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# TWO-STAGE SOUND FIELD LEARNING THROUGH HOLOMORPHIC NEURAL NETWORKS AND VEKUA OPERATORS

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## Abstract

Holomorphic neural networks have recently been proposed as two-dimensional universal function approximators constrained to produce holomorphic outputs. In this work, we combine holomorphic neural networks with Vekua operators to construct a machine learning architecture that inherently generates solutions to the Helmholtz equation. This approach is particularly advantageous for two-dimensional boundary value problems in room acoustics at low frequencies, as training can be restricted to enforcing only boundary conditions, resulting in substantial speed-ups and improved accuracy compared to traditional physics-informed neural networks. Possible applications include real-time sound field prediction, inverse design and surrogate modelling.

**Keywords:** *Vekua operators, Helmholtz equation, room acoustics, physics-informed neural networks*

## 1. Introduction

Artificial neural networks have proven particularly effective at embedding physical laws into the learning process [1], [2], [3]. Specifically, physics-informed neural networks (PINNs) enforce consistency with governing partial differential equations by incorporating their residuals as penalty terms in the loss function, evaluated at selected sampling points [4]. This approach leverages automatic differentiation (AD) to compute residuals exactly, and the backpropagation algorithm to train the networks until convergence. However, training often remains challenging and inefficient due to the large number of evaluation points required and the complexity of the resulting physics-based loss landscape [5].

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To overcome these difficulties, a new paradigm has emerged in which neural networks are constructed so that their outputs automatically satisfy the underlying physical constraints in a Trefftz-like manner [6]. In particular, Calafà et al. [7], [8] introduced holomorphic neural networks (HNNs) as a framework for inherently satisfying the linear elasticity equations in solid mechanics, demonstrating substantial performance speed-up over traditional physics-informed machine learning by training the model solely on boundary conditions. More recently, the approach has been expanded to a wider family of elliptic problems [9].
In this work, we present a rigorous methodology for applying the holomorphic framework to problems governed by the Helmholtz equation. Therefore, this approach has potential applications across a wide range of fields, including acoustics, optics, quantum mechanics, electromagnetism, and surface fluid mechanics [10], [11], [12], [13], [14]. However, in the present work, we primarily focus on acoustics, where the development of efficient sound simulation tools is of particular importance [15], [16], [17].

## 2. Method

We introduce holomorphic neural networks in the original multilayer perceptron architecture (MLP) [7], despite its Kolmogorov-Arnold alternative is also possible [9].
A complex-valued fully-connected feed-forward MLP neural network (CVNN) [18] can be defined as

$$\mathrm{MLP}(z) = \mathcal{T}_L \circ \sigma \circ \mathcal{T}_{L-1} \circ \cdots \circ \sigma \circ \mathcal{T}_1(z),$$

where $\mathcal{T}_l$ denotes the $l$-th linear layer, $\sigma : \mathbb{C} \to \mathbb{C}$ is a non-linear activation function and $L \in \mathbb{N}$ is the network depth. In particular, each layer is defined by a weight matrix $\mathbf{W}_l \in \mathbb{C}^{M_l, M_{l-1}}$ and a bias vector $\mathbf{b}_l \in \mathbb{C}^{M_l}$, such that $\mathcal{T}_l(\mathbf{z}_l) = \mathbf{W}_l \mathbf{z}_l + \mathbf{b}_l$, where $\mathbf{z}_l$ is the input vector to the $l$-th layer.
In standard CVNNs, $\sigma(\cdot)$ is typically chosen as split

