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UNCERTAINTIES AND EFFICIENCY OF BAYESIAN DECAY ANALYSIS

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ABSTRACT

Bayesian methods have been widely applied in vibro-acoustics, architectural acoustics and signal processing in acoustics. Bayesian framework is capable of not only parameter estimation, optimization (lower level of inference), but also model selection (higher level of inference). These two levels of inference have already substantiated their wide range of applicability in acoustics. Furthermore, Bayesian probabilistic analysis is naturally useful for quantification of uncertainties. This work focuses on analyzing sound energy decays often encountered in experimentally measured data. Several advanced methods, such as nonlinear regressions, Bayesian methods, and artificial neural networks have been developed to cope with energy decay analysis being challenging in vibro-acoustics and air-borne sound in enclosures. Using these methods, a wide range of data resolutions can meet the need of energy decay analysis. Yet for high efficiency, lower resolutions may be preferable, still adequately representing energy decay processes. This paper discusses conditions of representing energy decay processes by desirable, sufficiently less data points for higher efficiency of data analysis. At the same time, increased efficiency inevitably brings uncertainties. Using Bayesian decay analysis in foreground, the analysis uncertainties are investigated against those of experimental measurements. This paper quantifies uncertainties for leveraging adequate accuracies and the analysis efficiency.

Keywords: *Uncertainty quantification, Bayesian analysis, sound energy decay, probability theory*

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1. INTRODUCTION

Decay analysis is vital in structural acoustics and room-acoustics systems. Bayesian inference has long been developed and applied in energy decay analysis from beginning of 2000s [1] until more recently [2]. The wide-spread application has been impeded by seemingly challenging methodology, particularly computational concerns. In fact all the concerns are unnecessary. This paper first derives a necessary condition of energy decay data using Schroeder integrated method [3]. The condition ensures that a handful data points on order of hundreds discrete values are sufficiently accurate to represent the Schroeder decay functions. This dispels misconception and misunderstanding that the Bayesian decay analysis be computationally prohibitive. In opposite, Bayesian inferential method can be computationally efficient with acceptably low uncertainties. This work quantifies the uncertainties of Bayesian analysis using low enough temporal resolutions to achieve computational efficiency. It also compares with those associated with experimental uncertainties.

2. ENERGY DECAY FUNCTION AND MODEL

This work analyzes sound energy decays using the integrated impulse method by Schroeder [3] to obtain an energy decay function $D(t_k)$ by integrating the squared impulse response $h(t)$ between a source and a receiver:

$$D(t_k) = \sum_{\tau=t_k}^{t_K} h^2(\tau), \quad \text{for} \quad 0 \le k < K, \tag{1}$$

with K being the total number of discrete point of the resulting decay function $D(t_k)$ and t_k being discrete time. Integer index k also implicitly denotes the time increment with a finite temporal resolution.

2.1 Schroeder decay model

Xiang *et al.* [4, 5], among other, has applied a parametric model

$$H_S(\boldsymbol{\Theta}, t_k) = \theta_0(t_k - t_K) + \sum_{s=1}^{S} \theta_{(2s-1)} \left[\mathrm{e}^{-\theta_{2s}\, t_k} - \mathrm{e}^{-\theta_{2s}\, t_K} \right], \quad (2)$$

where the upper limit of the summation S represents the number of decay rates/slopes potentially contained in the Schroeder decay data, $A_i = \theta_{(2s-1)}$ with $s = 1, 2, \ldots$ represents the initial amplitudes of exponential decays, while $T_s = 13.8/\theta_{2s}$ represent decay time parameters. Sufficient representations of Schroeder decay functions are of vital importance for the analysis accuracy and computational efficiency. The following Section outlines the necessary condition for the sufficient representation of Schroeder decay functions.

2.2 Condition of the Integral resolution

The sufficient representations of Schroeder decay functions essentially rely on *Nyquist*'s sampling theory. So far it has not been documented in the major literature in room-acoustics, but a conference presentation [6]. The Schroeder decays contain predominantly a sum of exponential decays, while the first term in Eq. (2) is a linearly decaying function often with an insignificantly low value of θ_0. When examining decay process, this term represents a much slower changing rate, therefore, it can be ignored. Furthermore, for multi-slope decays, acousticians are primarily concerned with the ordered decay parameters $T_1 < T_2, \ldots$ with $A_1, A_2, \ldots$.

When *Fourier* transforms the decay model in Eq. (2), the frequency spectrum of the decay functions of the single-slope decay is sufficient for the condition [6]. The normalized magnitude spectrum leads to a simplified relation

$$\frac{2\pi\, T_1\, f}{13.8} \approx \frac{T_1\, f}{2} > 20, \quad (3)$$

where a ratio of 20 is based on 10 times of 'half-power' point along the magnitude spectra in Fig. 1 (see Ref. [6] and Sec 4 for more elaborated discussion). Equation (3) yields an upper limit of frequency f_{u} and the decay sampling rate f_{d} to discretise the Schroeder decay

$$f_{\mathrm{u}} \geq \frac{40}{T_1}, \quad \text{and} \quad f_{\mathrm{d}} \geq 2.5\, f_{\mathrm{u}} \geq \frac{100}{T_1}. \quad (4)$$

This represents the condition for the Schroeder integral to be accomplished at a sufficient temporal resolution $t_{\mathrm{d}} = 1/f_{\mathrm{d}}$. It allows the Schroeder integral in Eq. (1) in a relatively larger time step (low temporal resolution), much larger than the sampling frequency f_{s} when initially acquiring the impulse response $h(t)$.

