(x)\frac{\partial^2 w}{\partial x^2})+m(x)\frac{\partial^2 w}{\partial t^2}\\&-\vartheta\frac{\partial^2}{\partial x^2}\sum_{j=1}^{N}v_j\left(t\right)\left[H\left(x-x_j^L\right)-H\left(x-x_j^R\right)\right]=0\end{aligned}\tag{1a}$$

$$\begin{aligned}&C_{p,j}\dot{v}_j+\frac{1}{L_j}\int_0^t v_j\left(\tau\right)d\tau\\&+\frac{1}{L_{cj}}\left(\int_0^t v_j\left(\tau\right)d\tau\right)^3+\vartheta\int_{x_j^L}^{x_j^R}\frac{\partial^3 w}{\partial x^2\partial t}dx=0\end{aligned}\tag{1b}$$

where the $j-\mathrm{th}$ pair of piezoelectric patch spans the spatial positions $x_j^L$ to $x_j^R$. The transverse deflection $w(x,t)$ and the voltage in the $j-th$ shunt circuit $v_j$ are the unknown variables in Eq. (1). The distribution of mass $m(x)$, flexural rigidity $EI(x)$, electromechanical coupling $\vartheta$ and capacitance $C_{p,j}$ are dependent on the material and structural properties of the beam, while the linear and nonlinear inductance $L_j$ and $L_{cj}$, can be independently designed on the circuit side, allowing tunability of the system's response.

Previous analysis in [2] has shown that a continuous and homogeneous approximation can be applied to the foregoing integro-differential equations, converting the governing equations to a simpler nonlinear partial differential equation in the long wavelength limit:

$$\bar{m}\frac{\partial^2\bar{w}}{\partial t^2}+\bar{EI}\frac{\partial^4\bar{w}}{\partial x^4}-\bar{\vartheta}\frac{\partial^3\bar{\lambda}}{\partial x^2\partial t}=0\tag{2a}$$

$$\bar{C}_p\frac{\partial^2\bar{\lambda}}{\partial t^2}+\frac{1}{\bar{L}}\bar{\lambda}+\frac{1}{\bar{L}_c}\bar{\lambda}^3+\bar{\vartheta}\frac{\partial^3\bar{w}}{\partial x^2\partial t}=0\tag{2b}$$

where an overbar represents homogenized variables, and $\bar{\lambda}$ denotes the flux linkage, defined as the time integral of voltage over time. $\bar{L}$ and $\bar{L}_c$ are the linear and cubic nonlinear inductance scaled over an entire piezoelectric patch given by:

$$\begin{aligned}&\bar{L}=L_j\Delta\ell;\;\;\bar{L}_c=L_c\Delta\ell\\&\Delta\ell=x_j^R-x_{j-1}^R=x_j^L-x_{j-1}^L\end{aligned}\tag{3}$$

Under weakly nonlinear assumption, the method of multiple scales is introduced to solve the nonlinear partial differential equations given by Eq. (2):

$$t_i=\varepsilon^i t;x_i=\varepsilon^i x,i=0,1,2,...\tag{4}$$

where $\varepsilon$ is a small bookkeeping parameter. The leading-order solution of Eq. (2) is the linearized plane wave solution:

$$\begin{pmatrix}\bar{w}\\\bar{\lambda}\end{pmatrix}\approx\varepsilon A_1\left(x_1,t_1,x_2,t_2\right)\begin{pmatrix}C_{w1}\\C_{\lambda1}\end{pmatrix}e^{i(kx_0-\omega t_0)}+c.c.\tag{5}$$

where $(C_{w1},C_{\lambda1})^T$ denote the mode shape, and c.c. denotes the complex conjugate. $\omega$ and $k$ denote the frequency and wavenumber of the plane wave that obeys the dispersion relation. Past research [3] reveals that the linearized dispersion relation of the piezoelectric metamaterial exhibits a frequency bandgap near the resonant frequency of the shunt circuit.