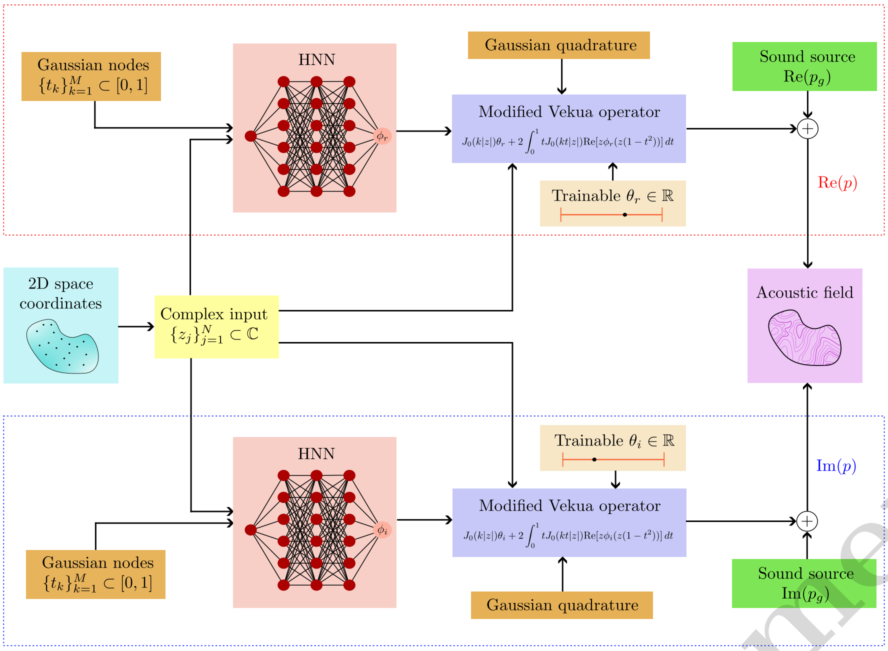


**Figure 1:** Visualization of the proposed neural network framework.

activation function $\sigma(z) = \tilde{\sigma}(\mathrm{Re}(z)) + i\tilde{\sigma}(\mathrm{Im}(z))$, being $\tilde{\sigma}(\cdot)$ an activation function commonly used for real-valued networks, such as ReLU or the sigmoid [19]. In contrast, HNNs are defined as CVNNs where activation functions are holomorphic, i.e. the complex differential

$$\lim_{z \to z_0} \frac{\sigma(z) - \sigma(z_0)}{z - z_0},$$

exists for every $z_0$. It has been proven that such architecture generates exclusively holomorphic functions and can approximate any such function to arbitrary accuracy on simply-connected domains, making HNNs universal holomorphic function approximators [20].

Let $\Omega \subset \mathbb{R}^2$ be a two-dimensional, compact and simply-connected domain. The Helmholtz problem can be stated as

$$\nabla^2 p + k^2 p = g, \quad \text{in } \Omega, \tag{1}$$

where $p : \Omega \to \mathbb{C}$ is the acoustic pressure, $g : \Omega \to \mathbb{C}$ is the sound source and $k = 2\pi f/c$ is the wave number associated to the frequency $f > 0$ and speed of sound $c > 0$.

Equation (1) reduces to the Laplace equation when $k = 0$, whose solutions for $g = 0$ are known as harmonic functions. In particular, it is well-known that for any harmonic function $u$ there exists a holomorphic function $\varphi$ such that $u(\mathbf{x}) = \mathrm{Re}(\varphi(z))$, where $\mathbf{x} = (x, y)$ denotes the 2D space coordinates associated to $z = x + iy$.

We introduce the *Vekua* operator

$$V_k[u](\mathbf{x}) = u(\mathbf{x}) - \int_0^1 \frac{J_1(k|\mathbf{x}|\sqrt{1-t})}{2\sqrt{1-t}} k|\mathbf{x}|u(t\mathbf{x})\,dt, \tag{2}$$

where $J_n(\cdot)$ is the $n$-th order Bessel function of the first kind. Such operator was originally introduced by Vekua [21] and, under certain hypotheses, is proven to be a bijection between the space of harmonic functions and that of homogeneous Helmholtz solutions [22].

By applying integration by parts to Equation (2), performing the change of variables $\tilde{t} = \sqrt{1-t}$, and formulating in terms of the potential $\varphi$, we obtain

$$\begin{aligned} p_0(\mathbf{x}) &= J_0(k|z|)\mathrm{Re}(\varphi(0)) \\ &\quad + 2\int_0^1 tJ_0(kt|z|)\mathrm{Re}(z\varphi'(z(1-t^2)))\,dt, \end{aligned} \tag{3}$$

which provides a complete representation of real-valued and homogeneous solutions to Equation (1). In particular, $\varphi'$ exists and is holomorphic due to the properties of $\varphi$. Consequently, $\varphi'$ can be modeled through the output of a HNN. Additionally, $\theta = \mathrm{Re}(\varphi(0))$ is a real constant independent from $\varphi'$, and may therefore be treated as a scalar trainable parameter.