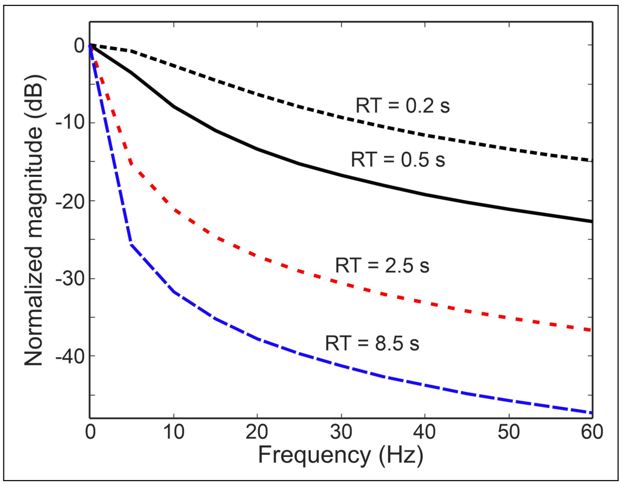

Figure 1. Normalized magnitude spectra of the simplified decay model for $T_1 = 0.2, 0.5, 2.5$ and 8.5 seconds.

3. MODEL-BASED BAYESIAN INFERENCE

Bayesian analysis has been applied to the energy decay analysis [5], where Bayes theorem plays a pivotal role in the process

$$\text{posterior} = \frac{\text{likelihood}\,(\mathcal{L}) \times \text{prior}}{\text{evidence}}. \quad (5)$$

Two probabilistic quantities in the nominator on the right-hand side, *likelihood function* $\mathcal{L}$, and *prior* are of prior probability in nature. They must be assigned according to available information /knowledge [7] at hand. Within Bayesian framework, the prior assignment applies *Shannon-Jaynes* entropy to the prior probabilities, since the entropy is a quantitative measure of uncertainties. To ensure the most non-committal assignment, the best way is to maximize the entropy, this is so-called the principle of maximum entropy (MaxEnt). The MaxEnt leads to a Gaussian likelihood [7], which takes the form of Student-

T distribution [1]

$$\mathcal{L} \propto \left(\sum_{k=0}^{K-1} \epsilon_k^2 \right)^{-K/2}, \quad (6)$$

where the sum inside $(\cdot)$ is the total error energy with

$$\epsilon_k = D(t_k) - H(\mathbf{\Theta}, t_k) \quad (7)$$

being the residual errors between the decay data $D(t_k)$ in Eq. (1) and the model $H(\mathbf{\Theta}, t_k)$ in Eq. (2) . The formulation features the fully parameterized decay model as in Eq. (2). For a S-number of decay slopes, it contains $2\,S + 1$ decay parameters, including S decay times being of primary interest in acoustics applications.

4. UNCERTAINTIES VS. EFFICIENCY

Within Bayesian framework, the probability theory readily provides probabilistic calculus to estimate mean values, variances and standard deviations to quantify uncertainties. As discussed in Sec. 2.2, the spectra slowly decay with increasing frequency as shown in Fig. 1, a ratio of 20 in Eq. (3) conservatively ensures more spectral content of decay functions being included in the analysis. Table 1 lists different temporal resolutions when evaluating an experimentally measured Schroeder decay function at 1 kHz (oct) as an example. The Schroeder decay functions are evaluated using different resolutions (see Table 1), a temporal resolution of 20 ms/pt still fulfills the condition as stated in Eq. (4). This temporal resolution corresponds to a set of discrete values with $K = 130$ data points. The lower the resolution, the smaller the number of data points involved in the decay analysis.

4.1 Uncertainties in Bayesian analysis

Within the Bayesian framework, a smaller number of data points leads to less computational expense, yet larger uncertainties of decay analysis. The Student-T distribution in Eq. (6) quantitatively reflects this relationship of the analysis uncertainty. Inside the exponent of the total error energy, the larger number of points K, the narrower the likelihood /posterior distribution becomes. The standard deviation $\sigma_{\rm T}$ represents a quantitative measure of the narrowness, therefore, the uncertainty. Using experimentally measured data in Ozawa Hall, Lenox, MA, USA, Figure 2 illustrates the posterior distributions at different data resolutions when the prior is assigned to a uniform distribution. [1] Figure 3 illustrates the mean reverberation time T_μ and relative standard deviations $\sigma'_{\rm T} = \sigma_{\rm T}/T_\mu$ at different temporal resolutions. When evaluation probabilistic quantities using Bayes theorem on order of a few hundreds data points (temporal resolutions ranging between 2 and 20 ms/pt), Bayesian decay analysis costs computational expense on the currently available laptop PCs only within a tiny fraction of second.