The modulation equation of the envelope $A_1$ can be solved by substituting Eq. (5) back into Eq. (2). Perturbation analysis in [2] shows that the modulation of $A_1$ follows the nonlinear Schrödinger equation (NLSE) that is well known to support solitary wave solutions [4]:

$$i\frac{\partial A_1}{\partial\tau_2}+P\frac{\partial^2 A_1}{\partial\xi_1^2}+QA_1^2A_1^*=0\tag{6}$$

where $\xi_i$ and $\tau_i$ are coordinate variables in the frame moving at the group velocity $v_g$,

$$\xi_i=x_i-v_g t_i;\;\tau_i=t_i,\tag{7}$$

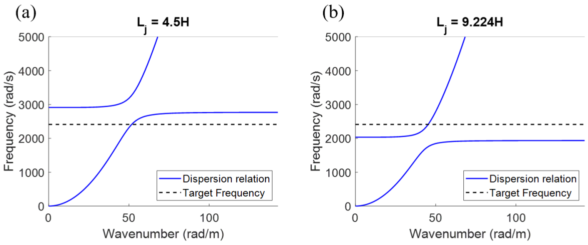


**Figure 2**. Linearized dispersion relation for different linear inductance values (a) $L_j = 4.5\mathrm{H}$ where the excitation frequency hits the acoustic band and (c) $L_j = 9.224\mathrm{H}$ where the excitation frequency hits the optical band.

Furthermore, $P$ and $Q$ quantify the effect of dispersion and nonlinearity:

$$P = \frac{1}{2}\frac{\mathrm{d}^2\omega}{\mathrm{d}k^2} \tag{8a}$$

$$Q = -\frac{3\left|C_{\lambda 1}\right|^4}{L_c \begin{pmatrix} C_{w1}^* & C_{\lambda 1}^* \end{pmatrix} \begin{pmatrix} 2\bar{m}\omega & i\bar{\vartheta}k^2 \\ -i\bar{\vartheta}k^2 & 2\bar{C}_p\omega \end{pmatrix} \begin{pmatrix} C_{w1} \\ C_{\lambda 1} \end{pmatrix}}, \tag{8b}$$

where the superscript $*$ denotes the complex conjugate.

## 3. RESULTS AND DISCUSSIONS

We investigate the tunability of solitons by numerically simulating a cantilever beam with 300 pairs of piezoelectric patches as shown in Fig. 1 inset. The excitation is applied at the free end of the cantilever beam with the profile shown in Fig. 1. The pulse has a sech-shaped envelope, characteristic of NLSE soliton solutions, and oscillates at a frequency close to 2500 $\mathrm{rad/s}$.

We first examine the dispersion curves for different values of linear inductance, which influence the frequencies of the LC (inductive-capacitive) circuit dynamics. Since the frequency bandgap of the metamaterial beam is close to the circuit resonant frequency, varying the linear inductance shifts the bandgap accordingly. In Fig. 2, we present the linearized dispersion curves of the metamaterial beam, as developed in [5], for linear inductance values of $4.5\mathrm{H}$ and $9.224\mathrm{H}$. The figure illustrates the tunability of the frequency bandgap, which can be shifted by varying the inductance—allowing the excitation frequency to lie within either the acoustic or the optical band.

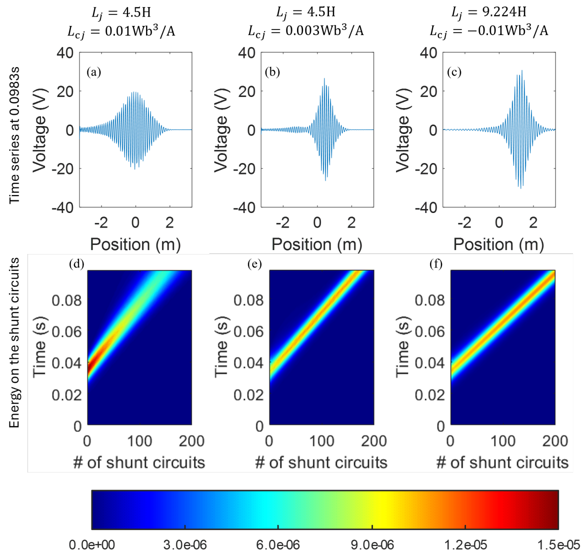


**Figure 3**. Distribution of the voltage across the metamaterial beam: (a,b,c) Time series of voltage across all entire shunt circuits at $t = 0.0983\mathrm{s}$ and (d,e,f) Spatio-temporal variation of the energy in each shunt circuits. The linear and nonlinear inductances are given by, (a,d) $L_j = 4.5\mathrm{H}$ and $L_{cj} = 0.01\mathrm{Wb}^3/\mathrm{A}$, (b,e) $L_j = 4.5\mathrm{H}$ and $L_{cj} = 0.003\mathrm{Wb}^3/\mathrm{A}$ and (c,f) $L_j = 9.224\mathrm{H}$ and $L_{cj} = -0.01\mathrm{Wb}^3/\mathrm{A}$.