The representation through the modified Vekua operator in Equation (3) is preferred over the original in Equation (2), as it allows for faster evaluation and often exhibits better convergence properties. Note that complex-valued Helmholtz solutions require the application of Equation (3) separately for the real and imaginary components, as illustrated in Figure 1. Furthermore, Gaussian quadrature is employed to numerically compute the integrals.

By linearity, non-homogeneous solutions can be generated by $p = p_0 + p_g$, where $p_0$ is given from Equation (3) and $p_g$ is any particular solution to the Helmholtz problem with $g \neq 0$. For instance, in the presence of a point source $g(\mathbf{x}) = -\delta(\mathbf{x}-\mathbf{x}_0)$, a convenient choice is the fundamental solution $p_g(\mathbf{x}) = iH_0^{(1)}(k|\mathbf{x}-\mathbf{x}_0|)/4$, where $\delta(\cdot)$ is the Dirac delta function and $H_0^{(1)}(\cdot)$ is the zero-order Hankel function of the first kind.

To summarize, the representation in Equation (3) combined with HNNs entails a universal function approximator constrained as solutions to the 2D Helmholtz problem. Consequently, boundary value problems (BVP) can be solved by training the networks only to satisfy the boundary conditions. Accordingly, we define the loss function as

$$\mathcal{L}(p) := \mathbb{E}_{\mathbf{x}\sim\mathcal{U}(\partial\Omega)}\left[\left(p(\mathbf{x}) - Z(\mathbf{x})v_{\mathbf{n}_\mathbf{x}}(\mathbf{x})\right)^2\right], \quad (4)$$

where $\partial\Omega$ is the boundary of the domain, $\mathbf{n}_\mathbf{x}$ is the unit vector at $\mathbf{x}$ that is outward normal to $\partial\Omega$, $Z(\cdot)$ is the prescribed acoustic surface impedance and the sound velocity can be obtained by $v_{\mathbf{n}_\mathbf{x}} = i(\nabla p \cdot \mathbf{n}_\mathbf{x})/(\rho c k)$, where $\rho$ is the density of the medium.

## 3. Results

The surrogate model and neural network framework introduced in the previous section are tested through a simple numerical benchmark.

The code has been implemented in Python 3.12. In particular, the original implementation of HNNs [7] has been readapted from PyTorch 2.2.2 [23] to JAX 0.7.0 [24] through the NNX Flax API. The benchmark test is run on a single NVIDIA GeForce RTX 5090, with 32 GB memory and CUDA 12.8.

We consider a squared domain $\Omega$ of lengths [1,1.4] m, in accordance to the Louden's ratio [25]. Air is selected as the sound medium, with density $\rho = 1.2$ kg/m$^3$ and associated speed of sound $c = 343$ m/s. The prescribed frequency-independent surface impedance is uniformly $Z = (10 - 10i)\rho c$ Pa·s/m on each wall, yielding an absorption coefficient of $\alpha \approx 0.18$, i.e. corresponding to rather acoustically hard materials. The room is excited with a sound point source $g(\mathbf{x}) = -\delta(\mathbf{x} - \mathbf{x}_0)$ at the coordinates $\mathbf{x}_0 = [0.2, 0.2]$ m with respect to the center of the domain.

1000 training points are uniformly distributed on the boundary of the square. Both HNNs contain two inner layers with 30 neurons each and employ complex exponential activation functions. Networks are initialized by following the ad-hoc algorithm proposed in [7]. The scalar parameters $\theta$ are instead initialized to zero. Adam [26] is employed as optimizer across the 2500 epochs with initial learning rate $5 \times 10^{-3}$. The number of Gaussian quadrature points is set to 20, as this is sufficiently high to yield a negligible quadrature error. In particular, the total input set is the cartesian product of the space coordinates and the quadrature nodes, contributing to $10^4$ complex numbers.

A reference solution is obtained analytically by the modal expansion described in [11]. However, modes are known explicitly only for perfectly reflecting walls. Because of this, a Newton's root-finding algorithm is applied to adjust them according to the prescribed surface impedance.