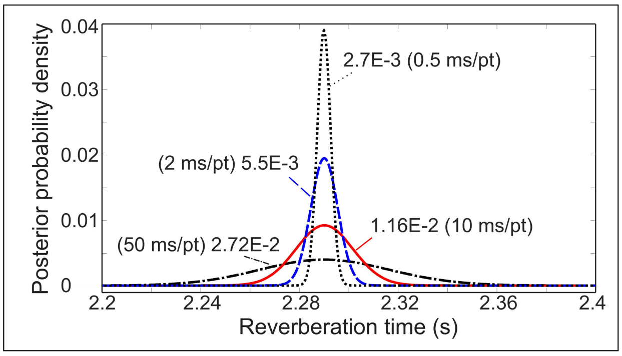

Figure 2. Posterior probability densities (PPDs) evaluated from experimentally measured decay functions in Seiji Ozawa Hall USA. Four different PPDs are quantified by the standard deviation.

Table 1. Time resolution, number of data points (K) used for Bayesian analysis, and the reverberation time in its mean value and standard deviation.

$t_{\rm d}$ (ms/pt)	# of points	Mean RT $\pm\,\sigma_{\rm T}$
50	54	2.264 ±1.82E-2
20	130	2.148± 1.1E-2
10	273	2.281± 6.6E-3
5	546	2.284± 4.8E-3
2	1365	2.283± 3.1E-3
1	2730	2.283± 2.2E-3
0.5	5461	2.283± 1.5E-3

[1] For ease of comparison, the distributions in Fig. 2 are aligned up at their peaks due to random nature of probabilistic distributions.

4.2 Analysis vs. Experimental Uncertainties

However, reducing the number of points (K) increases analysis uncertainties. From the third row in Table 1 downwards, the relative standard deviations fall below 5.1E-3, getting even higher with decreased resolutions.

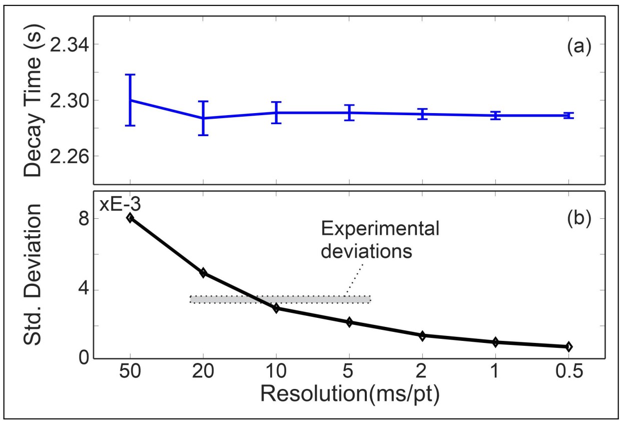

Figure 3. Means and standard deviations of reverberation time estimated from a Schroeder decay function experimentally measured as listed in Table 1. (a) Means with $\pm$ standard deviations by Bayesian decay analysis. (b) Relative standard deviations, $\sigma'_{\mathrm{T}} = \sigma_{\mathrm{T}}/T_{\mu}$. Experimental uncertainties fall within the shaded range.

Although the architectural acoustics practice often requires the analysis uncertainty of decay times no more than $1 - 2\%$, it is of practical significance to validate the sampling condition in Eq. (4) by comparing the analysis uncertainties with the experimental ones. For this purpose, ten repetitive measurements at the same receiver location in Ozawa Hall are consecutively conducted within a short period of time (less than 6 min). The experimental uncertainties of the reverberation time manifest itself in form of the relative standard deviations using the least-square fitting after a noise compensation (as single-slope).

In this example, the least square fit engages temporal resolutions between 1-4 ms /pt. The relative standard deviations of the experimental variations range between 2.97E-3 - 3.26E-3 as marked in Fig. 3 (b) for ease of comparison with the analysis uncertainties. This comparison indicates that the temporal resolutions of 5 - 20 ms/pt will result in the analysis uncertainties that are on similar order of the experimental ones, particularly at resolution of 10 ms/pt.

5. CONCLUDING REMARKS

This work investigates analysis uncertainties within Bayesian framework. Using single-slope decay model, and experimentally measured data in a performing arts venue, the analysis uncertainties are comparable with those experimental ones with the temporal resolutions in favor of high efficiency of Bayesian computations. Intentional selection of higher resolutions will favor higher accuracies, particularly for double- and multi-slope decay analysis, yet at cost of increased computational cost since more data points are involved. Even then, discrete data of a few hundreds values still cost computational expense within a fraction of second. This investigation dispels unnecessary concerns and clarifies misunderstanding that the Bayesian decay analysis be computationally prohibitive. In opposite, Bayesian inferential method can be computationally efficient with acceptably low uncertainties.

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