The excitation frequency falls in different branches of the dispersion curve, resulting in distinct solitary wave solutions. As shown in [4], the NLSE supports solitary wave solution only with the focusing nonlinearity condition $PQ > 0$, where the nonlinearity counterbalance the effect of dispersion. Eq. (8) shows that $P$ is proportional to the second derivative of the dispersion curve, while $Q$ always has the opposite sign of the cubic inductance $L_{cj}$. To initiate a solitary wave, a softening nonlinear inductance is required in the acoustic band where the dispersion relation is concave upwards, whereas hardening nonlinear inductance is required in the optical band where the dispersion relation is concave downward.

To confirm our analysis, we present finite-element simulation results of the entire metamaterial beam, considering the linear inductances $4.5\mathrm{H}$, and $9.224\mathrm{H}$, combined with various nonlinear inductance values $L_{cj} = \pm 0.01\mathrm{Wb}^3/\mathrm{A}$, and $0.003\mathrm{Wb}^3/\mathrm{A}$. The voltage and energy distributions across the shunt circuits of the entire metamaterial beam are presented in Fig. 3.

The simulation results confirm the theoretical analysis on soliton's tunability. Fig. 3a and 3d present the simulation results of $L_j = 4.5\mathrm{H}$ and $L_{cj} = 0.01\mathrm{Wb}^3/\mathrm{A}$. With an excitation frequency in the acoustic branch and a hardening nonlinearity, the wavepacket maintains a solitary-like bell shape. However, from Fig. 3d, we note that the energy gradually disperses as the wavepacket propagates along the metamaterial beam. A stronger nonlinear focusing effect is observed with a larger nonlinear coefficient. As shown in Fig. 3e, reducing the nonlinear inductance to $L_{cj} = 0.003\mathrm{Wb}^3/\mathrm{A}$ enhances the nonlinearity, resulting in reduced energy dispersion during propagation. In Fig. 3b, the wavepacket is much narrower compared with the one in Fig. 3a as the nonlinear coefficients are increased.

In addition, Figs. 3c and 3f present simulation results for $L_j = 9.224\mathrm{H}$ and a softening nonlinear inductance of $L_{cj} = -0.01\mathrm{Wb}^3/\mathrm{A}$. These results confirm the initiation of a solitary-like wavepacket when the excitation frequency lies within the optical branch and the system exhibits softening nonlinearity. Overall, the simulations highlight the high tunability of solitary wave responses—demonstrating that by varying the linear and nonlinear inductance, the frequency bandgap can be shifted and the nonlinearity can switch between hardening and softening, enabling the effective generation of solitons in both the acoustic and optical bands.

## 4. CONCLUDING REMARKS

To summarize, we propose a piezoelectric metamaterial beam coupled with nonlinear shunt Duffing circuits consisting of both linear and cubic nonlinear inductance. We investigate the effects of these inductance values on the solitary wave propagation within the metamaterial beam, demonstrating strong tunability of solitary waves. Specifically, varying the linear inductance shifts the linearized frequency bandgap, allowing excitation of either the acoustic band, optical band, or bandgap under identical external excitation. Since the dispersive effect is dependent on the second derivative of the dispersion curve, different types of nonlinearity supports distinct soliton solutions. In particular, hardening nonlinearity supports solitons in the acoustical band, whereas softening nonlinearity supports solitons in the optical band. Furthermore, since the soliton is formed due to the counterbalance of nonlinearity, we demonstrate that the steady-state profile of the soliton can be tuned by varying nonlinear inductance. With identical excitation, a larger nonlinear coefficient leads to stronger focusing effect, resulting in a narrower wavepacket. By tuning the linear and nonlinear inductance, solitons with varying amplitudes and widths can be achieved under the same excitation profile.

## 5. ACKNOWLEDGMENTS

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