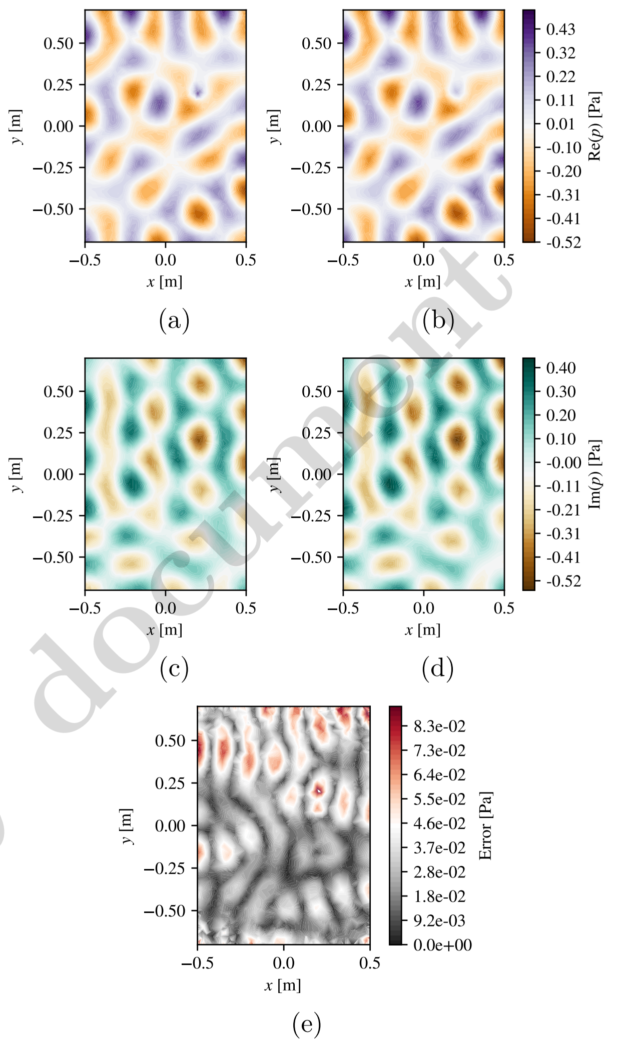


**Figure 2:** Contour plots of the acoustic field at 1200 Hz. (a) Predicted and (b) analytic Re($p$). (c) Predicted and (d) analytic Im($p$). (e) Absolute error between predicted and analytic solution. All color levels are in Pa.

A first test is run with $f = 1200$ Hz, which corresponds to $k \approx 22$ m$^{-1}$ and approximately 4 training points per wavelength (PPW). Contour plots are shown in Figure 2, where both the real and imaginary part of the acoustic pressure are nearly identical to the analytic reference. As expected, the only noticeable discrepancy occurs at the location of the point source, where the analytic solution diverges. In particular, the maximum relative error across the whole space is 10%.

Table 1 provides details on the computational cost of the proposed method. It can be noted that memory requirements are significantly low, mostly due to the limited number of network parameters and compact in-

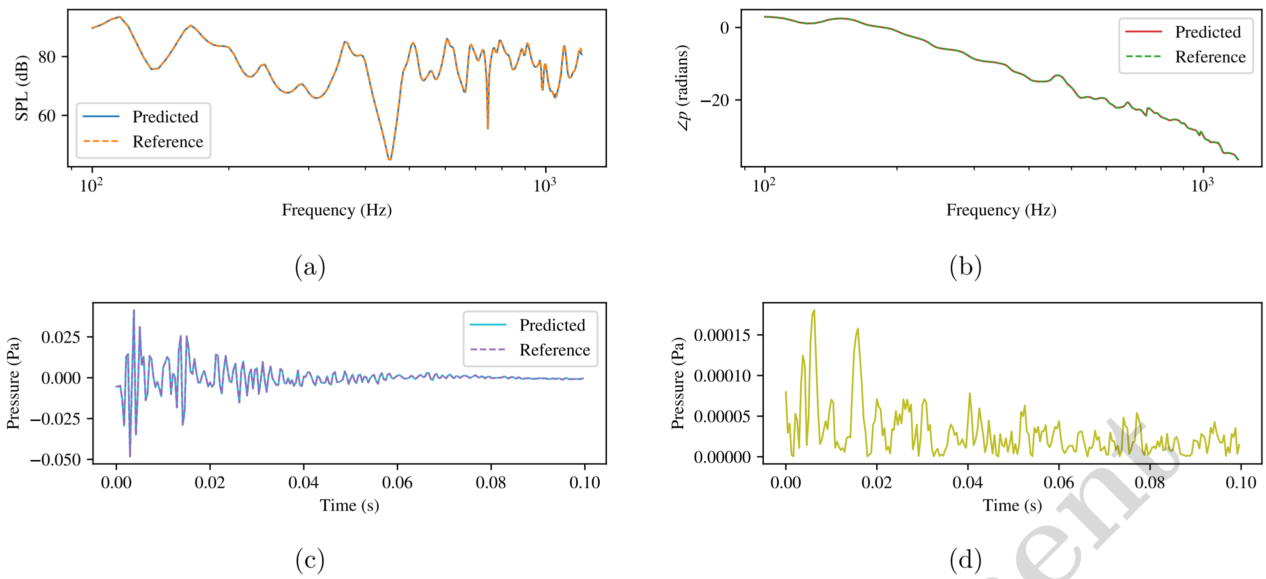


**Figure 3:** Acoustic signal for source at $[0.2, 0.2]$ m and receiver at $[-0.25, -0.4]$ m. Transfer function in terms of (a) SPL and (b) phase. (c) Impulse response. (d) Impulse response error.

put sizes. Moreover, training is completed within a few seconds, clearly outperforming conventional PINNs, which are often reported to require several minutes or even hours [27].

**Table 1:** Computational cost for acoustics simulation at 1200 Hz. $N_{param}$ denotes the total number of float trainable parameters, whereas $N_{train}$ is the total input size given by the space coordinates and quadrature points.

| Time (s) | Memory (GB) | $N_{param}$ | $N_{train}$ |
|---|---|---|---|
| 16 | 0.77 | 4086 | $1000 \times 20$ |

We additionally validate the quality of the approximation across a larger spectrum of frequencies. In particular, we consider a sweep from 100 Hz up to 1200 Hz, with intervals of 5 Hz. Neural networks are re-initialized at each step, and hyperparameters are set as above. Note that transfer learning could be used to accelerate the learning over the sweep, however we aim to investigate results independently on each frequency.

The acoustic pressure is evaluated at a receiver position approximately on the opposite side of the room with respect to the source at the coordinates $[-0.25, -0.4]$ m. Phase and sound pressure level (SPL) are obtained, where the latter is defined as $20 \log_{10}(|p|/p_{ref})$ with $p_{ref} = 2 \times 10^{-5}$ Pa, and both are displayed in Figure 3.

The simulated and reference results show a great fit in the entire range from 100 Hz to 1200 Hz, with a perfect match at the low frequencies. Specifically, the relative error remains below $10^{-3}$ and close to $10^{-4}$ in this regime. In addition, SPL dips are also perfectly reconstructed. On the other hand, small deviations between the two curves start to appear from 1000 Hz and additional tests indicate that errors become significant beyond 1500 Hz, requiring different network setups.

The considerable contrast of the method accuracy between low and high frequencies can be readily examined. Specifically, the Vekua operator from Equation (2) converges to the identity for $k \to 0$, as expected from the fact that the Helmholtz problem reduces to the Laplace equation. Consequently, the learning strategy transforms to the simpler application of HNNs alone. In contrast, high frequencies lead to a more involved operator characterization which significantly complicates the learning mechanism. In particular, the term $J_0(kt|z|)$ becomes highly oscillatory and requires a larger number of quadrature nodes. On the other hand, using an excessive number of quadrature nodes can complicate the loss landscape, increasing the tendency of convergence to local minima. The number of training points also increases at high frequencies as a minimum PPW is required, entailing a larger computational cost and further complicating the learning convergence.

## 4. Conclusions

A new machine learning framework using HNNs and Vekua operators has been proposed to learn functions under the constraint of solving the Helmholtz problem. As verified in the tests, imposing the physics in a hard manner entails a significant learning speed-up and reduced memory requirements. The method is particularly effective at low frequencies, where the associated errors are found to be negligible and the underlying rationale has been established. In this regime, some approaches such as geometric acoustics (GA) are typically unreliable, while others, such as the finite element method (FEM), are often computationally expensive. Consequently, the introduced strategy offers promising potential for the efficient and accurate learning of sound fields.

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